Longitudinal Sampling and Aliasing in Spiral CT

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Abstract—Although analyses of in-plane aliasing have been done for conventional computed tomography (CT) images, longitudinal aliasing in spiral CT has not been properly investigated. We propose a mathematical model of the three-dimensional (3-D) sampling scheme in spiral CT and analyze its effects on longitudinal aliasing. We investigated longitudinal aliasing as a function of the helical-interpolation algorithm, pitch, and reconstruction interval using CT simulations and actual phantom scans. Our model predicts, and we verified, that for a radially uniform object at the isocenter, the spiral sampling scheme results in spatially varying cancellation of the aliased spectral islands which, in turn, results in spatially varying longitudinal aliasing. The aliasing is minimal at the scanner isocenter, but worsens with distance from it and rapidly becomes significant. Our results agree with published results observed at the isocenter of the scanner and further provide new insight into the aliasing conditions at off-isocenter locations with respect to the pitch, interpolation algorithm, and reconstruction interval. We conclude that longitudinal aliasing at off-isocenter locations can be significant, and that its magnitude and effects cannot be predicted by measurements made only at the scanner isocenter.

Index Terms—Longitudinal aliasing, sampling, spiral CT.

NOMENCLATURE

We present in this section a summary of the parameters and notations used in the derivation of the mathematical model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\beta$</td>
<td>Projection-view angle.</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Projection-ray angle.</td>
</tr>
<tr>
<td>$\gamma'$</td>
<td>Projection-ray angle passing through point $(r, \phi)$.</td>
</tr>
<tr>
<td>$\gamma_m$</td>
<td>Half a fan angle.</td>
</tr>
<tr>
<td>$d_s$</td>
<td>Sampling period in conventional CT.</td>
</tr>
<tr>
<td>$d_{s1}$</td>
<td>Sampling period in spiral CT.</td>
</tr>
<tr>
<td>$d_{s2}$</td>
<td>Reconstruction interval in spiral CT.</td>
</tr>
<tr>
<td>$M$</td>
<td>Number of projection views acquired per $2\pi$ rotation.</td>
</tr>
<tr>
<td>$M'$</td>
<td>Number of projection view acquired per $2\pi + 4\gamma_m$ rotation.</td>
</tr>
<tr>
<td>$D_0$</td>
<td>Source to isocenter distance.</td>
</tr>
<tr>
<td>$D$</td>
<td>Detector collimator width.</td>
</tr>
<tr>
<td>$n_s$</td>
<td>Number of reconstructed sections per collimator width in spiral CT.</td>
</tr>
<tr>
<td>$R$</td>
<td>Number of revolutions per scan (in spiral CT).</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of reconstructed transaxial images.</td>
</tr>
<tr>
<td>$L$</td>
<td>Resampling factor $L = d_{s2}/d_{s1}$.</td>
</tr>
<tr>
<td>$f(r, \phi)(z)$</td>
<td>Continuous 1-D function of $z$ reconstructed at $(r, \phi)$.</td>
</tr>
<tr>
<td>$\tilde{f}_{s}(r, \phi)(z)$</td>
<td>A sampled version of the function $f(r, \phi)(z)$ reconstructed from noninterpolated data.</td>
</tr>
<tr>
<td>$f^h_{s}(r, \phi)(z)$</td>
<td>Same as above, but reconstructed from interpolated data, as in spiral CT.</td>
</tr>
<tr>
<td>$p_{\beta}(\gamma), p_{\beta}(\gamma, z)$</td>
<td>2-D and 3-D projection functions.</td>
</tr>
<tr>
<td>$g(\gamma)$</td>
<td>Tomographic reconstruction filter for fan-beam projections.</td>
</tr>
<tr>
<td>$a(z)$</td>
<td>The longitudinal area sampling kernel.</td>
</tr>
<tr>
<td>$L(r, \phi, \beta)$</td>
<td>Fan-beam weights.</td>
</tr>
<tr>
<td>$q_{\beta}(\gamma)$</td>
<td>Filtered-projection function at $\beta$ as a function of $\gamma$.</td>
</tr>
<tr>
<td>$q_{\beta}(\gamma, z)$</td>
<td>Filtered-projection function at $\beta$ as a function of $z$ and $\gamma$.</td>
</tr>
<tr>
<td>$q_{s, \beta}(\gamma, z)$</td>
<td>A projection sequence sampled from $q_{\beta}(\gamma, z)$; samples consist of noninterpolated data.</td>
</tr>
<tr>
<td>$d_{s, \beta}(\gamma, z)$</td>
<td>Same as above, but samples consist of interpolated data, as in spiral CT.</td>
</tr>
<tr>
<td>$h(\gamma, z)$</td>
<td>Helical-interpolation kernel.</td>
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Upper case functions correspond to the Fourier transforms of the corresponding lower case functions.

I. INTRODUCTION

SPIRAL CT has revolutionized conventional (or step-and-shoot) computed tomography by taking it from a two-dimensional (2-D) imaging modality to a fully three-dimensional (3-D) modality. The data-acquisition process, where the path of the source traces out a spiral trajectory along the scan axis, is 3-D and allows a larger volume to be scanned in a shorter amount of time. This form of data acquisition has both advantages and disadvantages. The full projection data set required to reconstruct a transaxial image at any longitudinal position is incomplete and needs to be interpolated from the available volumetric data, resulting in a broadened slice sensitivity profile (SSP). However, since there is no predetermined location for reconstruction, the sections...
can be reconstructed with overlap and at arbitrarily chosen intervals.

Many spiral-CT reconstruction algorithms have been proposed over the years [1]–[3]. The main difference between these algorithms lies in the method with which the projection data necessary for reconstructing transaxial images are interpolated. These algorithms aim at obtaining a good tradeoff between the SSP broadening, image noise, and reconstruction artifacts [1], [3]–[7]. Methods of measurement and analysis of the SSP in spiral CT have also been studied extensively [4], [5], [8]–[10]. However, most of the available literature deals only with measurements of the SSP at the isocenter and analyzes the longitudinal profile in the continuous domain. Analysis in the continuous domain ignores the aliasing that is inherent in discrete sampling systems such as CT. Although some studies have looked at the spiral artifacts that arise in the longitudinal direction [11], as well as the spatial dependence of SSP and noise in spiral CT [8], [12], they do not take into account the longitudinal sampling and the possible role that aliasing plays in these observed effects.

Given the 3-D nature of the data provided by spiral CT and the recent advances in volumetric data visualization, there is an increasing demand for isotropic resolution in the reconstructed data volumes [13]. Therefore, a clear understanding of the 3-D sampling scheme in spiral CT and its effects on the volumetric resolution is essential to the development of techniques for obtaining isotropic resolution.

In this paper we analyze the longitudinal sampling and reconstruction unique to spiral CT and discuss the spatially varying longitudinal aliasing that results from it. We limit our investigation to longitudinal aliasing and only briefly discuss its implications on longitudinal resolution, which is the subject of ongoing research. Section II presents the relevant background and our analysis of the longitudinal sampling. In particular, our theory predicts, and we verify that, for the special case of a radially uniform object concentric with the scanner isocenter, longitudinal aliasing is negligible at the isocenter, but worsens substantially with distance from it. Section III describes the experiments and simulations conducted to support our theory, and Section IV presents the results from these experiments. Finally, in Section V, we discuss the implications of our findings on the interpretation of the longitudinal aliasing in spiral CT and in Section VI we present our conclusions.

II. BACKGROUND AND THEORY

A. Continuous Model

In this section we derive the mathematical model using the filtered backprojection equation for fan-beam projections. Fig. 1(a) illustrates the fan-beam projection geometry and the relevant parameters for the reconstruction of a 2-D point \( f(r, \phi) \). Most equations in this paper will be expressed as either 3-D functions in \((r, \phi, z)\), or as one-dimensional (1-D) functions of \( z \) at fixed \((r, \phi)\) positions. This notation facilitates our analysis in the longitudinal direction.

