Distillation with Labelled Transition Systems

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Abstract
In this paper, we provide an improved basis for the “distillation” program transformation. It is known that superlinear speedups can be obtained using distillation, but cannot be obtained by other earlier automatic program transformation techniques such as deforestation, positive supercompilation and partial evaluation. We give distillation an improved semantic basis, and explain how superlinear speedups can occur.

Categories and Subject Descriptors D.3.4 [Programming Languages]: Processors—optimization; I.2.2 [Artificial Intelligence]: Automatic Programming—program transformation; F.3.2 [Logics and Meanings of Programs]: Semantics of Prog. Languages

General Terms Languages, performance, theory.

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1. Introduction
It is well known that programs written using functional programming languages often make use of intermediate data structures and this can be inefficient. Several program transformation techniques have been proposed to eliminate some of these intermediate data structures; for example partial evaluation [11], deforestation [25] and supercompilation [22]. Positive supercompilation [21] is a variant of Turchin’s supercompilation that was introduced in an attempt to study and explain the essentials of Turchin’s supercompiler. Although strictly more powerful than both partial evaluation and deforestation, Sørensen has shown that positive supercompilation (and hence also partial evaluation and deforestation) can only produce a linear speedup in programs [20]. For superlinear speedup, a more powerful transformation algorithm is needed.

Example 1.1. Consider the function call $nrev vs$ shown in Figure 1. This reverses the list $vs$, but the recursive function call $(nrev xs')$ is an intermediate data structure, so in terms of time and space usage, it is quadratic with respect to the length of the list $vs$. A more efficient function that is linear with respect to the length of the list $vs$ is the function $arev$ shown in Figure 1. A number of algebraic transformations have been proposed that can perform this transformation (e.g. [3, 24]) making essential use of “eureka” steps, requiring human insight and not easy to automate (for the given example above, this is done by making use of a specific law stating the associativity of the $app$ function). However, none of the automatic program transformation techniques mentioned above (deforestation, positive supercompilation and partial evaluation) are capable of performing this transformation.

The first author defined a transformation algorithm called distillation [6–8] to allow transformations such as the above to be performed. However, the correctness of this algorithm was not perfectly clear; it needed a more solid theoretical foundation. In more recent work by both authors [9], a theoretical framework using labelled transition systems was defined that can be used to prove the correctness of unfold/fold program transformations using bisimilarity, and the correctness of positive supercompilation was proved within this framework. In this paper, we define the distillation algorithm in the same framework and prove its correctness. We also investigate the efficiency improvements that can be obtained by distillation and explain how superlinear speedups can occur.

The paper is structured as follows. In Section 2 we define the higher-order functional language on which the described transformations are performed, and give an operational semantics for this language. In Section 3 we give an alternate characterisation of Turchin’s “driving” algorithm. This is done by translating, via symbolic execution, a functional program into an equivalent labelled transition system. The equivalence between source and target semantics uses weak bisimulation, an approach related to work by Howe and Gordon. In Section 4 we define the distillation algorithm and give an example for which a superlinear speedup is obtained. In Section 5 we consider the termination and correctness of the distillation algorithm and show how superlinear speedups are sometimes achieved. Section 6 concludes and discusses related work.

2. Language
In this section, we describe the higher-order functional language that will be used throughout this paper. It uses call-by-name evaluation.

Definition 2.1 (Language Syntax). The syntax of this language is as shown in Figure 2.
A program in the language consists of an expression to evaluate and a set of function definitions. For notational convenience we sometimes assume the set of function definitions have been collected into a function environment denoted by $\Delta$. An expression can be a variable, constructor application, function call, $\lambda$-abstraction, application or case. Variables introduced by $\lambda$-abstraction or case patterns are bound; all other variables are free. We write $e_1 \equiv e_2$ if $e_1$ and $e_2$ differ only in the names of bound variables.

Each constructor has a fixed arity; for example $\text{Nil}$ has arity 0 and $\text{Cons}$ has arity 2. In an expression $c \ e_1 \ldots \ e_n$, $n$ must equal the arity of $c$. We allow the usual notation $[]$ for $\text{Nil}$, $x:xs$ for $\text{Cons} \ x \ xs$ and $[e_1, \ldots, e_n]$ for $\text{Cons} \ e_1 \ldots \ e_n \ \text{Nil}$. The patterns in case expressions may not be nested. No variable may appear more than once within a pattern. We assume that the patterns in a case expression are non-overlapping and exhaustive. It is also assumed that erroneous terms such as $(c \ e_1 \ldots \ e_n) e$ where $c$ is of arity $n$ and case $(\lambda x.e) \ p$ does not reduce.

**Definition 2.2** (Substitution). $\theta = \{x_1 \mapsto e_1, \ldots, x_n \mapsto e_n\}$ denotes a substitution. If $e$ is an expression, then $e \theta = e(x_1 \mapsto e_1, \ldots, x_n \mapsto e_n)$ is the result of simultaneously substituting the expressions $e_1, \ldots, e_n$ for the corresponding variables $x_1, \ldots, x_n$, respectively, in the expression $e$ while ensuring that bound variables are renamed appropriately to avoid name capture.

