On the MGF and BER of Linear Diversity Schemes in Nakagami Fading Channels with Arbitrary Parameters

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Abstract—In this paper, we explore the relationship between the Lauricella hypergeometric functions \( F_a, F_b, \) and \( F_0 \) and the performances of linear diversity combining schemes in Nakagami fading channels with arbitrary parameters. We consider maximal ratio combining, selection combining, and equal gain combining of an arbitrary number of independent Nakagami faded diversity branches. Specifically, we show that the moment generating function and the average bit error rate of these combining schemes can all be expressed in terms of the Lauricella hypergeometric functions.

Index Terms—Lauricella Hypergeometric Function, Nakagami Fading Channel, Diversity Combining.

I. INTRODUCTION

Multi-path fading seriously degrades the performances of digital wireless communication systems. Therefore, the performance analysis of such systems has been extensively done by many researchers. To this end, many statistical distributions are available in the literature to model the fading process in order to predict the system performance. Of these, the Nakagami-\( m \) distribution [1] has attracted considerable interest as a fading channel model. This is because it provides a very good fit to measured data in a variety of fading environments.

It is well known that diversity reception can significantly improve the performances of digital wireless communication systems in the presence of fading and co-channel interference. The performances of commonly used linear diversity combining techniques such as maximal ratio combining (MRC), equal gain combining (EGC), and selection combining (SC) have also been studied extensively for a variety of fading environments, including the Nakagami fading channel [2]. In MRC, the received signals on the diversity branches are co-phased and weighted (that is, both the amplitude and phase are used) to obtain the output of the combiner. In the EGC systems, on the other hand, the received signals are co-phased and combined with equal weights (that is, only the signal phase is used) to obtain the combiner output. Finally, in SC, only the branch with the largest instantaneous amplitude is connected to the output of the combiner.

The average bit error rate (BER) performance of a binary digital receiver operating in a Nakagami-\( m \) fading channel with arbitrary real fading parameters is usually expressed in terms of the Gauss hypergeometric function, \( _2 F_1(a, b; c; x) \). This function is analytic for all the parameters and can be easily computed since it is in the library of several commonly available computer software packages such as Matlab and Mathematica. When \( M \)-ary signaling is employed or when linear diversity reception is used to combat Nakagami fading, the average error rate can usually be expressed in terms of the Lauricella functions. For example, Shin and Lee [3] express the average symbol error rate (SER) in terms of the Lauricella functions, \( F_a(n) \). Recently, the authors also expressed the average BER for MRC systems in a Nakagami fading environment in terms of this function [4]. On the other hand, Ugweje [5] and Annavajjala, et al. [6] derive the moment generating function (MGF) of the output SNR and the average BER of a coded SC system, respectively, in terms of the Lauricella function \( F_a(n) \). Furthermore, in this paper we show for EGC, the MGF of the output SNR and the average BER can be expressed in terms of the Lauricella function \( F_a(n) \). It should be noted that these multivariate hypergeometric functions all reduce to the Gauss hypergeometric function when the order \( n = 1 \) [7]. The main aim of this paper is to show that, in fact, several important performance measures (such as MGF of output SNR and average BER) of the three main linear diversity combining schemes (MRC, SC, EGC) can be expressed in terms of the Lauricella functions.

II. MGF OF OUTPUT SNR

In a Nakagami-\( m \) fading channel, the PDF of the received SNR, \( \gamma_i \), on each diversity branch is given by

\[
 f_{\gamma_i}(\gamma_i) = \left( \frac{m_i}{\bar{\gamma}_i} \right)^{m_i} \frac{1}{\Gamma(m_i)} \gamma_i^{m_i-1} \exp \left[ -\frac{m_i\gamma_i}{\bar{\gamma}_i} \right], \quad \gamma_i \geq 0
\]  

where \( m_i \geq 1/2 \) is a parameter that determines the severity of the fading and \( \bar{\gamma}_i \) is the average SNR on the \( i \)-th diversity branch [2].