The backprojection equation for fan-beam projections \( p_\beta(\gamma) \) in conventional CT is given by [14]

\[
\int_0^{2\pi} \frac{1}{L^2(r, \phi, \beta)} q_\alpha(\gamma') d\beta
\]

where

\[
q_\alpha(\gamma) = (p_\beta(\gamma) D_0 \cos \gamma) \ast g(\gamma).
\]

\( \gamma' \) is the angle of the fan-beam ray that passes through the point \((r, \phi)\), \( L^2(r, \phi, \beta) \) is the fan-beam weight at \((r, \phi)\) for the projection angle \( \beta \), and \( g(\gamma) \) is the tomographic reconstruction filter for fan-beam geometry. The location of the point \((r, \phi)\) and the projection angle \( \beta \) completely determine \( L(r, \phi, \beta) \) and

\[
L(r, \phi, \beta) = \sqrt{D_0^2 + r \sin(\beta - \phi)^2 + r \cos(\beta - \phi)^2}
\]

and

\[
\gamma = \tan^{-1} \frac{r \cos(\beta - \phi)}{D_0 + r \sin(\beta - \phi)}
\]

where \( L^2(0, \phi, \beta) = D_0 \). We can extend (1) for the reconstruction of a 3-D point \( f(r, \phi, z) \) from fan-beam projection functions \( p_\beta(\gamma, z) \) as follows:

\[
f(r, \phi, z) = f_{(r, \phi)}(z) = \int_0^{2\pi} \frac{1}{L^2(r, \phi, \beta)} q_\alpha(\gamma', z) d\beta
\]

where

\[
q_\alpha(\gamma, z) = (p_\beta(\gamma, z) D_0 \cos \gamma) \ast g(\gamma).
\]

The projection function \( p_\beta(\gamma, z) \), evaluated at fixed \( \gamma \) values, is a 1-D projection of the object taken at \((\beta, \gamma)\) as
a function of \( z \), as shown in Fig. 1(b). The reconstructed function \( f(\gamma, z) \) is a line (or profile) in \( z \), located at \( (\gamma, \phi) \).

This way of representing the data does not reflect the manner in which the data are acquired, but it presents the projection data as functions of a longitudinal variable \( z \), thus facilitating the analysis of the 3-D volume along the longitudinal axis.

B. Discrete Model

In this section we introduce the sampled version of the model presented above. Since the sampling in conventional CT is well understood, the comparison between these two sampling schemes will help illustrate the unique sampling paradigm in spiral CT.

1) Sampling and Reconstruction in Conventional CT: In conventional CT using fan-beam projections the data acquired consists of projections collected over a \( 2\pi \) interval, where each projection in turn consists of rays that span a specific fan angle.

The full set of projection data required to reconstruct an image can be represented by a plane in \((\beta, \gamma)\), oriented perpendicular to the longitudinal axis as illustrated in Fig. 2(a). These sets of projection data are acquired at discrete table positions; a process equivalent to sampling along the longitudinal axis. As shown in Fig. 2(a), the spacing between the projection data planes \( d_c \) gives the longitudinal sampling period in conventional CT. Furthermore, we can form sampled sequences in \( z \) at each projection ray \((\beta, \gamma)\) and reconstruct a line \( f(\gamma, z) \) from the appropriate projection sequences.

The reconstruction of a line at \((\gamma, \phi)\) in the 3-D volume from the sampled projection sequences \( q_{\beta, \gamma}(\gamma, z) \) [shown in Fig. 2(a)] can be modeled as

\[
f_{\gamma}(\gamma, \phi)(z) \approx \Delta \beta \sum_{m=0}^{M-1} \frac{1}{L^2(\gamma, \phi, \beta)} q_{\beta, \gamma}(\gamma, z)
\]

where

\[
q_{\beta, \gamma}(\gamma, z) = \sum_{n=0}^{N-1} \delta(z - nd_c) q_{\beta, \gamma}(\gamma, z).
\]

\( M \) is the number of \( \beta \) projections acquired per \( 2\pi \) rotation and \( N \) is the number of reconstructed sections.

Equations (7) and (8) form the basic equations for our longitudinal analysis of the sampled 3-D volume. The approximation in (7) comes from the discretization of the continuous equation (5). Our analysis focuses on sequences \( q_{\beta, \gamma}(\gamma, z) \) sampled in \( z \) at discrete \((\beta, \gamma)\) positions and the reconstructed line \( f_{\gamma}(\gamma, \phi)(z) \).

The finite-collimator width along the longitudinal axis acts as a low-pass filter and smooths the acquired projections in the \( z \) direction. We call this low-pass filter the area sampling kernel \( a(z) \). For an ideal collimator of width \( D \), the area sampling kernel is given by

\[
a(z) = \frac{1}{D} \text{rect} \left( \frac{z}{D} \right) = \begin{cases} 1, & \text{for } z \leq \frac{D}{2} \\ 0, & \text{otherwise}. \end{cases}
\]

The low-pass filtered-projection sequence can then be written as

\[
d_{\beta, \gamma}(\gamma, z) = \left( \sum_{n=0}^{N-1} \delta(z - nd_c) q_{\beta, \gamma}(\gamma, z) \ast a(z) \right).
\]

The corresponding longitudinal spectrum is given by

\[
F_z\{d_{\beta, \gamma}(\gamma, z)\} \approx \sum_{n=\infty}^{\infty} \left\{ \frac{1}{d_c} \delta \left( f_z - \frac{n}{d_c} \right) \ast (Q_{\beta, \gamma}(f_z, D) \text{sinc}(f_z D)) \right\}
\]

\[
= \frac{1}{d_c} \sum_{n=\infty}^{\infty} \left[ Q_{\beta, \gamma}(f_z, D) \text{sinc} \left( \frac{n}{d_c} D \right) \right] \times \text{sinc} \left( f_z - \frac{n}{d_c} D \right)
\]

where the approximation accounts for the change from a finite-impulse train to an infinite-impulse train.

The summation in (11) produces the replicated islands in the spectral domain that result from sampling. It should be clear from this equation that inadequate sampling in the longitudinal direction may result in aliasing and loss of resolution. In imaging systems with rectangular aperture functions, the effective system bandwidth \( BW_{\text{eff}} \) is generally defined by the width of the mainlobe of the Fourier transform of the detector aperture function \[14\]. In a CT system, the finite-collimator
width described in (9) is equivalent to the rectangular aperture function. If we assume the corresponding effective bandwidth $BW_{eff}$ a sampling rate $\geq 2$ samples per collimator width will be needed to avoid aliasing in the longitudinal direction. For conventional-CT scans with nonoverlapping sections, no more than one sample per collimator width is acquired, resulting in significant spectral aliasing in the longitudinal direction.

2) Sampling and Reconstruction in Spiral CT: In spiral CT, the continuous data acquisition results in a linear relationship between the projection angle $\beta$ and the scanning axis $z$. The planes representing the acquired projection data now lie at an angle with respect to the $z$ axis. For a given projection angle, the distance between these planes $d_{s1}$ determines the longitudinal sampling period, as shown in Fig. 2(b). Again, we can form sampled sequences in $z$ at each projection ray ($\beta_k$, $\gamma_j$), but in the case of spiral CT the linear relationship between $\beta$ and $z$ introduces a linear shift between the sequences. This shift is proportional to the distance traveled by the source, per projection measurement $d_{s1}/M$, where $M$ is the number of projections acquired per $2\pi$ gantry rotation, as illustrated in Fig. 2(b). We can see from Fig. 2(b) that the projection data required to reconstruct a 2-D section centered at a given location (represented by planes perpendicular to the longitudinal axis) is incomplete and must be interpolated from the available volumetric data. Unlike conventional CT, however, the spacing between reconstructed sections can be chosen arbitrarily. In this section we will show that the arbitrarily chosen section spacing (or the reconstruction interval) represents a resampling period and, unlike the section spacing in conventional CT, does not uniquely determine the longitudinal aliasing in the final 3-D volume.

We model the reconstruction of a point $f_{(r,\phi)}(z)$ from the interpolated filtered-projection functions $q_{\beta_k}^{\phi}(\gamma, z)$ as follows. Using the same example of the fan-beam reconstruction equation we have

$$f_{(r,\phi)}^{\phi}(z) \approx \Delta \beta \sum_{i=0}^{M-1} \frac{1}{L^2(r, \phi, \beta_k)} q_{\beta_k}^{\phi}(\gamma, z)$$ \hspace{1cm} (12)

where

$$q_{\beta_k}^{\phi}(\gamma, z) = \left[ (s_{\beta_k}(\gamma, z) * h(\gamma, z)) D_0 \cos(\gamma) \right] * g(\gamma)$$ \hspace{1cm} (13)

where $*$ indicates a 1-D convolution along the $z$ axis, and $s_{\beta}$ a 1-D convolution along the $\gamma$ axis. $q_{\beta_k}^{\phi}(\gamma, z)$ is a continuous function obtained from convolving the sampled projection sequence $s_{\beta_k}(\gamma, z)$ with the helical-interpolation function $h(\gamma, z)$ in the longitudinal direction, and then applying the tomographic reconstruction filter $g(\gamma)$ to the interpolated projections. $s_{\beta_k}(\gamma, z)$ is the sampled projection sequence defined by

$$s_{\beta_k}(\gamma, z) = f_{\beta_k}(\gamma, z) \sum_{k=0}^{n-1} \delta \left( z - \left( \frac{d_{s1}}{M} \cdot k + (d_{s1}/M) \right) \right)$$ \hspace{1cm} (14)

where $R$ equals the number of revolutions in the acquisition scan, $d_{s1}$ is the distance traveled by the source per $2\pi$ gantry rotation, and $d_{s1}/M$ is the $z$ increment per projection $\beta_k$ [see Fig. 2(b)] and

$$f_{\beta_k}(\gamma, z) = p_{\beta_k}(\gamma, z) * a(z)$$ \hspace{1cm} (15)

$p_{\beta_k}(\gamma, z)$ is the set of projections at $\beta_k$ as functions of $z$ and $\gamma$, and $a(z)$ is the normalized area sampling kernel as defined in (9). The superscripts in the notation $f_{(r,\phi)}^{\phi}(z)$ and $q_{\beta_k}^{\phi}(\gamma, z)$ are used to emphasize the fact that these functions are obtained through interpolation with $h(\gamma, z)$.