**Definition 2.3** (Renaming). $\sigma = \{x_1 \mapsto x'_1, \ldots, x_n \mapsto x'_n\}$, where $\sigma$ is a bijective mapping, denotes a renaming. If $e$ is an expression, then $e(x_1 \mapsto x'_1, \ldots, x_n \mapsto x'_n)$ is the result of simultaneously replacing the variables $x_1, \ldots, x_n$ with the corresponding variables $x'_1, \ldots, x'_n$, respectively, in the expression $e$.

**Definition 2.4** (Context). A context $C$ is an expression with a “hole” $[]$ in the place of one sub-expression. $C[e]$ is the expression obtained by replacing the hole in context $C$ with the expression $e$. Free variables within $e$ may become bound within $C[e]$; if $C[e]$ is closed then we call it a closing context for $e$.

**Definition 2.5** (Evaluation Contexts, Redexes and Observables). Evaluation contexts, redexes and observables are defined as shown in Figure 3, where $E$ ranges over evaluation contexts, $R$ ranges over redexes and $O$ ranges over observables.

**Lemma 2.6** (Unique Decomposition Property). For every expression $e$, either $e$ is an observable or there is a unique evaluation context $E$ and redex $e'$ such that $e = E[e']$.

The call-by-name operational semantics of our language is standard: define an evaluation relation $\Downarrow$ between closed expressions and values, where values are expressions in weak head normal form (i.e. constructor applications or $\lambda$-abstractions). We define a one-step reduction relation $\Downarrow^*$ inductively as shown in Figure 4, where the reduction $r$ can be $f$ (unfolding of function $f$), $c$ (elimination of constructor $c$) or $\Downarrow e$ (substitution of the expression $e$).

We use the notation $e \Downarrow^*$ if the expression $e$ reduces, $e \Downarrow$ if $e$ diverges, $e \Downarrow^*$ if $e$ converges and $e \Downarrow v$ if $e$ evaluates to the value $v$. These are defined as follows, where $\Downarrow^*$ denotes the reflexive transitive closure of $\Downarrow$:

- $e \Downarrow^* v$, iff $\exists e' \Downarrow v$;
- $e \Downarrow v$, iff $\forall e' \Downarrow^* v \wedge (v \Downarrow^*)$

**Definition 2.7** (Observational Equivalence). Observational equivalence, denoted by $\simeq$, equates two expressions if and only if they exhibit the same termination behaviour in all closing contexts i.e. $e_1 \simeq e_2$ iff $\forall C, C[e_1] \Downarrow \Downarrow C[e_2] \Downarrow$.

## 3. Labelled Transition Systems

In this section, we use a labelled transition system to characterise the run-time behaviour of a functional program. In the spirit of Gordon [5], we define a particular labelled transition system, in general infinite, that characterises the immediate observations that can be made on expressions to determine their observational equivalence. We extend [5] by allowing free variables in expressions and thus also in actions. Observational equivalence will therefore require that the free variables in actions match up in addition to the bound variables. A similar extension has also been done for environmental bisimulations by Sangiorgi et al. [19].

**Definition 3.1** (Driven LTS). The driven LTS associated with program $e_0$ is given by $t = (E, e_0, \rightarrow, \text{Act})$ where:

- $E$ is the set of states of the LTS. Each is an expression, or the end-of-action state $0$.
- $t$ always contains as root the expression $e_0$. Notation: root$(t)$.
- $\rightarrow \subseteq E \times \text{Act} \times E$ is a transition relation that relates pairs of states by actions according to Figure 5.
- If $e \in E$ and $e \Downarrow^* e'$ then $e' \in E$.
- $\text{Act}$ is a set of actions that can be silent or non-silent. A non-silent action may be: $x$, a variable; $c$, a constructor; $\#i$, the $i$th argument in an application; $\text{let}$, an abstraction.

Within the actions of a LTS, $\lambda$-abstracted variables, case pattern variables and let variables are bound; all other variables are free. We use $fv(t)$ and $bv(t)$ to denote the free and bound variables respectively of LTS $t$. No let transitions will appear in the driven LTS; these are only introduced later as a result of generalization.
\[ (f = e) \in \Delta \]

\[ f \sim_e e \]

\[ (\lambda x.e_0) e_1 \sim^{\Delta^1} (e_0 \{ x \mapsto e_1 \}) \]

\[ e_0 \sim_{e_0'} \]

\[ \text{case } (c e_1 \ldots e_n) \text{ of } p_1 : e_1' \ldots \| p_k : e_k' \sim (c_0 \{ x_1 \mapsto e_1, \ldots, x_n \mapsto e_n \}) \]

\[ e_0 \sim_{e_0'} \]

\[ \text{case } e_0 \text{ of } p_1 : e_1 \ldots \| e_0' \}

\[ \text{case } e_0' \text{ of } p_1 : e_1 \ldots \| p_k : e_k \]