Next, we consider the performances of three commonly employed linear combining schemes, namely; MRC, EGC, and SC, operating in a Nakagami fading channel with arbitrary fading parameters on the diversity branches. We assume throughout the paper that the diversity branches are sufficiently separated from each other for them to be statistically independent.
A. Maximal Ratio Combining

It is well known that the output SNR in an MRC scheme is the sum of the SNR on each diversity branch; that is,

\[ \gamma_{\text{MRC}} = \sum_{i=1}^{N} \gamma_i. \]  

(2)

The MGF of \( \gamma_{\text{MRC}} \) may be expressed as [2], [4]

\[ M_{\gamma_{\text{MRC}}}(-t) = \prod_{i=1}^{N} \left( \frac{m_i}{m_i + \beta \gamma_i t} \right)^{m_i} \times F_{\gamma_i}^{(N)} \left( 0, m_1, \ldots, m_N; \beta; \frac{m_i}{m_i + \beta \gamma_i t} \right) \]  

(3)

where we have used the fact that \( F_{\gamma_i}^{(N)} (0, \beta, \ldots, \beta; c; x_1, \ldots, x_N) = 1 \) [7]. It follows that the probability density function (PDF) of the output SNR is given by taking the inverse Laplace transform of (3). Therefore, the PDF of the SNR at the MRC output is given by [4]

\[ f_{\gamma_{\text{MRC}}} (\gamma) = \frac{1}{2\pi t} \int \frac{e^{\gamma t} M_{\gamma_{\text{MRC}}}(-t)}{\gamma} dt \]

\[ = \frac{1}{\Gamma \left( \sum_{i=1}^{N} m_i \right)} \prod_{i=1}^{N} \left[ \frac{m_i}{m_i + \beta \gamma_i t} \right]^{m_i} \frac{1}{n_i!} \sum_{a_n} \frac{\prod_{i=1}^{N} \left( m_i \right)_{a_n} \left( -m_i \gamma_i t \right)^{n_i} \Gamma \left( a_n + m_i \right)}{\Gamma \left( a_n \right) \Gamma \left( m_i \right) \Gamma \left( m_i + 1 \right)} \]  

(4)

where the Pochhammer symbol is defined as \( (a)_n = \frac{\Gamma (a + n)}{\Gamma (a)} \), with \( (a)_0 = 1 \) [8].

B. Selection Combining

In a selection combining scheme, only the branch with the largest instantaneous SNR is connected to the output of the combiner. Therefore, the output SNR is the maximum SNR on the diversity branches;

\[ \gamma_{\text{SC}} = \max \{ \gamma_1, \gamma_2, \ldots, \gamma_N \}. \]  

(5)

The cumulative distribution function (CDF) of the SNR at the output of the SC operating in a Nakagami fading channel with independent fading along the diversity branches is given by [5]

\[ F_{\gamma_{\text{SC}}} (\gamma) = \prod_{i=1}^{N} \frac{1}{\Gamma \left( m_i \right)} \mathcal{G} \left( m_i, \frac{m_i \gamma_i}{\beta \gamma_i} \right) \]  

(6)

where \( \mathcal{G} (\alpha, z) = \int_0^z x^{\alpha-1} e^{-x} \, dx \) is the incomplete gamma function, which may also be written in terms of the confluent hypergeometric function [8, eq. (6.5.12)]. Thus the CDF is given by

\[ F_{\gamma_{\text{SC}}} (\gamma) = \left[ \sum_{m_i} \prod_{i=1}^{N} \left( \frac{m_i \gamma_i}{\beta \gamma_i} \right)^{m_i} \frac{1}{m_i \Gamma \left( m_i \right)} F_{\gamma_i} \left( m_i, m_i + 1; -m_i \gamma_i \right) \right]^{N} \]  

(7)