The resulting continuous line at $(r, \phi)$, $f_{(r,\phi)}^{\phi}(z)$, can be subsequently sampled at the chosen reconstruction interval $d_{s2}$ to obtain a discrete representation of the 3-D volume

$$f_{(r,\phi)}^{\phi}(z) = \sum_{i=0}^{N-1} \delta (z - i d_{s2}) f_{(r,\phi)}^{\phi}(z)$$

$$= \sum_{i=0}^{N-1} \delta (z - i d_{s2}) \left\{ \Delta \beta \sum_{k=0}^{M-1} \frac{d_{s1}^{\phi}(\gamma, z)}{L^2(r, \phi, \beta_k)} \right\}$$ \hspace{1cm} (16)

Now, let us pause and examine the parallel between the longitudinal sampling equations derived for conventional CT and spiral CT thus far. Equation (16) parallels (7) in that both result in a reconstructed sequence in $z$ at $(r, \phi)$. However, in (16) the sampling function $\sum_{i=0}^{N-1} \delta (z - i d_{s2})$ takes samples from an interpolated function and the reconstruction interval $d_{s2}$ can be chosen arbitrarily, whereas in (7) the samples are taken from the original projection function and the reconstruction interval is fixed at $d_c$ unless further interpolation is applied.

The reconstruction interval $d_{s2}$ has been treated as the sampling interval in spiral CT [9], [15], perhaps because of the parallel that can be observed between (7) and (16). However, we know that it is possible to obtain sections with arbitrary spacing in conventional CT by interpolating between the reconstructed sections. It is clear in that case that the retrospectively chosen reconstruction interval is not the sampling period and there is certainly no improvement in the longitudinal resolution to speak of. The confusion arises because, in spiral-CT, interpolation is required to attain satisfactory image quality and there is an improvement in longitudinal resolution if the reconstruction interval is chosen to be finer than the sampling interval $d_{s1}$. We will see that the key to resolving this apparent paradox lies in the shifts introduced in the sampled sequences in spiral CT. As we will show in the next section, the sampling process in spiral CT introduces an additional phase component in the Fourier spectrum of each projection sequence. During the reconstruction process, the summation of these phase terms has a favorable effect upon aliasing for a radially symmetric object at the isocenter. However, as we shall see, reduction in aliasing lessens and, as a result, aliasing increases with distance from the isocenter.

3) Longitudinal Analysis of Filtered-Backprojection Reconstruction in Spiral CT: The filtered-backprojection equation reconstructs each point $f_{(r,\phi)}$ independently by a summation of the appropriately weighted filtered rays. For our analysis,
we will take these projection sequences in \( p_{\gamma}(\gamma, z) \) and study the sampled line reconstructed from these sequences. We repeat (12) here in its expanded form

\[
\begin{align*}
\tilde{f}^{h}_{(r,\phi)}(z) &= \Delta \beta D_{b} \sum_{r=0}^{M-1} \left\{ \frac{1}{L^2(r, \phi, \beta)} \delta \left( z - \left( \frac{d_{s1}}{M} + k d_{s1} \right) \right) \right. \\
& \left. \cdot \left( \sum_{k=0}^{R-1} x_{2} h(\gamma, z) \cos \gamma \right) \ast_{\gamma} g(\gamma) \right\}.
\end{align*}
\]

Appendix A derives the Fourier transform of (17) as

\[
\mathcal{F}_{z}\{\tilde{f}^{h}_{(r,\phi)}(z)\} = \Delta \beta D_{b} R \sum_{r=0}^{M-1} \left\{ \frac{1}{L^2(r, \phi, \beta)} \right. \\
\left. \cdot \left[ \left( \sum_{k=-\infty}^{\infty} P_{\beta}(\gamma, f_{z} - \frac{k}{d_{s1}}) e^{-j2\pi i \frac{k}{M}} \right) \right. \\
\left. \ast H(\gamma, f_{z}) \cos \gamma \right] \ast_{\gamma} g(\gamma) \right\}.
\]

Equation (18) explicitly shows the replicated islands in the Fourier domain and the phase term \( e^{-j2\pi i \frac{k}{M}} \) which results from the consecutive shifts in the sampled projection sequences \( s_{\beta}(\gamma, z) \). We can further simplify (18) by considering the case of a radially uniform object, i.e., one with variations only in the longitudinal direction centered at the isocenter of the scanner. This assumption allows us to focus on longitudinal effects independent of transaxial variations.

Assuming no shift between the sampled sequences \( s_{\beta}(\gamma, z) \) the projected ray at the isocenter \( (\gamma = 0) \) is independent of the projection angle \( \beta \) for a radially uniform object. Therefore, at the isocenter of this object we have

\[
s_{\beta}(\gamma = 0, z) = s_{\beta, \gamma = 0}(z) = P_{\beta, \gamma = 0}(z) \sum_{k=-\infty}^{\infty} \delta(z - k d_{s1}).
\]

Now, introducing the shift between the sampling functions, we have

\[
P_{\beta, \gamma = 0}(z) \sum_{k=-\infty}^{\infty} \delta(z - \left( \frac{d_{s1}}{M} + k d_{s1} \right))
\]

the Fourier transform of which is

\[
P_{\beta, \gamma = 0}(f_{z}) = \frac{1}{d_{s1}} \sum_{k=-\infty}^{\infty} \delta(f_{z} - \frac{k}{d_{s1}}) e^{-j2\pi i \frac{k}{M}}
\]

\[
= \frac{1}{d_{s1}} \sum_{k=-\infty}^{\infty} P_{\beta, \gamma = 0}(f_{z} - \frac{k}{d_{s1}}) e^{-j2\pi i \frac{k}{M}}.
\]

The Fourier transform of the projections remains independent of the projection angle \( \beta \) and the shift is taken care of by the phase term \( e^{-j2\pi i \frac{k}{M}} \). In addition, at the isocenter the helical-interpolation filter is no longer \( \gamma \) dependent and \( g(\gamma = 0) = g_{0} \). Substituting back into (18) gives us

\[
\mathcal{F}_{z}\{f_{(r,\phi)}(z)\} = \frac{\Delta \beta g_{0} R}{D_{b}} H(f_{z}) \sum_{k=-\infty}^{\infty} P_{\beta, \gamma = 0}(f_{z} - \frac{k}{d_{s1}})
\]

Note that the phase term is the only factor in the equation that retained its angular dependence, and

\[
\sum_{k=-\infty}^{\infty} e^{-j2\pi i \frac{k}{M}} = \begin{cases} 1, & \text{for } k = 0, \pm M, \pm 2M, \ldots, \\ 0, & \text{otherwise}. \end{cases}
\]

In this case, (22) reduces to

\[
\mathcal{F}_{z}\{f(0,0)(z)\} = \frac{\Delta \beta g_{0} R}{D_{b}} H(f_{z}) \sum_{k=-\infty}^{\infty} P_{\beta, \gamma = 0}(f_{z} - \frac{kM}{d_{s1}}),
\]

for \( k = 0, \pm 1, \pm 2, \ldots \).

Equation (24) shows that, for a radially uniform object, the effective-sampling period is given by \( \frac{d_{s1}}{M} \) at the isocenter of the scanner. On a real scanner \( M \) is approximately 1000. At this effective-sampling period, the spectral islands are spread so far apart that there is negligible aliasing.

This special case of aliasing cancellation at the isocenter is important since the phantoms that have been used to measure SSP’s are radially uniform and most of the measurements were done at the isocenter. Fig. 3(a) illustrates the spectral cancellation for the case \( M = 4 \) where the sampling period \( d_{s1} \) is equal to the collimator width \( D \) (sampling is done at half the required Nyquist rate). We can see that whereas the spectrum...
at each projection sequence contained significant aliasing, aliasing in the final spectrum was practically eliminated by virtue of the spectral islands being separated by \( \frac{1}{d_{4x}} \), giving an effective-sampling rate of four times \( 1/d_{4x} \). Note that oppositely oriented phasors \( e^{2\pi \frac{j}{4}} \) and \( e^{2\pi \frac{j}{3}} \) are multiplied by the magnitude of the spectral islands which is constant at all phase angles at the isocenter resulting in perfect cancellation of the spectral islands between the isocenter and off-isocenter locations. Fig. 3(b) shows the spectral cancellation for an off-isocenter case. Note that each sequence is weighted by the corresponding fan-beam weight \( L^2(r, \phi, \beta_k) \) [see (18)], which is a function of position \( (r, \phi) \) and projection angle \( \beta_k \). In this case, oppositely oriented phasors are weighted by different fan-beam weights and together they apply different weights to the spectral islands. The summation results in a set of slightly attenuated, but nonzero spectral islands between the isocenter and off-isocenter locations.