\[ \begin{array}{l}
D[e] = D'[e] \emptyset \\
D'[e = x e_1 \ldots e_n] \theta = \{ e \rightarrow (\tau_0; \theta(x), D'[\theta(x) e_1 \ldots e_n] \theta), \text{ if } x \in \text{dom}(\theta) \\
D'[e = \lambda x.e] \theta = e \rightarrow (c, 0), (#1, D'[e_1] \theta), \ldots, (#n, D'[e_n] \theta) \}
D'[e = E[f]] \theta = e \rightarrow (\lambda x, D'[e] \theta) \}
D'[e = E[\text{case } x \text{ of } p_1 \Rightarrow e_1 \ldots \| p_k \Rightarrow e_k]] \theta = \{ e \rightarrow (\tau_0; \theta(x), D'[\theta(x) e_1 \ldots e_n] \theta), \text{ if } x \in \text{dom}(\theta) \\
D'[e = E[\text{case } (c e_1 \ldots e_n) \text{ of } p_1 \Rightarrow e_1' \ldots \| p_k \Rightarrow e_k]] \theta = e \rightarrow (\text{case}, D'[x e_1 \ldots e_n] \theta, (p_1, D'[E[e_1]] \theta), \ldots, (p_k, D'[E[e_k]] \theta) \text{ if } x \rightarrow p_k) \}
D'[e = E[\text{case } (c e_1 \ldots e_n) \text{ of } p_1 \Rightarrow e_1' \ldots \| p_k \Rightarrow e_k]] \theta = e \rightarrow (\tau_n; D'[E[e_1]] \theta), \ldots, (p_k, D'[E[e_k]] \theta) \}
\end{array} \]

\[ \begin{array}{l}
D'[e = x e_1 \ldots e_n] \theta = \{ e \rightarrow (\tau_0; \theta(x), D'[\theta(x) e_1 \ldots e_n] \theta), \text{ if } x \in \text{dom}(\theta) \\
D'[e = \lambda x.e] \theta = e \rightarrow (c, 0), (#1, D'[e_1] \theta), \ldots, (#n, D'[e_n] \theta) \}
D'[e = E[f]] \theta = e \rightarrow (\lambda x, D'[e] \theta) \}
D'[e = E[\text{case } x \text{ of } p_1 \Rightarrow e_1 \ldots \| p_k \Rightarrow e_k]] \theta = \{ e \rightarrow (\tau_0; \theta(x), D'[\theta(x) e_1 \ldots e_n] \theta), \text{ if } x \in \text{dom}(\theta) \\
D'[e = E[\text{case } (c e_1 \ldots e_n) \text{ of } p_1 \Rightarrow e_1' \ldots \| p_k \Rightarrow e_k]] \theta = e \rightarrow (\text{case}, D'[x e_1 \ldots e_n] \theta, (p_1, D'[E[e_1]] \theta), \ldots, (p_k, D'[E[e_k]] \theta) \text{ if } x \rightarrow p_k) \}
D'[e = E[\text{case } (c e_1 \ldots e_n) \text{ of } p_1 \Rightarrow e_1' \ldots \| p_k \Rightarrow e_k]] \theta = e \rightarrow (\tau_n; D'[E[e_1]] \theta), \ldots, (p_k, D'[E[e_k]] \theta) \}
\end{array} \]

\[ \begin{array}{l}
\text{DEFINITION 3.3 (LTS Substitution). } \theta = \{ x_1 \rightarrow t_1, \ldots, x_n \rightarrow t_n \} \text{ denotes a LTS substitution. If } t \text{ is a LTS, then } \theta t = \{ x_1 \rightarrow t_1, \ldots, x_n \rightarrow t_n \} \} \}
\end{array} \]

\[ \begin{array}{l}
\text{DEFINITION 3.4 (Pure LTS). We define a pure LTS to be one that contains no let transitions. An impure LTS can be converted into a behaviourally equivalent pure LTS using the purity operation } P \text{ which is defined as follows:} \\
P[e \rightarrow (\text{let}, t_0), (x_1, t_1)] = P[t_0; (x \rightarrow t_1)] \\
P[e \rightarrow (\alpha_1, t_1), \ldots, (\alpha_n, t_n)] = e \rightarrow (\alpha_1, P[t_1], \ldots, \alpha_n, P[t_n])
\end{array} \]

\[ \begin{array}{l}
\text{DEFINITION 3.5 (Weak Simulation). Binary relation } R \subseteq E_1 \times E_2 \text{ is a weak simulation of pure LTS } (E_1, e_0, \rightarrow_1, Act_1) \text{ by pure LTS } (E_2, e_0', \rightarrow_2, Act_2) \text{ if } e_0, e_0' \in R \text{, and for every pair } (e_1, e_2) \in R \text{, } \alpha \in Act_1, e_1 \in E_1:} \\
\text{if } e_1 \not\sim e'_1 \text{ then } \exists e_2 \in E_2 \text{ such that } e_2 \not\sim e_2 \wedge (e_1, e_2) \in R
\end{array} \]

\[ \begin{array}{l}
\text{DEFINITION 3.6 (Weak Bisimulation). A weak bisimulation is a binary relation } R \text{ such that both } R \text{ and its inverse } R^{-1} \text{ are weak simulations.}
\end{array} \]
DEFINITION 3.7 (Weak Bisimilarity). If there exists a weak bisimulation \( R \) between pure LTSs \((E_1, e_1^0, \rightarrow_1, \text{Act}_1)\) and \((E_2, e_2^0, \rightarrow_2, \text{Act}_2)\), then there exists a unique maximal one, henceforth denoted by \( \sim \). A notation: we also write \( e_1^0 \sim e_2^0 \) in place of \((e_1^0, e_2^0) \in \sim\).