The corresponding PDF is obtained by performing the term-by-term differentiation to give

\[ f_{\gamma_{\text{SC}}} (\gamma) = \frac{dF_{\gamma_{\text{SC}}} (\gamma)}{d\gamma} \]  

\[ = \left[ \sum_{m_i} \prod_{i=1}^{N} \left( \frac{m_i \gamma_i}{\beta \gamma_i} \right)^{m_i} \frac{1}{m_i \Gamma \left( m_i \right)} F_{\gamma_i} \left( m_i, m_i + 1; -m_i \gamma_i \right) \right] \times \sum_{a_n} \frac{\prod_{i=1}^{N} \left( m_i \right)_{a_n} \left( -m_i \gamma_i \right)^{n_i} \Gamma \left( a_n + m_i \right)}{\Gamma \left( a_n \right) \Gamma \left( m_i \right) \Gamma \left( m_i + 1 \right)} \]  

(8)

The MGF at the output of the SC may also be obtained in terms of the Lauricella function by performing the term-by-term integration of (8), as

\[ M_{\gamma_{\text{SC}}} (-t) = \int_0^\infty e^{-\gamma t} f_{\gamma_{\text{SC}}} (\gamma) \, d\gamma \]  

\[ = \Gamma \left( \sum_{i=1}^{N} m_i \right) \prod_{i=1}^{N} \left[ \frac{1}{\Gamma \left( m_i + 1 \right)} \left( \frac{m_i \gamma_i}{\beta \gamma_i} \right)^{m_i} \right] \times \sum_{a_n} \frac{\prod_{i=1}^{N} \left( m_i \right)_{a_n} \left( -m_i \gamma_i \right)^{n_i} \Gamma \left( a_n + m_i \right)}{\Gamma \left( a_n \right) \Gamma \left( m_i \right) \Gamma \left( m_i + 1 \right)} \]  

\[ = \Gamma \left( \sum_{i=1}^{N} m_i \right) \prod_{i=1}^{N} \left[ \frac{1}{\Gamma \left( m_i + 1 \right)} \left( m_i \gamma_i \right)^{m_i} \right] \times \frac{1}{\sum_{i=1}^{N} m_i} \sum_{a_n} \frac{\prod_{i=1}^{N} \left( m_i \right)_{a_n} \left( -m_i \gamma_i \right)^{n_i} \Gamma \left( a_n + m_i \right)}{\Gamma \left( a_n \right) \Gamma \left( m_i \right) \Gamma \left( m_i + 1 \right)} \]  

(9)

where we have used [7, eq. A.2.1] to obtain the last line in (9), which satisfies the convergence criteria for \( F_{\gamma_i} \) [7].

C. Equal Gain Combining

In EGC systems, the received signals on the diversity branches are co-phased and combined with equal weights to obtain the combiner output. It can be shown that the output SNR in an equal gain combining scheme is given by [10]

\[ \gamma_{\text{EGC}} = \frac{1}{N} \left( \sum_{i=1}^{N} \sqrt{\gamma_i} \right)^2. \]  

(10)

The MGF of \( R_{\gamma} = \sqrt{\gamma} \) is given by

\[ M_{R_{\gamma}} (-t) = \frac{2}{\Gamma \left( m_i \right)} \left( \frac{m_i \gamma_i}{\beta \gamma_i} \right)^{m_i} \int_0^\infty e^{-x} x^{m_i-1} \exp \left( \frac{m_i x^2}{\beta \gamma_i} \right) \, dx. \]  

(11)