We simulated our analysis equation (17) using Matlab (version 5.01). For \( p^t_{\beta_k}(\gamma, z) \) we used projection data generated by the CT simulator (see Section III-A) and for \( h(\gamma, z) \) we used two different helical-interpolation kernels corresponding to the 180°-LI and 360°-LI algorithms (constant speed helical, fullscan with interpolation (CSH-HI) and constant speed helical, halfscan with interpolation (CSH-HH) in [11]), respectively. The object we simulated was a 6.35-mm bar-pattern phantom [see Fig. 4(a)] scanned with a longitudinal translation of 5 mm/rotation (5 mm collimation at pitch one). The Fourier transform of each reconstructed longitudinal profile was inspected for energy at the fundamental and predicted aliased frequencies, and the results are shown in Fig. 5.

We can see from Fig. 5 that the spectral energy of the aliased frequency is negligible at the isocenter as predicted by our theory. At off-isocenter locations, however, the spectral energy increases as a function of distance from the isocenter for both interpolation methods. The energy of the fundamental frequency had a comparatively small increase as a function of distance from the isocenter. Fig. 5(a) and (b) shows that the relative contribution of the two frequencies is a function of the helical-interpolation filter \( H(\gamma, f_z) \). We will discuss this in more detail in Section IV-B.

4) Interpolation in Spiral-CT Reconstruction Algorithms:
In our derivations we have used a generic interpolation kernel \( h(\gamma, f_z) \) which is both a function of the projection ray \( \gamma \) and the longitudinal variable \( z \). For algorithms using the full set of projection data (e.g., the 360°-LI algorithm, which uses \( 4\pi \)-rotation of data), the interpolation kernel is independent of the projection ray \( \gamma \). Equation (18) reduces to

\[
F_z\{f^H_{(\gamma, \phi)}(z)\} = \Delta z D_0 R H(f_z) \sum_{k=-M/2}^{M/2-1} \left[ \frac{1}{L^2(r, \phi, \beta_k)} \right] \cdot \sum_{k=-\infty}^{\infty} \left( P^t_{\beta_k}(\gamma, f_z - \frac{k}{d_{4x}}) e^{-j2\pi \frac{k}{M} \cos \gamma} \right) \gamma' g(\gamma) \right\}_{\gamma'}
\]

where \( H(f_z) \) is taken out of the summation since it contains no \( \gamma \) dependence.

For algorithms using fan-beam reconstruction from a limited number of views (e.g., the 180°-LI algorithm) the interpolation becomes \( \gamma \) dependent. In that case, corresponding rays which are oppositely oriented are interpolated. In addition, nonlinear halfscan weights are applied to the interpolated projections to avoid artifacts arising from multiply acquired data [1]. In Appendix B we derive the Fourier transform of a line \( f^H_{(\gamma, \phi)}(z) \) reconstructed using the 180°-LI algorithm and the halfscan weights described in [1]. The resulting equation is shown below.

\[
F_z\{f^H_{(\gamma, \phi)}(z)\} = \Delta z D_0 R \sum_{k=0}^{M/2-1} \left[ \frac{1}{L^2(r, \phi, \beta_k)} \right] \cdot \sum_{k=-\infty}^{\infty} \left( P^t_{\beta_k}(\gamma, f_z - \frac{k}{d_{4x}}) e^{-j2\pi \frac{k}{M} H(\gamma, f_z) \cos \gamma} \right) \gamma' g(\gamma) \right\}_{\gamma'}
\]
Equation (26) can be interpreted as interlacing two sequences sampled at the same rate $d_{\Delta}$ to reconstruct an overall signal that is better sampled. The 180°-LI algorithm uses projection data equivalent to $2\pi + 4\pi r_M$ rotations. Note, however, that the summation over the projection angles in (26) still ranges from $i = 0, \ldots, M - 1$ equivalent to a full $2\pi$ rotation. This is due to an interesting wrap around effect that eliminates the redundancy in the projection data when the interpolation is done through a convolution in the longitudinal direction. (Please see Appendix B for detailed derivations.)

III. METHODS

We used CT simulations and actual phantom-CT scans to assess the validity and explore the implications of our mathematical model.

A. CT Simulations

Aliasing effects are usually hard to discern in natural scenes or where no high-frequency edges are present to bring out the effects of aliasing. Therefore, we designed a mathematical phantom that simulated a square wave of a fixed frequency in the longitudinal direction, in order to allow us to examine closely the effects of sampling on the aliased spectrum under controlled conditions.\(^1\) We performed computer-simulated scans of this phantom using CT-simulation software written by Dr. Carl Crawford. The algorithms used in the simulator are described in [1]. The mathematical phantom consisted of cylindrical disks of varying widths and densities used to form a bar pattern along the longitudinal axis. The attenuation coefficients used to simulate the different densities were 0.43 and 0.21 for Teflon and acrylic, respectively. The diameter of the cylinder was 400 mm in a scan field of view (SFOV) of 466 mm [see Fig. 4(a)].

Partial-volume effects were simulated by subdividing the source and detector rectangular apertures into a matrix of subunits called lets and casting pencil rays from each let in the source to each let in the detector. The projection value is taken to be the natural log of the average contribution (exponentiated line integrals) from all the rays. This operation simulates the nonlinear partial-volume effects generated by actual CT scanners. In our experiments, both linear (without the exponentiated line integrals) and nonlinear partial-volume effects were simulated, and will be noted for each experiment.

1) Aliasing Reduction as a Function of Radius (Experiment 1): We simulated scans of the cylindrical bar-pattern phantom with periods ranging from 12 mm to 18 mm (with 90% overlap). A filter equivalent to that used for the 360°-LI algorithm was convolved with the resulting volume to simulate the smoothing effects of the helical interpolation. The resulting volume was then compared to a spiral-CT simulation of the same phantom done at pitch values of 1, 1.5, and 2, all reconstructed at 1-mm intervals with the 360°-LI algorithm, and with only linear partial-volume effects. The results are shown in Section IV-A.

2) Effects of the Helical-Interpolation Algorithm on Longitudinal Aliasing (Experiment 2): The CT simulation described in Section III-A was repeated. However, this time the projection data were reconstructed using the 360°-LI algorithm. This experiment was done with only linear partial-volume effects. Results are presented in Section IV-B.

3) Effects of Reconstruction Interval on Longitudinal Aliasing (Experiment 3): We again used the 6.35-mm period bar-pattern phantom for this set of simulations. Scanning was simulated at pitch = 1 using only linear partial-volume effects. The reconstruction algorithm used was the 180°-LI algorithm, with reconstruction intervals of 5 mm, 2.5 mm, and 1 mm. Results were analyzed both in the spatial and spectral domains and presented in Section IV-C.

4) Comparison with Conventional CT Scans (Experiment 4): A conventional-CT scan of an 18-mm period bar-pattern phantom was simulated at a high sampling rate to generate an alias-free standard. The simulation parameters used were the same as those listed in Table I except for the collimator width, which was 10 mm. The reconstruction interval was 1 mm (with 90% overlap). A filter equivalent to that used for the 360°-LI algorithm was convolved with the resulting volume to simulate the smoothing effects of the helical interpolation. The resulting volume was then compared to a spiral-CT simulation of the same phantom done at pitch values of 1, 1.5, and 2, all reconstructed at 1-mm intervals with the 360°-LI algorithm, and with only linear partial-volume effects. The results are shown in Section IV-D.

5) Percentage of Aliasing as a Function of Pitch, the Helical-Interpolation Algorithm, and Distance from the Isocenter (Experiment 5): We performed a series of CT simulations using a range of discrete input frequencies (bar-pattern phantoms with periods ranging from 12 mm to 18 mm) and scanned at pitch values of 1, 1.5, and 2, with 10-mm collimation using both the 180°-LI and 360°-LI algorithms. We chose the range of input frequencies (1/18 mm\(^{-1}\) to 1/12 mm\(^{-1}\)) to lie within the mainlobe of the detector-response function. For each simulation, we measured the percentage of aliasing defined by

\[
\text{Percentage aliasing} = \frac{\text{spectral energy at the aliased frequency}}{\text{spectral energy at the fundamental frequency}}
\]  
\(27\)
as a function of frequency, pitch, and the helical-interpolation algorithm at three radial positions: \( r = 0 \) mm, \( r = 100 \) mm, and \( r = 200 \) mm. The results are presented in Section IV-E.

6) Validation Using CT Scans (Experiment 6): We performed CT simulations using the mathematical phantom illustrated in Fig. 4(b) to match an actual phantom-CT scan. The helical-extrapolative algorithm [1] was used in the simulator to match the helical interpolator used in the GE CTI scanner. The scanning parameters are summarized in Table I. This experiment was performed to correlate results obtained from two independent sources: simulations using the CT simulator and actual phantom-CT scans (Section III-B). The results are presented in Section IV-F.