A central property: the weak bisimilarity relation \( \sim \) is a congruence, and coincides with observational equivalence.

THEOREM 3.8 (Congruence). \( \forall C . e \sim e' \Rightarrow C[e] \sim C[e'] \)

PROOF 3.9. Similar to that of Howe [10].

THEOREM 3.10 (Operational Extensionality). \( \simeq = \sim \)

PROOF 3.11. The proof that \( \sim \subseteq \simeq \) follows from the congruence of \( \sim \). The reverse inclusion follows by co-induction after showing that \( \sim \) is a bisimulation on \( D[e] \).

4. Distillation

In this section, we define the distillation program transformation algorithm within our LTS framework. The algorithm is actually very similar to the positive supercompilation algorithm [21]. The distillation algorithm computes \( R[D[e]] \). This can be viewed graphically as follows:

Given a program \( e \in PGM \), the driving rules \( D \) in Figure 5 are applied to construct the LTS \( D[e] \) with root \( e \). Transformation rules \( T \) are then applied to the LTS \( D[e] \) to transform it into a LTS that contains a finite number of states.

Although the LTS \( D[e] \) can be infinite in general, the transformation rules \( T \) will traverse only finite portions of \( D[e] \) by working lazily from the root. If a danger of non-termination is detected (“the whistle is blown”), then generalization is performed to convert the current LTS into a new version without danger of non-termination. Overall, the effect of generalization is to ensure that a renaming of a previously encountered LTS will eventually be encountered, at which point folding can be applied to convert the current LTS into a new version with only a finite num-
4.1 Generalization

Generalization is performed when an LTS is encountered in which a previously encountered LTS is embedded. The embedding relation on LTSs is defined as follows. Its purpose is to guarantee termination of distillation; more on this in Section 5.

**Definition 4.1 (LTS Embedding).** LTS \( t_1 \) is embedded within LTS \( t_2 \) if there is a renaming \( \sigma \) such that \( t_1 \sqsubseteq_\sigma t_2 \), where the reflexive, transitive and anti-symmetric relation \( \sqsubseteq_\sigma \) is defined as shown in Figure 7.

One LTS is embedded within another by this relation if either diving (denoted by \( \ll_\sigma \)) or coupling (denoted by \( \bowtie_\sigma \)) can be performed. In the rules for diving, a transition can be followed from the current state in the embedding LTS that is not followed from the current state in the embedded one. In the rules for coupling, the same transitions are possible from each of the current states. Matching transitions may contain different free variables; in this case the transition labels should respect the renaming \( \sigma \). Matching transitions may also introduce bound variables (\( \lambda \), case and let); in these cases the corresponding bound variables are added to the renaming \( \sigma \). The parameter \( \rho \) is used to ensure termination of the rules; a record is kept in \( \rho \) of previously encountered function unfolding and substitution transitions. If one of these recorded transitions is encountered again no further comparison is done.

In the diagram above, distillation compares a current LTS with previously encountered LTSs. The main difference between distillation and positive supercompilation [9, 21] is that generalization and folding are performed with respect to LTSs in distillation, while they are performed with respect to expressions in positive supercompilation.

Distillation compares a current LTS with previously encountered ones to determine whether to generalize or to fold. This is particularly tricky as the objects being compared are potentially infinite. However, as any infinite sequence of transitions within either LTS must contain either a function unfolding or a substitution, a record is kept of previously encountered function unfolding and substitution transitions. If one of these recorded transitions is encountered again no further comparison is done.

\[ G[e \rightarrow (\tau_f, t)] [e' \rightarrow (\tau_f, t')] \theta \sigma \]

\[ = (G'[e' \rightarrow (\tau_f, t)] \theta', 0) \]

where

\[ (tg, \theta') = G[t][t'] \theta \sigma \]

\[ = (G'[e' \rightarrow (\tau_{i+1}, t)] \theta', 0), \text{if } e_1 \sigma = e_2 \]

where

\[ G[e \rightarrow (\alpha_1, t_1), \ldots, (\alpha_n, t_n)] [e' \rightarrow (\alpha_1', t_1'), \ldots, (\alpha_n', t_n')] \theta \sigma \]

\[ = (fv(t') \cap rng(\sigma)) \]

\[ t_2 = \text{root}(t') \overrightarrow{\lambda_2 \cdots \lambda_n} t' \]

\[ G'[e \rightarrow (\alpha, t)] = e \rightarrow (\alpha, t) \]

\[ G'[e \rightarrow (\alpha, t)] (\{ x \mapsto t' \} \cup \theta) = e \rightarrow (\text{let}, G'[e \rightarrow (\alpha, t)] \theta), (x, t') \]

One LTS is embedded within another by this relation if either diving (denoted by \( \ll_\sigma \)) or coupling (denoted by \( \bowtie_\sigma \)) can be performed. In the rules for diving, a transition can be followed from the current state in the embedding LTS that is not followed from the current state in the embedded one. In the rules for coupling, the same transitions are possible from each of the current states. Matching transitions may contain different free variables; in this case the transition labels should respect the renaming \( \sigma \). Matching transitions may also introduce bound variables (\( \lambda \), case and let); in these cases the corresponding bound variables are added to the renaming \( \sigma \). The parameter \( \rho \) is used to ensure termination of the rules; a record is kept in \( \rho \) of previously encountered function unfolding and substitution transitions and if one of these transitions is re-encountered no further rules are applied. The rules for diving and coupling are applied in top-down order, where the final rule is a catch-all.