The integral in (11) may be expressed in closed form in terms of the parabolic cylinder function [9], [10]. However, here, we use
the series expansion for the exponential function, 
\[ e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \]
and using the multiplication theorem for the gamma function [8, eq. (8.335)]
\[ \Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma(x + \frac{1}{2}) \],
we rewrite (11) as
\[ M_R(-t) = \frac{\Gamma(m + 1/2)}{\sqrt{\pi}} \left( \frac{m}{\overline{\gamma}^2} \right)^m
\times \sum_{k=0}^{\infty} \binom{m}{k} \gamma^k \left( \frac{-m}{(\overline{\gamma}^2)} \right)^k \]  
(13)
The series in (13) is recognized as the confluent hypergeometric function of the second kind [7], [8]. Since the diversity branches are independent, the MGF of \( R = \sum_{i=1}^{N} R_i \) is given by computing the \( N \) product of (13) as
\[ M_R(-t) = \prod_{i=1}^{N} \left[ \frac{\Gamma(m_i + 1/2)}{\sqrt{\pi}} \left( \frac{m_i}{\overline{\gamma}^2} \right)^{m_i} \right]
\times \sum_{n_1, n_2, \ldots, n_N = 0} \binom{m_1}{n_1} \gamma^{n_1} \binom{m_2}{n_2} \gamma^{n_2} \cdots \binom{m_N}{n_N} \gamma^{n_N} \]
\[ \prod_{i=1}^{N} \frac{1}{n_i!} \]  
(14)
The PDF of \( R \) is then obtained by evaluating the inverse Laplace transform of (14), to give
\[ f_R(r) = \sqrt{\pi} \left[ \prod_{i=1}^{N} \Gamma \left( \frac{1}{2} + m_i \right) \left( \frac{m_i}{\overline{\gamma}^2} \right)^{m_i} \right]
\times \sum_{n_1, n_2, \ldots, n_N = 0} \binom{m_1}{n_1} \gamma^{n_1} \binom{m_2}{n_2} \gamma^{n_2} \cdots \binom{m_N}{n_N} \gamma^{n_N} \]
\[ \prod_{i=1}^{N} \Gamma \left( \frac{1}{2} + \sum_{j=1}^{N} m_j \right) \]
\[ \prod_{i=1}^{N} \frac{1}{n_i!} \]  
(15)
It follows from (10) that the PDF of the SNR at the output of the EGC is given by
\[ f_{SNR}(\gamma) = \frac{N f_R(\sqrt{N} \gamma)}{\sqrt{N} \gamma} u(\gamma) \]
\[ = \sqrt{\pi} \left[ \prod_{i=1}^{N} \Gamma \left( \frac{1}{2} + m_i \right) \left( \frac{m_i}{\overline{\gamma}^2} \right)^{m_i} \right]
\times \sum_{n_1, n_2, \ldots, n_N = 0} \binom{m_1}{n_1} \gamma^{n_1} \binom{m_2}{n_2} \gamma^{n_2} \cdots \binom{m_N}{n_N} \gamma^{n_N} \]
\[ \prod_{i=1}^{N} \frac{1}{n_i!} \]  
(16)
Thus the PDF of the output SNR in EGC systems is expressed as a multivariate hypergeometric function [7]. The MGF of the output SNR for EGC is then obtained from (16) as
\[ M_{\gamma_{EGC}}(-t) = \int_0^\infty e^{-\gamma t} f_{\gamma_{EGC}}(\gamma) d\gamma \]
\[ = \sqrt{\pi} \left[ \prod_{i=1}^{N} \Gamma \left( \frac{1}{2} + m_i \right) \left( \frac{m_i}{\overline{\gamma}^2} \right)^{m_i} \frac{1}{\sqrt{\pi}} \right]
\times \prod_{i=1}^{N} \Gamma \left( \frac{1}{2} + \sum_{j=1}^{N} m_j \right)
\times \sum_{m_1, m_N = 1, \ldots, \gamma; m_1 + \frac{1}{2}, \ldots, m_N + \frac{1}{2}, \ldots} \sum_{i=1}^{N} \frac{m_i}{\overline{\gamma}^2} \cdots \frac{m_N}{\overline{\gamma}^2 n} \]
(17)
in terms of Lauricella function of the second kind [7], [11].

III. ERROR RATE ANALYSIS
The average BER is given by
\[ P_e = \int_0^\infty P_e(\gamma) f_{\gamma_{EGC}}(\gamma) d\gamma \]
(18)
where \( P_e(\gamma) = \frac{\Gamma(b, a \gamma)}{2 \Gamma(b)} \) is the conditional BER in an AWGN channel and \( f_{\gamma_{EGC}}(\gamma) \) is the PDF of the output SNR, which depends on the type of diversity combining scheme employed [2]. The parameter \( a \) depends on the type of binary modulation (\( a = 1 \) for binary phase shift keying and \( a = 1/2 \) for binary frequency shift keying), while the parameter \( b \) depends on the type of demodulation (\( b = 1 \) for noncoherent FSK or differentially coherent PSK and \( b = 1/2 \) for coherent PSK or FSK).