B. Phantom Experiments

We performed actual CT scans of a phantom to verify findings from the simulated results (see Section III-A6). We built the cylindrical bar-pattern phantom by interleaving 3.175-mm disks of Teflon [approximately 1095 Hounsfield units (HU)] and acrylic (approximately 115 HU). Fig. 6 shows the phantom used in the scans. Its dimensions and location with respect to the SFOV of the CT scanner are shown in Fig. 4(b). The relatively small size of the phantom was designed to facilitate its construction and transport. The phantom was scanned both at the isocenter of the scanner, and offset by \( \pm 76.4 \) mm with respect to the scanner isocenter. The phantom was attached to a GE phantom holder and care was taken to align the phantom so that the disks were perpendicular to the scanner axis. All phantom scans were performed with a GE CTI scanner. The scanning parameters used in the phantom scans are summarized in Table I.

IV. RESULTS

A. Aliasing Reduction as a Function of Radius (Experiment 1)

The results of the analysis in Section II-B3 predict that for a radially uniform object concentric with the scanner FOV there is no observable longitudinal aliasing at the isocenter of the object. The analysis also predicts that, at off-isocenter locations, longitudinal aliasing is nonnegligible and increases as a function of distance from the isocenter.

Fig. 7 shows longitudinal profiles of the bar-pattern phantom generated by the CT-simulation software. Fig. 7(a) shows the longitudinal profile at the isocenter \( (r = 0 \) mm) and the corresponding Fourier spectrum. We can see that at the isocenter the 6.35-mm period waveform is completely resolved both in the spatial and the Fourier domains. The presence of these spurious frequencies makes analysis in the Fourier domain difficult, and it is for this reason that we performed most of our Fourier-domain analyses from CT simulations done with only the linear partial-volume effects. Note, however, the trends of interest in Fig. 7, that is, negligible aliasing at the isocenter and increased aliasing with distance from the isocenter, are still be observed in Fig. 8.

Finally, we illustrate the effect that longitudinal aliasing has on transaxial images. Fig. 9 shows a series of reconstructed images at distinct longitudinal positions. A point near the isocenter is labeled B and two points located on either side of the isocenter are labeled A and C, respectively. Their corresponding longitudinal profiles are plotted below. From the images, we can see that the longitudinal low-frequency interference component in the waveform. Our theory assumes that the sampling frequency for this experiment is given by \( 1/d_{41} = 1/5 \) mm\(^{-1} \), and predicts aliased frequencies at \( \pm 1/23.5 \) mm\(^{-1} \), for a 1/6.35 mm\(^{-1} \) input waveform. Fig. 7(b) shows the peaks of the interfering low-frequency component at \( \pm 1/23.5 \) mm\(^{-1} \), the predicted aliased frequency, thereby confirming that the sampling period is indeed determined by the distance moved per gantry rotation \( (d_{41} = 5 \) mm) and not the reconstructed section spacing \( (d_{20} = 0.5 \) mm).
of phase, as shown in the longitudinal profiles. This indicates a transaxial radial symmetry, that is, longitudinal profiles at the same radial location will suffer the same amount of longitudinal aliasing, but the phase of the aliased component will be a function of the angular position of the longitudinal profiles. Longitudinal aliasing can have detrimental effects on the resulting image quality of the reconstructed transaxial images, i.e., aliasing can impact low-contrast detectability in transaxial images and can introduce spiral artifacts in volume-rendered images [11], or zebra artifacts in maximum intensity projection (MIP) images [12].

B. Effects of the Helical-Interpolation Algorithm on Longitudinal Aliasing (Experiment 2)

As mentioned in Sections II-B3 and II-B4, the helical-interpolation algorithm used in the reconstruction process has a significant effect on the aliasing content in the resulting 3-D volume. In this section we will show examples of the relative spectral energy between the fundamental and the aliased frequencies for two commonly used helical-interpolation algorithms: the $180^\circ$-LI algorithm and the $360^\circ$-LI algorithm.

Fig. 10(a) and (b) plots the spectral energy of the fundamental and aliased frequencies as a function of distance from the isocenter for CT simulations reconstructed with the $180^\circ$-LI and $360^\circ$-LI algorithms, respectively. The bottom portion of Fig. 10 shows the corresponding effect on the longitudinal profiles at the isocenter and at an off-isocenter location (at $r = 180$ mm) for each reconstruction algorithm.

These results explain what has been noted in the literature [1], [5], i.e., that volumes reconstructed with the $180^\circ$-LI algorithm have better longitudinal resolution than volumes reconstructed with the $360^\circ$-LI algorithm. This is due to the fact that the $180^\circ$-LI algorithm uses fewer projections for its interpolations (see Appendix B and [1]), resulting in a narrower filter kernel in the spatial domain or, equivalently, a broader passband in the Fourier domain. At the isocenter, where the aliasing is negligible, a broader passband implies better frequency resolution, and we can see that the input frequency is much better resolved for the $180^\circ$-LI algorithm than for the $360^\circ$-LI algorithm [see Fig. 10(a) and (b) (center)]. At off-isocenter locations, however, aliasing is nonnegligible. In fact, if we look at Fig. 10 (top), we can see that the amount of aliased energy is approximately the same for both algorithms. However, the broader passband in the $180^\circ$-LI filter resulted in a less attenuated main frequency component so that the final longitudinal profile was not overwhelmed by the aliased frequency, thus allowing us to still resolve the input (main) frequency. On the other hand, the relatively small amplitude in the spectral energy of the main frequency for the $360^\circ$-LI algorithm resulted in a longitudinal profile [Fig. 10(b) (bottom)] that was dominated by the aliased frequency. Note that the results shown in Fig. 10 (top) agree with the results of our analysis, illustrated in Fig. 5(a) and (b) for the $180^\circ$-LI and $360^\circ$-LI algorithms, thereby confirming the validity of our approach.
Fig. 9. Reconstructed images showing the effect of longitudinal aliasing in transaxial images. These images are reconstructed at 37.5, 50, 62.5, and 75 mm (as measured from the origin along the \( z \) axis). The longitudinal profile and the spectral plots at points A, C (located 180 mm from the isocenter) and B (at isocenter) are plotted below the images and illustrate the longitudinal aliasing at these transaxial locations.

C. Effects of Reconstruction Interval on Longitudinal Aliasing (Experiment 3)

The results of our analysis (Section II-B3) showed that the reconstruction interval \( d_{22} \) in spiral CT, which may be chosen arbitrarily, is a resampling period. Therefore, the choice of reconstruction interval cannot remove any longitudinal aliasing from the resulting volume. Figs. 11 and 12 show simulated CT reconstructions of the 6.35-mm period bar-pattern waveform, reconstructed at (a) 5-mm, (b) 2.5-mm, and (c) 1-mm intervals at the isocenter \((r = 0)\) and off-isocenter \((r = 180\) mm), respectively. Recall that for these CT simulations we have used a collimator width of 5 mm at a pitch of one. Therefore, at a reconstruction interval of 5 mm, the resampling period is longer than the required Nyquist period (2.5 mm for two samples per collimator width). At a reconstruction interval of 2.5 mm the resampling period equals the required Nyquist period, and at 1 mm the resampling period is higher than the required Nyquist period.

First, if we look at the longitudinal profiles and the corresponding spectra at the off-isocenter locations (Fig. 12) for each of the reconstruction intervals, we can see the low-frequency interfering signal in the longitudinal profiles and the aliased frequency peaking at \(+1/23.5\) mm\(^{-1}\), regardless of the reconstruction interval. That is, at all \( d_{22} \) values there was observable aliasing at the off-isocenter locations. This aliased frequency of \(+1/23.5\) mm\(^{-1}\) corresponds to the aliased frequency predicted from sampling theory for an input frequency of 1/6.35 mm\(^{-1}\) sampled with a period of \( d_{11} = 5\) mm (1/23.5 mm\(^{-1}\) = 1/5 mm\(^{-1}\) = 1/6.35 mm\(^{-1}\)). This clearly shows the reconstruction interval as a resampling period and that the ability to choose arbitrarily small reconstruction intervals does not result in suppression of aliasing. Changing the sampling period of the system would have affected the spectral aliasing by shifting the location of the replicated islands and, consequently, the location of the observed aliased frequencies.