**Definition 4.2 (Generalization of LTSs).** Figure 8 defines function \( G[t][t'] \theta \sigma \) that generalizes LTS \( t' \) with respect to LTS \( t \), where \( \theta \) is the set of previous generalizations that can be reused and \( \sigma \) is a renaming s.t. \( t \bowtie_\sigma t' \).

Figure 8 performs generalization incrementally from the roots of the two LTSs, the “increment” being the interval between function unfolding and substitution transitions. The generalizations that are performed within each of these intervals are extracted using let's at the start of the interval. These generalizations are then passed down for use in further generalizations. If the same generalization is subsequently encountered again, the same generalization variable is reused. The let's that are introduced will therefore appear just
The generalization rules $G$ are applied in top-down order. Generalization is performed as follows. If two corresponding states have the same transitions, these transitions remain in the resulting generalized LTS, and the corresponding targets of these transitions are then generalized. Matching transitions may contain different free variables; in this case the transition labels should respect the renaming $\sigma$. Matching transitions may also introduce bound variables ($\lambda$, case and let); in these cases the corresponding bound variables are added to the renaming $\sigma$. Unmatched states are replaced by variable applications. The arguments of these applications are the variables of the second LTS which differ from those of the first LTS according to the renaming $\sigma$; this also ensures that bound variables are not extracted outside their binders. It is assumed that the new variables introduced are all different and distinct from the original program variables.

**Example 4.3.** In the driven LTS of Figure 6, it can be seen that the LTS $\dagger$ is embedded in the LTS $\ddagger$. Figure 9 shows a portion of the LTS resulting from generalizing LTS $\ddagger$ with respect to LTS $\dagger$. Within the result, one can see a further embedding: LTS $\ast$ is embedded in the LTS $\ast\ast$. Next step: Figure 10 shows a portion of the result of generalizing the LTS $\ast\ast$ with respect to LTS $\ast$.

### 4.2 Folding

Folding is performed when a LTS is encountered that is a renaming of a previously encountered LTS. The renaming relation on LTSs is defined as follows.

**Definition 4.4 (LTS Renaming).** LTS $t_1$ is a renaming of LTS $t_2$ iff there is a renaming $\sigma$ such that $t_1 \sim_\sigma^0 t_2$, where the reflexive, transitive and symmetric relation $\sim_\sigma^0$ is defined in Figure 11.

The rules for renaming are applied in top-down order, where the final rule is a catch-all. Two LTSs are renamings of each other if the

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**Figure 9.** Portion of Generalized LTS for $nrev \, xs$ before function unfolding and substitution transitions and will be distributed throughout the generalized LTS.

The generalization rules $G$ are applied in top-down order. Generalization is performed as follows. If two corresponding states have the same transitions, these transitions remain in the resulting generalized LTS, and the corresponding targets of these transitions are then generalized. Matching transitions may contain different free variables; in this case the transition labels should respect the renaming $\sigma$. Matching transitions may also introduce bound variables ($\lambda$, case and let); in these cases the corresponding bound variables are added to the renaming $\sigma$. Unmatched states are replaced by variable applications. The arguments of these applications are the variables of the second LTS which differ from those of the first LTS according to the renaming $\sigma$; this also ensures that bound variables are not extracted outside their binders. It is assumed that the new variables introduced are all different and distinct from the original program variables.
same transitions are possible from each corresponding state (modulo variable renaming according to the renaming \( \sigma \)). The definition also handles transitions that introduce bound variables (\( \lambda, \text{case} \) and \( \text{let} \)); in these cases the corresponding bound variables are added to the renaming \( \sigma \). The parameter \( \rho \) is used to ensure termination of the rules in the same manner as the rules for embedding.

**Definition 4.5** (Folding of LTSs). Folding with respect to the previously encountered LTS \( t \) with the renaming \( \sigma \) is performed by the function \( \mathcal{F}[t] \sigma \), which is defined as follows.

\[
\mathcal{F}[t] \emptyset = t
\]

\[
\mathcal{F}[t] (\{x \mapsto x'\} \cup \sigma) = \text{root}(t) \rightarrow \langle \text{let}, \mathcal{F}[t] \emptyset \rangle, (x, x' \xrightarrow{t} 0)
\]