A. Maximal Ratio Combining
The average BER for MRC is obtained by substituting (4) in (18) to obtain
\[ P_{e_{MRC}} = \frac{1}{2 \Gamma(b)} \Gamma \left( \sum_{i=1}^{N} m_i \right) \left[ \prod_{i=1}^{N} \left( \frac{m_i}{\overline{\gamma}^2} \right)^{m_i} \right]
\times \sum_{n_1, n_2, \ldots, n_N = 0} \binom{m_1}{n_1} \gamma^{n_1} \binom{m_2}{n_2} \gamma^{n_2} \cdots \binom{m_N}{n_N} \gamma^{n_N} \]
\[ \prod_{i=1}^{N} \frac{1}{n_i!} \]  
(19)
Thus the average BER for both coherent and noncoherent differentially coherent binary modulations may be expressed in terms of the Lauricella function. Using the transformation in [8, eq. A.2.1.1], we may rewrite (19) as

$$\tilde{P}_{E_{MC}} = \frac{\Gamma \left( b + \sum_{i=1}^{N} m_i \right)}{2\Gamma (b)} \prod_{i=1}^{N} \left( \frac{m_i}{m_i + a\gamma} \right)^{m_i} \times F_D^{(\infty)} \left( 1 - b; m_1, ..., m_N; 1 + \sum_{i=1}^{N} m_i; m_i + 1, ..., m_N + 1; -m_i, ..., -m_N \right),$$

which satisfies the convergence criteria for $F_D(\ldots)$ [7]. Note that in the special case when the diversity branches are independent and identically distributed, we may use the following reduction formula of the Lauricella function [12, p. 61],

$$F_D^{(\infty)} (\alpha, \beta, ..., \beta; c; x, ..., x) = z F_1 \left( m \sum_{i=1}^{N} \beta_i c_i x \right),$$

show that (20) can be easily simplified to give the more familiar result

$$\tilde{P}_{E_{MC}} = \frac{\Gamma (Nm + b)}{2\Gamma (b)\Gamma (Nm + 1)} \left( \frac{Nm}{Nm + a\gamma} \right)^{Nm} \times \left( \frac{a\gamma}{Nm + a\gamma} \right)^{\gamma} F_D \left( 1, Nm + b, Nm + 1; \frac{Nm}{Nm + a\gamma} \right),$$

which agrees with [2].

### B. Selection Combining

In this case, we substitute (8) in (18) to obtain

$$\tilde{P}_{E_{SC}} = \frac{\sum_{i=1}^{N} m_i}{2\Gamma (b)} \prod_{i=1}^{N} \left( \frac{m_i}{m_i + 1} \right)^{m_i} \times \sum_{n_1, n_2, ..., n_N = 0} \frac{\prod_{i=1}^{N} \left( m_i \right)_{n_i} \left( -m_i - n_i \right) \cdots \prod_{i=1}^{N} \left( m_i + 1 \right)^{n_i} \gamma^{n_i} y}{\prod_{i=1}^{N} \left( m_i + 1 \right)_{n_i}} \int_0^\infty \Gamma (b, a\gamma) d\gamma.$$

With the help of [8, (6.5.37)], and using the fact that

$$\frac{a}{a + n} = \frac{(a)_n}{(a)_n},$$

(23) simplifies to

$$\tilde{P}_{E_{SC}} = \frac{\Gamma (b + \sum_{i=1}^{N} m_i)}{2\Gamma (b)} \prod_{i=1}^{N} \left( \frac{m_i}{m_i + a\gamma} \right)^{m_i} \times \sum_{n_1, n_2, ..., n_N = 0} \frac{\prod_{i=1}^{N} \left( m_i \right)_{n_i} \left( -m_i - n_i \right) \cdots \prod_{i=1}^{N} \left( m_i + 1 \right)^{n_i} \gamma^{n_i} y}{\prod_{i=1}^{N} \left( m_i + 1 \right)_{n_i}} \int_0^\infty \Gamma (b, a\gamma) d\gamma.$$