Next, let us focus on the longitudinal profiles and the corresponding spectra at the isocenter for the chosen reconstruction intervals (Fig. 11). Recall that at the isocenter there is negligible aliasing due to the high effective-sampling rate of \( M/d_{11} \) [see (24)]. Thus, the reconstruction interval \( d_{22} \) resamples the longitudinal profile under conditions of negligible aliasing. At \( d_{22} = 5\) mm (under downsampled conditions), the aliased component was reintroduced as can be seen in Fig. 11(a). For \( d_{22} = 2.5\) mm and \( d_{22} = 1\) mm, where the resampling period was equal to and less than the Nyquist period, respectively, there was no aliasing at the isocenter. However, in the spatial domain, we can see that the longitudinal profile in Figs. 11(c) is better resolved (greater modulation depth) than the corresponding profile in Fig. 11(b). This effect is mirrored by an equivalent increase in the amplitude of the frequency components in the spectral plot.
TABLE II

PERCENTAGE OF ALIASING FOR TWO DIFFERENT HELICAL INTERPOLATION ALGORITHMS AS A FUNCTION OF PITCH VALUE AND RADIAL LOCATION. THE RANGE OF PERCENTAGE VALUES ARE GIVEN FOR FREQUENCIES RANGING FROM 1/12 mm\(^{-1}\) to 1/18 mm\(^{-1}\).

<table>
<thead>
<tr>
<th>Radial position</th>
<th>180°-LI</th>
<th>360°-LI</th>
</tr>
</thead>
<tbody>
<tr>
<td>pitch=1</td>
<td>0 %</td>
<td>6-21 %</td>
</tr>
<tr>
<td></td>
<td>14-48 %</td>
<td>43-73 %</td>
</tr>
<tr>
<td>pitch=1.5</td>
<td>0 %</td>
<td>22-89 %</td>
</tr>
<tr>
<td></td>
<td>48-181 %</td>
<td>776-805 %</td>
</tr>
<tr>
<td>pitch=2</td>
<td>0 %</td>
<td>56-770 %</td>
</tr>
<tr>
<td></td>
<td>123-552 %</td>
<td>102-832 %</td>
</tr>
</tbody>
</table>

Fig. 11. Plots of CT simulations of the 6.35-mm period bar-pattern cylinder, reconstructed using different reconstruction intervals. The plots on the left column are the longitudinal profiles at isocenter (r=0 mm) and the plots on the right column show the corresponding spectra. (a) Images are reconstructed at 5-mm intervals. (b) Images are reconstructed at 2.5-mm intervals. (c) Images are reconstructed at 1-mm intervals. (Note that the nonzero spectral content at f_z = 0 is introduced by spectral leakage from the windowing performed prior to taking the Fourier transform in each case.)

D. Comparison with Conventional-CT Scans (Experiment 4)

Fig. 13(a) and (b) shows a longitudinal cross section through the isocenter of the simulated volume reconstructed using conventional CT and spiral CT, respectively, and Fig. 13(c) shows the difference image for scans simulated at pitch values of 1, 1.5, and 2. We can see that at the isocenter, the longitudinal profile of the conventional-CT scan matches that of the spiral-CT scan, confirming again that the aliasing is negligible at the isocenter. The spatially varying longitudinal aliasing can be seen in the difference image as an increasingly stronger low-frequency component in the longitudinal profile with distance from the isocenter. The frequency of the bar pattern simulated was 18 mm, the sampling period was 10 mm (pitch 1), 15 mm (pitch 1.5), and 20 mm (pitch 2), producing aliased waveforms with periods of 22.5 mm, 90 mm, and 180 mm, respectively. It can be seen from the difference image that aliased waveforms are indeed at these periods and that their magnitudes increase with distance from the isocenter.

Fig. 12. Plots of CT simulations of the 6.35-mm period bar-pattern cylinder, reconstructed using different reconstruction intervals. The plots on the left column are the longitudinal profiles at an off-isocenter location (r = 180 mm) and the plots on the right column show the corresponding spectra. (a) Images are reconstructed at 5-mm intervals. (b) Images are reconstructed at 2.5-mm intervals. (c) Images are reconstructed at 1-mm intervals.

E. Percentage of Aliasing as a Function of Pitch, the Helical-Interpolation Algorithm and Distance from the Isocenter (Experiment 5)

Table II gives a range of aliasing percentages for two reconstruction algorithms: the 180°-LI and the 360°-LI algorithms, at three pitch values. Measurements are taken at three radial locations (r = 0 mm, r = 100 mm, and r = 200 mm). We can see that for all pitch values there was negligible aliasing at the isocenter. However, at off-isocenter locations the percentage of aliasing was significant and indicates that measurements taken at the isocenter of the scanner are not sufficient to characterize the aliasing conditions at off-isocenter locations.

From Table II we can also see that the percentage of aliasing is lower for the 180°-LI algorithm, an observation which is consistent with the conclusions arrived at in Section IV-B. Also, for both reconstruction algorithms the percentage of aliasing increases with increasing pitch (longer sampling period), as expected.
Fig. 13. CT-simulated longitudinal cross sections of a 18-mm period bar-pattern phantom by spiral- and conventional-CT scan. (a) The spiral-CT cross sections at pitch of 1, 1.5, and 2. (b) the corresponding conventional-CT cross sections convolved with a filter equivalent to the interpolating filter used in the spiral reconstruction. (c) The difference image [(a)–(b) at each pitch]. Note that in the difference images the alternating dark and bright bands show the aliased waveforms at periods of 22.5, 90, and 180 mm, for pitches of 1, 1.5, and 2, respectively. This aliased waveform is negligible at the isocenter and its amplitude increases as a function of distance from the isocenter. Also, note that the phase of the aliased waveforms on either side of the isocenter is exactly 180° apart, as was noted in Fig. 9. The display window is 270 HU for all the images, and the levels are chosen to be the mean of the waveform at isocenter: 742 HU for the images in the first two columns, and zero for the difference images.

F. Validation Using CT Scans (Experiment 6)

In this section we show the correlation between the results obtained from CT simulations and results from actual phantom-CT scans. Fig. 14(a) shows the spectral energy of the fundamental frequency plotted as a function of distance from the isocenter for an off-centered bar-pattern cylinder. Fig. 14(b) shows the same plot for the spectral energy at the aliased frequency. These results parallel those shown in Figs. 5 and 10 (top) for a centered bar-pattern cylinder.

Fig. 14 shows a good correspondence between results obtained from the CT simulator and those obtained from the CT scanner. This confirms the presence of spatially varying longitudinal aliasing as predicted by our theory.

V. DISCUSSION

A. Our Use of a Radially Uniform Phantom

Throughout our simulations and phantom-CT scans we have employed a cylindrical radially uniform phantom consisting only of variations along the longitudinal axis. This bar pattern phantom takes the place of the impulse phantom that is widely used in the literature [1], [4], [5] by exciting single frequencies independently, rather than a range of frequencies as with the impulse phantom. We have simulated a range of individual frequencies that span the mainlobe of the detector-response function and obtained consistent results at all the tested frequencies. While this allowed us to focus our analysis in the longitudinal direction, it is a highly contrived model and can only give us an estimate of the amount of longitudinal aliasing that may be expected. Favorable phase cancellation requires that the spectral components in each projection (from each view angle) be comparable. Most realistic objects will not have radial symmetry. Therefore, the phase cancellation is not expected to be as good as that for the simple radially uniform phantom used in our analysis. At the same time, the aliasing present in more realistic objects may not be as obvious or distracting as the artifacts shown in our results. However, even with a more realistic object we would expect to see spatially varying longitudinal aliasing and the trend for increased aliasing with increasing pitch to hold true.

Although it is hard to assess the clinical significance of the longitudinal-aliasing artifacts as shown with our phantom, the theoretical implication of the results that we have been able to observe with this phantom are significant. This phantom enabled us to conduct simple experiments that confirmed our theory and provided crucial information about the longitudinal sampling and aliasing that would not have been obvious with the commonly used impulse phantom. For example, the single-frequency input it provided allowed the observation that the foldover frequency does not change between the 360°-LI and 180°-LI algorithms and, likewise, does not change with an increase in the number of reconstructed images per collimator width. The clinical relevance of longitudinal aliasing is currently being researched by us and others and is beyond the scope of this paper.

We have chosen to show CT-scan results for an off-centered cylinder (Experiment 6) since the relatively small size of the phantom combined with the low aliased spectral-energy content at locations near the isocenter resulted in a aliased spectral-energy content which was at or below the noise floor for the cylinder when centered and accurate measurements of
the aliased-frequency content were not possible. We therefore displaced the cylinder to show the trend of increase in the energy of the aliased frequency at off-isocenter locations. We noted, however, that displacing the cylinder from the isocenter also increased the amount of aliased energy observed. This increase in the aliased-energy content is expected, as moving the cylinder off center disturbs the delicate cancellation of the aliased spectral islands that was necessary for the results described in Section II-B3.

Though Fig. 14 shows good agreement between the CT simulations and actual CT scans of an equivalent phantom, some small differences can be observed. These may have resulted from a mismatch in the reconstruction filters and/or various scanner-specific effects that were not implemented in the CT-simulation software. The discussion of these effects are beyond the scope of this paper and will be the subject of further studies.