**Example 4.6.** The LTS + in Figure 10 is a renaming of the LTS + in Figure 9, so folding will be applied at this point.
\[
T[e] = T[e] \emptyset \emptyset
\]

\[
T'[t = e \rightarrow (\tau_f, t')] \pi \theta = \begin{cases} 
  F[t'][\sigma], & \text{if } \exists t'' \in \pi, \sigma.t' \leadsto_0 t \\
  T'[G[t'][\theta] \pi \phi, \text{ otherwise if } \exists t'' \in \pi, \sigma.t' \leadsto_0 t \\
  \end{cases}
\]

\[
T'[t = e \rightarrow (\tau_{\text{new}}, t')] \pi \theta = \begin{cases} 
  F[t'][\sigma], & \text{if } \exists t'' \in \pi, \sigma.t' \leadsto_0 t \\
  T'[G[t'][\theta] \pi \phi, \text{ otherwise if } \exists t'' \in \pi, \sigma.t' \leadsto_0 t \\
  \end{cases}
\]

\[
T'[t = e \rightarrow (\text{let}, t_0, (x, t_1)) \pi \theta = \begin{cases} 
  T'[t_0 (x \mapsto x')] \pi \theta, & \text{if } \exists (x' \mapsto t_2) \in \theta.t_1 \leadsto_0 t_2 \\
  (t, T'[t_1] \pi \theta), \text{ otherwise if } \exists (x, T'[t_1] \pi \theta) \\
  \end{cases}
\]

\[
T'[e \rightarrow (\alpha_1, t_1), \ldots, (\alpha_n, t_n)] \pi \theta = e \rightarrow (\alpha_1, T'[t_1] \pi \theta), \ldots, (\alpha_n, T'[t_n] \pi \theta)
\]

**Figure 12. Transformation Rules**

\[
\mathcal{R}[e] = \mathcal{R}'[e] \emptyset \emptyset
\]

\[
\mathcal{R}'[e \rightarrow (x, 0), (\#1, t_1), \ldots, (\#n, t_n)] \epsilon = x (\mathcal{R}'[t_1] \epsilon, \ldots, (\mathcal{R}'[t_n] \epsilon)
\]

\[
\mathcal{R}'[e \rightarrow (c, 0), (\#1, t_1), \ldots, (\#n, t_n)] \epsilon = c (\mathcal{R}'[t_1] \epsilon), \ldots, (\mathcal{R}'[t_n] \epsilon)
\]

\[
\mathcal{R}'[e \rightarrow (\lambda x.(\mathcal{R}'[t] \epsilon)] \epsilon = \lambda x.(\mathcal{R}'[t] \epsilon)
\]

\[
\mathcal{R}'[e \rightarrow \text{case } (\pi, \sigma, t) \pi \phi \text{ of } \pi_1 \Rightarrow (\mathcal{R}'[t_1] \epsilon) \cdots | \pi_k \Rightarrow (\mathcal{R}'[t_k] \epsilon)
\]

\[
\mathcal{R}'[e \rightarrow \text{let } (\pi, t_0, \pi_1, \ldots, (\#n, t_n)] \epsilon = \begin{cases} 
  f' x_1 \ldots x_n \text{ if } \exists f' x_1 \ldots x_n = e \epsilon & \epsilon \\
  f x_1 \ldots x_n \text{ where } f = \lambda x_1 \ldots x_n.(\mathcal{R}'[t] \epsilon \cup \{ f' x_1 \ldots x_n = e \epsilon \}) \text{, otherwise (}f' \text{ is fresh, } \{ x_1 \ldots x_n = f'(t) \} \\
  f x_1 \ldots x_n \text{ where } f = \lambda x_1 \ldots x_n.((\mathcal{R}'[t] \epsilon \cup \{ f x_1 \ldots x_n = e \epsilon \}) \text{, otherwise (}f \text{ is fresh, } \{ x_1 \ldots x_n = f(t) \}
\end{cases}
\]

\[
\mathcal{R}'[e \rightarrow (\tau_f, t)] \epsilon = \begin{cases} 
  f' x_1 \ldots x_n \text{ where } f = \lambda x_1 \ldots x_n.(\mathcal{R}'[t] \epsilon \cup \{ f' x_1 \ldots x_n = e \epsilon \}) & \epsilon \\
  f x_1 \ldots x_n \text{ where } f = \lambda x_1 \ldots x_n.((\mathcal{R}'[t] \epsilon \cup \{ f x_1 \ldots x_n = e \epsilon \}) \text{, otherwise (}f \text{ is fresh, } \{ x_1 \ldots x_n = f(t) \}
\end{cases}
\]

\[
\mathcal{R}'[e \rightarrow (\tau_{\text{new}}, t)] \epsilon = \mathcal{R}'[t] \epsilon
\]

**Figure 13. Rules For Residualization**

### 4.3 Transformation Rules

The rules to transform a driven LTS are shown in Figure 12. It is assumed that rules \(T'\) are applied in top-down order, where the final rule is a catch-all. The driven LTS is traversed lazily from the root. As any infinite sequence of steps must include function unfolding or substitution, all LTSs in which there is such a transition from the root are recorded in the environment \(\pi\). If a previously recorded LTS is a renaming of the current one then folding is done, creating a transition back to a previously encountered state. If a previously recorded LTS is coupled with the current one, the current LTS is generalized, and we continue on the generalized LTS. In these rules, the parameter \(\theta\) is used to record all components previously extracted by generalization. If the same component is subsequently extracted, then it is replaced by the same variable.

### 4.4 Residualization

**Definition 4.7 (Residual Program Construction).** A residual program can be constructed from a folded LTS using rules \(R\) in Figure 13. In these rules, the parameter \(e\) contains the set of new function calls that have been created, and associates them with the expressions they replace. On re-encountering an associated expression, it is also replaced by the corresponding function call.

**Example 4.8.** The program constructed from the LTS resulting from the distillation of \(\text{new } x\), via Figure 10 and Example 4.6, is shown in Figure 14. We can see that the distilled program is a super-linear improvement over the original.