Using the transformation in [7, eq. A.2.1.2], we may rewrite (24) as

$$F_{E_{SC}}^{(\infty)} = \frac{\Gamma (b + \sum_{i=1}^{N} m_i)}{2\Gamma (b)} \prod_{i=1}^{N} \left( \frac{m_i}{m_i + 1} \right)^{m_i} \times \int_0^\infty \Gamma (b, a\gamma) d\gamma.$$
\[
\sqrt{\pi} \Gamma \left( b + \sum_{i=1}^N m_i \right) \left[ \prod_{i=1}^N \Gamma \left( \frac{1}{2} + m_i \right) \right] \frac{\left( m_i / (a \gamma) \right)^{m_i}}{\sqrt{\pi}} \\
= 2\Gamma(b) \Gamma \left( 1 + \sum_{i=1}^N m_i \right) \Gamma \left( \frac{1}{2} + \sum_{i=1}^N m_i \right) \\
\sum_{n_1, n_2, \ldots, n_N = 0} \left( \frac{m_1 + \sum_{i=1}^N m_i}{n_1 + \ldots + n_N} \right) \frac{1}{\sqrt{\pi}} \left( \frac{1}{2} + \sum_{i=1}^N \frac{m_i}{n_i} \right) \frac{(-m_i / (a \gamma))^{n_i}}{n_i!}
\]
\[
\sqrt{\pi} \prod_{i=1}^N \Gamma \left( \frac{1}{2} + m_i \right) \frac{\left( m_i / (a \gamma) \right)^{m_i}}{\sqrt{\pi}} \\
= 2\Gamma(b) \Gamma \left( \frac{b}{2} + \sum_{i=1}^N m_i \right) \\
\times f_{\tilde{b}}^{(N)} \left( m_1, \ldots, m_N; \frac{1}{2} + \sum_{i=1}^N m_i; \frac{-m_i}{a \gamma}, \ldots, \frac{-m_N}{a \gamma} \right)
\]
(28)

where \( \tilde{b} = \left\{ \begin{array}{ll} 1, & \text{if } b = 1/2 \\ 2, & \text{if } b = 1 \end{array} \right. \). We note that the result in (28) can be easily expressed in terms of the simplex integrals [13] using [7, eq. 2.4.6], so that the average BER for EGC may also be expressed in the forms given in [14]. However, we are unable to find a transformation similar to [8, eq. A.2.1.2 or eq. A.2.2.1] that guarantees the convergence of the Lauricella function \( F_{\tilde{b}}(\ldots) \) in (28). Therefore in the numerical analysis, we restrict the fading parameter on each diversity branch to the range \( 1/2 < m_i < a \gamma \).

IV. NUMERICAL RESULTS

In this section, we present some selected numerical results to illustrate the use of the Lauricella functions to evaluate the performances of linear diversity combining schemes operating in a Nakagami fading environment with arbitrary fading and power parameters. Numerical results are plotted for a four-branch diversity system with arbitrary Nakagami fading parameters \( \{m_1, m_2, m_3, m_4\} = \{2.0, 1.5, 1.0, 0.5\} \) and for two cases of exponential power profile with decay rate \( \delta \), i.e., \( \gamma = \gamma_0 \exp[-\delta(i-1)] \), \( i = 1, \ldots, 4 \). The average BER for all combining schemes versus the average SNR of the first diversity branch is plotted in Fig. 1. As expected, MRC offers the biggest diversity gain, followed by EGC and SC, while the system with balanced average SNRs (\( \delta = 0 \)) outperforms the one with unbalanced average SNRs (\( \delta = 0.5 \)). Finally, note that the MGF based approach can be used to compute the SER of these diversity combining schemes for M-ary modulations.

V. CONCLUSION

In this paper, we have studied the performances of maximal ratio combining, selection combining, and equal gain combining systems operating over a flat-fading Nakagami-\( m \) channel with independent fading along the diversity branches. The MGF of the output SNR and the BER of binary modulations were derived in terms of the well known Lauricella functions.

REFERENCES