B. Longitudinal Aliasing and Resolution

A number of articles in the literature report improvements in the longitudinal resolution measured at the isocenter of the scanner with an increase in the number of reconstructed sections $n_s$ [1], [4]–[6], [9]. Furthermore, experimental results have shown that a continuous domain formulation of the SSP [6], [9] based on the assumption of bandlimited input spectrum and a sampling rate high enough to result in negligible aliasing, is a good first-order approximation to the longitudinal system response function. The effects of improved resolution at the isocenter under these conditions can be explained using discrete-time-sampling theory [16] which states that when a bandlimited signal is sampled at or above the Nyquist rate (no aliasing), upsampling by interpolation may improve the resolution of the signal by providing better modulation depth. For imaging systems with rectangular apertures, it is commonly assumed that the spectrum is bandlimited by the mainlobe of the corresponding sinc function. Under these assumptions, the Nyquist rate is satisfied only for the 180° LI algorithm at the isocenter at a pitch $\leq 1$. On the other hand, when these conditions are not met, upsampling (by interpolating) an aliased signal does not recover details lost in the original sampling. In other words, upsampling an aliased spectrum does not reduce the aliasing or increase the Nyquist frequency which can result only from an increase in the original sampling rate.

One might question then the validity of our results that show negligible aliasing at the isocenter when the Nyquist rate is not satisfied. Wang et al. [9] derived a continuous-domain equation for estimating the longitudinal SSP (at the isocenter), where aliasing could be neglected by assuming a bandlimited spectrum and a sufficiently high sampling rate. The analysis in our paper points to a different interpretation, which relies on a cancellation of the spectral islands that is best at the isocenter and worsens at off-isocenter locations. We make no assumptions about a bandlimited input or Nyquist sampling. We believe that the negligible aliasing at the isocenter results from the spectral cancellation described in Section II-B3 and the subsequent upsampling by reconstructing more sections per collimator width improves the full-width at tenth-maximum (FWTM) values observed experimentally [5] in accord with discrete sampling theory. In summary, the reconstruction interval $d_{\text{eff}}$ is a resampling rate. At the isocenter, where aliasing is negligible, this corresponds to upsampling a minimally aliased signal and therefore can be treated as an effective-sampling rate: an observation consistent with the literature. However, at off-isocenter locations, upsampling resamples an already aliased signal and therefore cannot be considered an effective-sampling rate.

Note that the negligible aliasing at the isocenter can also be interpreted by treating the distance between adjacent projection angles $[d_{\text{eff}}/M$ in (17)] as the effective-sampling rate. In other words, the effective-sampling rate is $M$ times better than the original sampling rate $d_{\text{eff}}$. This implies that one could use a scanning pitch close to $M$ when scanning a radially uniform and centered object and encounter minimal aliasing at the isocenter. In practice, however, the blurring caused by the interpolation kernel at such high pitch values would offset any advantage that could be obtained from aliasing cancellation at the isocenter.

Also, we observed that whereas the magnitude of the aliased frequency component depended on the choice of the reconstruction interval and the radial location of the longitudinal line being analyzed, the location of the aliased frequency components remained unaffected. In other words, whereas we normally relate changes in aliasing to changes in the sampling conditions (the spectral islands are moved closer or further apart, introducing more or less aliasing), in spiral CT the location of the replicated spectral islands are fixed. However, their magnitudes change in relation to the spectral-aliasing cancellation at that radial location and in relation to the interpolating filters that are applied.

Another consequence of the fact that the longitudinal aliasing is determined by the fixed sampling rate $d_{\text{eff}}$ of each sampled projection sequence is that the position of the aliased islands are related only to the table-speed-per-gantry rotation and not, for example, to the choice of helical-interpolating algorithm. However, a popular interpretation regards the 180° LI algorithm as having twice the sampling rate as the 360° LI algorithm. Our results are consistent with this interpretation at the isocenter in the following way. We formulate the 180° LI algorithm as an interlacing of two sequences sampled at the same sampling rate as that of the 360° LI algorithm. At the isocenter, the offset between the two sequences to be interlaced is given by $a = d/2$ where $d$ is the distance between the original samples. The offset $a = d/2$ introduces an $e^{-j\pi n_0}$ phase factor in the transform, where $n_0$ is the index of the replicated spectral islands. In this manner, every other spectral island in the second sequence is multiplied by a factor of $-1$ ($e^{-j\pi n_0} = -1$ for $n_0 = \pm 1, \pm 3, \ldots$). When the two sequences are summed, the spectral islands which would have introduced aliasing are canceled, resulting in a spectrum equivalent to that obtained by sampling at twice the original period $d$. Therefore, at the isocenter, the two interpretations are equivalent. Off-isocenter, however, it is necessary to further understand the interlacing approach to explain the presence of aliasing for both the 360° LI and the 180° LI algorithms. Fig. 3(b) shows an example spectrum for an off-isocenter location. Note that
the aliased spectral islands are not completely canceled, so that some residual aliasing is present at the theoretically predicted location for the original sampling period. This interpretation has important implications for the analysis of aliasing because if the sampling rates were indeed different for the 180°-LI and the 360°-LI algorithms, it would have resulted in different locations for the replicated spectral islands and hence, different aliased frequencies.

We showed in Section IV-B that the volume reconstructed with both algorithms have the same aliased frequency. Fig. 10 shows, in fact, that the poor performance of the 360°-LI algorithm comes from the suppression of the fundamental frequency by the narrower spectral response of its corresponding interpolating function. By comparison, the magnitude of the aliased frequency is hardly changed by the choice of interpolating algorithm in this case. Thus, it is the spectral response of the interpolation filter and the interleaving of the sampled sequences that resulted in better performance for the 180°-LI algorithm.

VI. CONCLUSION

In this paper, we presented an analysis of the sampling and the resulting spectral behavior along the longitudinal direction in spiral CT. We derived a mathematical model that enabled us to study the unique sampling scheme in spiral CT. In particular, we showed that at each projection angle, the acquired samples form sequences in $z$ sampled at a rate determined by the pitch, with a constant shift between consecutive projection sequences. For the special case of a radially symmetric phantom at the scanner isocenter, the shift between the projection sequences resulted in a spatially varying cancellation of the aliased spectral islands, which led to a spatially varying longitudinal aliasing that is least at the isocenter and increased with distance from the isocenter.

We verified our mathematical model through both computer simulations and actual phantom-CT scans. Our results showed that the negligible aliasing observed at the isocenter through measurements performed with an impulse phantom can be viewed as a consequence of the spectral cancellation scheme in spiral CT. We conclude from our findings that the SSP measured at the isocenter with an impulse phantom provides a good first-order approximation to the longitudinal resolution at the isocenter, but is a poor indicator of longitudinal aliasing at other transaxial locations.

Using our mathematical model, we also showed the relationship between the sampling rate (determined by the pitch) and the resampling rate (determined by the spacing between reconstructed section images) and how the interplay between these two parameters affects the longitudinal aliasing. In particular, we showed that the improved longitudinal resolution observed by reconstructing more section images per collimator width comes from an improved modulation depth, and not an extension of the bandwidth through a shift in the Nyquist frequency.

From our discussion we can see that although spiral CT has brought us closer to isotropic data volumes, we are not really as close to it as previously believed. We have shown that aliasing varies spatially, an effect that has not been thoroughly analyzed in the literature. Awareness of this spatially varying aliasing in spiral CT and insight into the longitudinal sampling pattern may help to understand previous reports of longitudinal resolution in spiral CT [4], [5], [9], [10] and may also aid in future development of reconstruction algorithms that would take better advantage of the spectral-aliasing cancellation property in spiral CT.

APPENDIX A

LONGITUDINAL ANALYSIS EQUATION FOR THE 360°-LI ALGORITHM

In this section we derive the analysis equation (18) for reconstructions from fan-beam projections. We start from the sampling equation (17) which is repeated here

$$f_{y=0}(z) = \Delta \beta \sum_{i=0}^{M-1} \left\{ \frac{1}{L^2(r, \phi, \beta)} \left[ \left( P_{\beta, \gamma}^{\phi}(\gamma, f_{z}) \right)_z \cdot \sum_{k=0}^{R-1} \delta \left( z - \frac{k}{M} \right) \right] \ast_{\gamma} \cdot D_{0} \cos \gamma \right\} \ast_{y} \cdot g(\gamma) \right\}$$

where the rectangle of width $d_{s1} R$ is added to account for the finite scan length.