### 5. Termination, Correctness and Efficiency

In this section, we consider the termination and correctness of the distillation algorithm and the efficiency gains that can be obtained.

**Figure 14. Result of Applying Distillation to \(\text{new } x\)**

### 5.1 Termination

**Theorem 5.1 (Termination of Distillation).** The distillation algorithm always terminates.

**Proof.** We firstly need to show that in any infinite sequence of LTSs encountered during transformation \(t_0, t_1, \ldots\) there definitely exists some \(i < j, \sigma\) where \(t_i \triangleright 0^\sigma t_j\), so an embedding must eventually be encountered and transformation will not continue indefinitely without the need for generalization. This amounts to proving that the embedding relation \(t_i \triangleright 0^\sigma t_j\) is a well-quasi order.

**Definition 5.3 (Well-Quasi Order).** A well-quasi order on a set \(S\) is a reflexive, transitive relation \(\leq\) such that for any infinite sequence \(s_1, s_2, \ldots\) of elements from \(S\) there are numbers \(i, j\) with \(i < j\) and \(s_i \leq s_j\).

**Lemma 5.4.** \(\triangleright 0^\sigma\) is a well- quasi order on any sequence of LTSs that are encountered during transformation.
Proof 5.5. Nash-Williams has previously shown that the infinite trees are well-quasi ordered under the topological minor relation \cite{15}. The sequence of LTSs that are encountered during transformation obviously belong to the topological minor relation, as each successive LTS is a sub-tree of the previous one. The embedding relation $\leq_{\text{top}}$ is therefore a well-quasi order. To show that $\leq_{\text{top}}$ is also a well-quasi order, we additionally need to show that in the sequence of LTSs encountered during transformation, the transition from the root state must eventually re-occur. This must be the case since in any infinite sequence of transitions, either a function unfolding or a substitution transition must re-occur.

When we compare LTSs for possible embeddings or renamings, only a finite portion of the LTSs are inspected, namely the portion up to a re-occurrence of a function unfolding or substitution transition. We call this portion the core component of the LTS. In order to prove that the distillation algorithm terminates, we need to ensure that there is a bound on the size of the core components that are used to compare for possible embeddings. If there is such a bound, then a renaming of a previous core component must eventually be encountered at which point folding can be applied and the transformation will terminate.

Definition 5.6 (Size of Core Component of LTS). The size of the core component of LTS $t$, denoted by $|t|$, is defined as follows:

$$|t| = \mathcal{S}[t] \emptyset$$

$$\mathcal{S}[e \rightarrow \tau_f t] = \begin{cases} 0, & \text{if } f \in \rho \\ 1 + \mathcal{S}[t] (\rho \cup \{f\}), & \text{otherwise} \end{cases}$$

$$\mathcal{S}[e \rightarrow \tau_{\alpha e} t] = \begin{cases} 0, & \text{if } (\downarrow \alpha e') \in \rho \\ 1 + \mathcal{S}[t] (\rho \cup \{\downarrow e'\}), & \text{otherwise} \end{cases}$$

$$\mathcal{S}[e \rightarrow (\alpha_1 t_1), \ldots, (\alpha_n t_n)] = 1 + \sum_{i=1}^{n} \mathcal{S}[t_i] \rho$$

Lemma 5.7. There is a bound on the size of the core components of the LTSs that are encountered during transformation.

Proof 5.8. We define an extraction operation $\Lambda'$ which removes lets from a LTS as follows:

$$\Lambda'[e \rightarrow \text{let}, t_0, (x, t_1)] = \Lambda[t_0]$$

$$\Lambda'[e \rightarrow (\alpha_1 t_1), \ldots, (\alpha_n t_n)] = e \rightarrow (\alpha_1 \Lambda'[t_1]), \ldots, (\alpha_n \Lambda'[t_n])$$

If a LTS $t$ is encountered where $t' \equiv_{\rho, t}$ for a previously encountered LTS $t_0$, then generalization will be performed yielding a generalized LTS $t_0$, where $|\mathcal{X}[\text{let}]| \leq |t'|$. There will therefore be a bound on the size of the core components of the LTSs that are encountered during transformation if a point is reached after which no new lets are introduced by successive generalization passes. This will be the case if there is a bound on the size of the components that are extracted in each successive generalization pass and the same generalization variables will be reused without introducing new lets. This must be the case as extracted components which would otherwise grow in size will be generalized with respect to previously occurring extracted components, and can therefore be no larger than them.

5.2 Correctness

In order to show that the LTS resulting from transformation is observationally equivalent to the original LTS, we need to show that they are weakly bisimilar. Weak bisimilarity is used because silent transitions in the original LTS may have been removed by transformation. To prove this, the transformed LTS has to be purified using the operation $\mathcal{P}$ as given in Definition 3.4. The correctness of the algorithm can now be stated as follows.

Theorem 5.9 (Correctness of Distillation).

$\forall t \in \text{LTS} . \mathcal{P}[t] \sim \mathcal{P}[\mathcal{T}[t]]$

Proof 5.10. The proof is by induction on the transformation rules $\mathcal{T}$. The rules that do not involve generalization or folding can be proved straightforwardly by application of the inductive hypothesis. For the rules involving generalization and folding, we need to prove the following:

Lemma 5.11 (Correctness of Generalization).

$\forall t, t', \alpha, \sigma . \mathcal{P}[t] \sim \mathcal{P}[\mathcal{G}(t', \alpha, \sigma)]$

Lemma 5.12 (Correctness of Folding).

$\forall t, t', \sigma . ((t') \rightarrow_{\rho, t} \sigma) \Rightarrow \mathcal{P}[t] \sim \mathcal{P}[\mathcal{F}(t', \sigma)]$

These can be shown by straightforward induction on the LTS $t$.

5.3 Efficiency

We now look at the efficiency gains that can be obtained using distillation, and try to give some intuition as to how superlinear speedups are obtained. Speedups are obtained in distillation through two mechanisms; removing silent transitions from the LTS representation of a program, and the identification of extracted components resulting from generalization. Sorensen has previously shown that removing silent transitions can only produce a linear speedup in programs \cite{20}. This is because there will only be a constant number of such silent transitions between each recursive call of a function. Hence, only linear speedups are possible. It is therefore the identification of extracted components resulting from generalization that produces superlinear speedups.

Definition 5.13 (Run-Time). We define $t(\theta)$, the run-time of program $e$ with ground data given by the substitution $\theta$, as the number of states in the driven LTS $D[e[\theta]]$.

Theorem 5.14 (Superlinearity of Speedups).

$\exists e \in \text{Exp} . R[t'[D[e]]] = e' \Rightarrow \exists n . \forall \theta . (t(\theta) \geq t(\theta))$

Proof 5.15. The components that are extracted from between successive function calls and subsequently identified can be of greater than constant complexity. The higher complexity of these components may be entirely due to silent transitions that will be removed by transformation, producing components that can be subsequently identified. For example, the components that are extracted from between successive calls in the naive reverse example running through this paper correspond to the terms $\text{Cons} \times \text{Nil}$, $\text{app} \text{Nil} (\text{Cons} \times \text{Nil})$, $\text{app} \text{Nil} (\text{app} \text{Nil} (\text{Cons} \times \text{Nil}))$, and are therefore of linear complexity with respect to the size of the input list. However, all of these components can be transformed to a component corresponding to the term $\text{Cons} \times \text{Nil}$ and subsequently identified.

If components are extracted from between $N$ recursive calls of a function on input data of size $N$, and the run-time of the extracted and remaining components within each function call are given by $f_1(N)$ and $f_2(N)$ respectively, then the run-time of the function before generalization will be $t_1 = \sum_{i=0}^{n-1} (f_1(i) + f_2(i))$. If the extracted components are of greater than constant complexity then $\exists n . \forall i . f_1(i) \leq n$. If these components are transformed to become of complexity $f_1(0)$ and subsequently identified, then the run-time of the function after generalization will be $t_2 = f_1(0) + \sum_{i=0}^{n-1} f_2(i)$. Therefore $\exists n . \forall i . f_2(i) \leq n$. Therefore $\exists n . \forall i . f_2(i) \leq n$.

6. Conclusion and Related Work

We have presented a semantic basis for the distillation program transformation algorithm and explained why superlinear speedups can be achieved that cannot be obtained by other fully automatic
program transformation techniques such as deforestation, positive supercompilation and partial evaluation.

Previous works [1, 2, 13, 20, 26] have noted that the unfold/fold transformation methodology is incomplete; some programs cannot be synthesized from each other. This is partly due to the linear relationship that must exist between the original and transformed programs. This paper has shown how to overcome this restriction.

Sand’s theory of local improvement [18] was seminal in proving correctness of unfold/fold program transformations. However, it is complicated by the use of folding, which causes a loss of efficiency locally, but not globally if always done in conjunction with a corresponding unfold. Also, the correctness of transformations to an improvement in efficiency restricts this approach to particular program semantics and transformation techniques.

Distillation can be considered as a method that works on the meta-level above supercompilation, where instead of sets of states (“configurations”) the results of supercompilation (“residual graphs”) are manipulated. There have been several attempts to do this, the first one by Turchin himself using walk grammars [23]. In this approach, traces through residual graphs are represented by regular grammars that are subsequently analysed and simplified. This approach is also capable of achieving superlinear speedups, but no automatic procedure is defined for it; the outlined heuristics and strategies may not terminate.

The most recent work on building a meta-level above supercompilation is by Klyuchnikov and Romanenko [12]. They construct a hierarchy of supercompilers in which lower level supercompilers are used to prove lemmas about term equivalences, and higher level supercompilers utilise these lemmas by rewriting according to the term equivalences (similar to the “second order replacement method” defined by Kott [14]). This approach is also capable of achieving superlinear speedups, but again no automatic procedure is defined for it; the need to find and apply appropriate lemmas introduces infinite branching into the search space, and various heuristics have to be used to try to limit this search.

Logic program transformation is closely related, and the equivalence of partial deduction and driving has been argued by Glück and Sørensen [4]. Superlinear speedups can be achieved in logic program transformation by goal replacement [16, 17]: replacing one logical clause with another to facilitate folding. Techniques similar to the notion of “higher level supercompilation” [12] have been used to prove correctness of goal replacement, but have similar problems regarding the search for appropriate lemmas.

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References