The Fourier transform of the line at $f_{y=0}(z)$ is given by

$$F_{y}\{f_{y=0}(z)\} = \Delta \beta D_{0} \sum_{k=0}^{M-1} \left\{ \frac{1}{L^2(r, \phi, \beta)} \left[ \left( P_{\beta, \gamma}^{\phi}(\gamma, f_{z}) \right)_z \cdot d_{s1} R \sin(\alpha_{s1} R f_{z}) e^{-\alpha_{s1} R f_{z}} \right] \right\} \left( \gamma, f_{z} \right)$$

$$\cdot \sum_{k=-\infty}^{\infty} \delta \left( f_{z} - \frac{k}{d_{s1}} \right) e^{-2\pi i \frac{k}{d_{s1}}} \left( H(\gamma, f_{z}) \right) \ast_{\gamma} \cdot g(\gamma) \right\} \right\}_{\gamma, f_{z}}$$

$$= \Delta \beta D_{0} R \sum_{i=0}^{M-1} \left\{ \frac{1}{L^2(r, \phi, \beta)} \left[ H(\gamma, f_{z}) \left( P_{\beta, \gamma}^{\phi}(\gamma, f_{z}) \right)_z \right] \ast_{\gamma} \cdot g(\gamma) \right\} \right\} \right\}_{\gamma, R}$$

$$= \Delta \beta D_{0} R \sum_{i=0}^{M-1} \left\{ \frac{1}{L^2(r, \phi, \beta)} \left[ H(\gamma, f_{z}) \left( P_{\beta, \gamma}^{\phi}(\gamma, f_{z}) \right)_z \right] \ast_{\gamma} \cdot g(\gamma) \right\} \right\} \right\}_{\gamma, R}$$

$$(28)$$

(29)
where we have made the approximation
\[ P'_{\beta}(\gamma, f_z) \approx P'_{\beta}(\gamma, f_z) \ast \text{sinc}(d_{s1} R f_z) e^{-j \pi d_{s1} R f_z}. \]

By using this approximation we are assuming that \( P'_{\beta}(\gamma, f_z) \) has infinitely high-frequency resolution, whereas in reality its frequency resolution is limited by the extent of the scan in the spatial domain \( d_{s1} R \). Within the context of our analysis, this approximation is a mathematical convenience that does not affect the conclusion of our analysis. The results presented via CT simulations show the results at the true frequency resolution.

For the simple case of interpolation with the 360°-LI algorithm, the helical filter has no \( \gamma \) dependence and can be described by
\[ H(\gamma, z) = \nabla \left( \frac{z}{2d_{s1}} \right). \] (30)

**APPENDIX B**

**LONGITUDINAL ANALYSIS EQUATION FOR THE 180°-LI ALGORITHM**

In this section, we derive the analysis equation (26) for the 180°-LI algorithm. For the 360°-LI algorithm, corresponding rays at the same orientation are used to interpolate the data that required to reconstruct a transaxial image. Therefore, two full revolutions (\( 4\pi \)) of data are used to interpolate data in the range \( \beta \in [0, 2\pi] \) and \( \gamma \in [-\gamma_m, \gamma_m] \). In (18), the summation over the projection angles \( \beta z \) runs from \( i = 0 \) to \( M - 1 \) (a full \( 2\pi \) rotation) since the filter kernel used in the 360°-LI algorithm has width \( 2d_{s1} \) and applies weights to two corresponding rays at the same projection angle. In this case both projections can be weighted at the same time and their contributions added to produced the interpolated data.

In the 180°-LI algorithm oppositely oriented rays are used to interpolate the necessary data, where the corresponding rays \((\beta_1, \gamma_1)\) and \((\beta_2, \gamma_2)\) are given by
\[ \beta_2 = \beta_1 + \pi + 2\gamma_1 \]
and
\[ \gamma_2 = -\gamma_1. \]

The helical-interpolation filters are derived from these set of equations: details are given in [1].

In this case, data acquired in a \( 2\pi + 4\gamma_m \) revolution are used to interpolate data in the range \( \beta \in [0, \pi + 2\gamma_m] \) and \( \gamma \in [-\gamma_m, \gamma_m] \). We can rewrite (18) for the 180°-LI algorithm case as
\[ F_z \{ f_{\gamma, \phi}^h(z) \} = \Delta \beta D_0 R \sum_{i=0}^{M'-1} \left\{ \frac{1}{I^2(r, \phi, \beta_{2i})} \cdot \left[ \left( \sum_{k=-\infty}^{\infty} f_{\beta_1, i}^k(\gamma, f_z - \frac{k}{d_{s1}} e^{-2\pi i \Delta p M} \right) \right] \right\} \]
\[ \ast g(\gamma) \left| \right. \gamma_1 \]
\[ + \Delta \beta D_0 R \sum_{i=0}^{M'-1} \left\{ \frac{1}{I^2(r, \phi, \beta_{2i})} \cdot \left[ \left( \sum_{k=-\infty}^{\infty} f_{\beta_1, i}^k(\gamma, f_z - \frac{k}{d_{s1}} e^{-2\pi i \Delta p M} \right) \right] \right\} \]
\[ \ast g(\gamma) \left| \right. \gamma_1 \]
\[ + \Delta \beta D_0 R \sum_{i=0}^{M'-1} \left\{ \frac{1}{I^2(r, \phi, \beta_{2i})} \cdot \left[ \left( \sum_{k=-\infty}^{\infty} f_{\beta_1, i}^k(\gamma, f_z - \frac{k}{d_{s1}} e^{-2\pi i \Delta p M} \right) \right] \right\} \]
\[ \ast g(\gamma) \left| \right. \gamma_1 \]

where we have made explicit the weighted interpolation between corresponding rays \( \beta_{2i} \) and \( \beta_{2i} \) for \( i = 0 \) to \( M' - 1 \). \( M' \) is the number of projections equivalent to \( \pi + 2\gamma_m \) projections, given by
\[ M' = \frac{\pi + 2\gamma_m}{2\pi} M. \] (32)

Fig. 15 shows the Radon-space representation of the data acquired for the 180°-LI algorithm. The dark shaded region and the lighter shaded region represent regions of corresponding data. The halfscan weights derived in [1] apply a zero weight to regions of redundant data and a weight of one to regions of nonredundant data, resulting in the nulled regions shown in the graph. We will show in this section that the 3-D representation with longitudinal filtering results in an interpolated volume that contains no redundant data. From Fig. 15(a) we can see that at each \( \gamma \) value, the range of \( \beta \) (nonredundant data) is constant at \( 2\pi \). This is important, as in our 3-D representation (see Fig. 2) \( \beta \) also ranges from 0 to \( 2\pi \).

Fig. 15(a) and (b) shows that the Radon-space representation in (a) can be mapped to the Radon-space representation shown in (b) if we allow a wrap around at \( 2\pi \). In essence, Fig. 15(a) shows the reconstruction in 2-D Radon space (if in-
individual section images are reconstructed), whereas Fig. 15(b) shows the reconstruction in 3-D Radon space. This mapping results in a completely filled Radon-space representation spanning $\beta \in [0, 2\pi]$ and $\gamma \in [-\gamma_m, \gamma_m]$ without nulled or redundant regions. (Note that this mapping applies to any chosen plane of reconstruction (POR) although we have only shown it for the case of POR at $\pi + 2\gamma_m$.) This wrap around at $2\pi$ is a natural consequence of the periodicity of the projection angle. In our formulation, the sampled sequence at $\beta_0$ will contain samples at $\beta_0 + n 2\pi$ where $n = 0, \ldots, R - 1$ and $R$ equals the number of rotations in the acquisition.

From Fig. 15(a) we can see that $\beta_1, \gamma$ and $\beta_2, \gamma$ can be indexed sequentially and that the $\gamma$ value will be uniquely determined by the location of the line $f(\gamma, \beta)$ to be reconstructed. The sinusoid superimposed on the figure shows one such example. We can therefore rewrite (31) as

$$
F \{ f(\gamma, \beta) \}(\gamma) = \frac{1}{2\pi} \sum_{l=0}^{2\pi} \left\{ \int_{-\gamma_m}^{\gamma_m} f(\gamma, \beta) e^{-j2\pi l \gamma} d\gamma \right\} \delta(\beta - \beta_0).
$$

With this formulation, some projection values are redundant, and need to be zeroed out to avoid reconstruction artifacts.

We can further simplify (34) by taking into account the mapping discussed above and shown in Fig. 15(b). Note that the nonzero portions of the sinusoid are preserved in the mapping and that the range of $\beta$ values in the new representation ranges from 0 to $2\pi$. In this case (34) reduces to

$$
F \{ f(\gamma, \beta) \}(\gamma) = \frac{1}{2\pi} \sum_{l=0}^{2\pi} \left\{ \int_{-\gamma_m}^{\gamma_m} f(\gamma, \beta) e^{-j2\pi l \gamma} d\gamma \right\} \delta(\beta - \beta_0).
$$

where we have changed the range of the summation over $\beta$ from $2(M' - 1)$ to $M - 1$, corresponding to $\beta \in [0, 2\pi]$ range.

A comparison between (18) and (35) reveals that they differ only in the $\gamma$-dependent filter $H(\gamma, \beta)$. Equation (35) is the reconstruction equation for the 180°-LI algorithm given in (26). The helical reconstruction approach described in the Appendix for the 180°-LI and the 360°-LI algorithms were used to generate the results shown in Fig. 5.

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