A cold system with two different components and a single vacation of the repairman

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ABSTRACT

In this paper, we model a cold standby repairable system with one repairman. Suppose that the system consists of two dissimilar components and one repairman, in which component 1 is the major working unit and component 2 is the cold stand-by unit, component 1 has priority in use and repair; then the repairman takes a single vacation. Under these assumptions, we derive a model of partial differential equations. Furthermore, using the functional analysis method, we discuss the dynamic asymptotical behavior of the system and get some reliability indices. Finally we analyze a variety for indices of reliability and profit of the system with change of other parameters of the system by numerical simulation.

1. Introduction

With the development of modern technology and the world economy, the reliability problem of a system attracts more and more attentions. More and more people are working on the reliability of systems including the economic system, in particular, on the reliability of repairable systems. In the repairable system aspect, there are two points being noted: one is the deteriorating system, for instance, see [1–5]; the other is that the system can be repaired “as good as new” but the repairman takes the vacation, here we refer to [6–10]. In fact, the former considers a profile of long term running of the system while the latter is based on consideration of the present economic case. It is well known that with increasing time, the system fails easily. In order to enhance the reliability of the system, some people propose a strategy consisting of increasing standby components for example, see [11–15]. Obviously, it has a larger cost. So it is more economical to use a cold standby system that consists of a main component, a standby component and a repairman on vacation. We note that the cold standby component is in the waiting state for a long time, it need not to be the same as the main component from an economic point of view. On the other hand, the vacation rate of the repairman in [10] is assumed to be a constant; in order to retain the profits of the system the vacation rate should be changed. Based on the above reasons, we consider in the present paper a cold standby system with two different components and the repairman under a single vacation. A single vacation means that the repairman returns to the system after each vacation; if all the components are in good condition, he should stay in the system until some component is invalid, then he would take a vacation after the repair. If there is a broken component, the repairman begins to repair the component immediately, then he takes his new vacation. Herein we assume that if some component is invalid during the vacation of the repairman, it will be in the state of waiting for repair until the repairman returns. The detailed description of the system will be given in Section 2. We shall establish a mathematical model for such a system and then study the reliability indices of the system.

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The rest of this paper is organized as follows: in Section 2, we describe the system in detail and establish a mathematical model. In Section 3, we prove the existence and uniqueness of nonnegative time-dependent solution of the system via $C_0$ semigroup theory of the bounded linear operator. In Section 4, we give the stability results of the system based on the spectral analysis of the system operator. Further, in Section 5, we study the exponential stability of the steady state of the system. In Section 6, we discuss some indices of the reliability of the system including the availability and the failure frequency and analyze the profit of the system. In addition, we study if the change of these indices is dependent on other parameters, and give some numerical simulation. Finally in Section 7, we conclude the results of the present paper.

2. Mathematical modeling

In this section we shall model a cold standby system under consideration. Firstly we give a detailed description of the system, and then establish a model via partial differential equation.

2.1. Description of system

Here we consider a cold standby system consisting of two components and one repairman. Component 1 is the main unit and component 2 is the standby unit. The repairman can take a vacation. The system obeys the following rules:

(1) At time $t = 0$, the two components are both in good condition, component 1 is in a working state while component 2 is in a standby state. The repairman is waiting in the system until component 1 fails. After its repair, he will begin to take his vacation.

(2) If both components fail, then the system fails.

(3) Both components will be repaired as good as before.

(4) Component 1 has priority in working and repair. When we divide the proof into the following several, component 1 has a higher use priority than component 2. Even though component 2 is in working condition, it must be switched immediately into the standby state, as soon as component 1 gets repaired and becomes the working state. When the system fails, the component 1 has higher repair priority than component 2. Even if the repairman is repairing component 2, he has to switch to repair component 1 if it fails. Only after that, he continues to repair component 2.

(5) When the repairman comes back after his vacation, there are three cases: one is that component 1 fails and is waiting for repair, in this case he begins to repair it; two is that both components are in a state of waiting for repair, in this case he begins to repair component 1 immediately, then he repairs the other one after completing the repair on component 1; the third is that the machine is working, he will stay in the system until there is one component failure. In the first and second cases, after the repair, the repairman begins his vacation immediately.

(6) If the machine fails during the vacation of the repairman, the system is in the state of waiting for repair.

(7) The standby component will not fail in the standby time.

(8) The failure and repair times of both components follow the exponential distribution and the general distribution respectively. $\lambda_j$ denotes the failure rate of the component $j$ ($j = 1, 2$).

(9) Component 1 is better than component 2, that means that $\lambda_2$ is bigger than $\lambda_1$.

(10) Both components have different repair rates, denoted by $\mu_{x}(X)$ and $\mu_{y}(Y)$, respectively.

(11) The vacation time of the repairman follows the general distribution with a vacation rate $\eta(u)$.

(12) At any time, there is only one state changing in the system. It is impossible that there are two states changing at the same time.

(13) All above random variables are independent.

(14) Each switch-over of the states is perfect and each switch-over time is instantaneous.

2.2. Mathematical modeling

Under the above assumptions, we can divide the system into the following states:

1: Component 1 is working while component 2 is in a standby state. And the repairman is working in the system.

2: The repairman is repairing component 1 and the standby component is working.

3: The repairman is repairing component 2 while component 1 is working.

4: The repairman is repairing component 1 while component 2 is waiting for repair. In this case, component 2 has been repaired for a period of time.

5: Component 1 is working while component 2 is in a standby state. And the repairman is taking his vacation.

6: Component 1 is waiting for repair while the standby component is working, and the repairman is taking his vacation.

7: Both components are waiting for repair while the repairman is taking his vacation.
8: The repairman is repairing component 1 while component 2 is waiting for repair. In this case, component 2 has not been repaired before the repairman begins repairing component 1.

Now we introduce random variable $S(t)$ at time $t$ valued in states set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. $S(t) = 1$ means the system is in state 1 at $t$; the random variable $X(t), Y(t), U(t)$ at time $t$ valued in $[0, \infty)$, they denote the elapsed time of repairing component 1, component 2 and the vacation of the repairman respectively. Thus, the probability or probability density functions of the system in all the states are

$$
P_1(t) = P\{S(t) = 1\},$$
$$p_2(t,x)dx \text{ represents the probability that the repairman is dealing with component 1 with the elapsed time lying in } [x,x + dx) \text{ and component 2 is in working state at time } t,$$
$$p_3(t,y)dy \text{ represents the probability of the system in state 3 dealing with component 2 with the elapsed time lying in } [y, y + dy],$$
$$p_4(t,x,y)dx \text{ represents the probability of the system in state 4 and the repairman is dealing with component 1 with the elapsed time lying in } [x, x + dx) \text{ and component 2 is working at time } t,$$
$$p_5(t,u)du \text{ represents the probability of the system in state 5 and the repairman is taking a vacation at the elapsed time lying in } [u, u + du) \text{ at time } t,$$
$$p_6(t,u)du \text{ represents the probability of the system in state 6 and the repairman is taking a vacation with the elapsed time lying in } [u, u + du) \text{ at time } t,$$
$$p_7(t,u)du \text{ represents the probability of the system in state 7 and the repairman is taking a vacation with the elapsed time lying in } [u, u + du) \text{ at time } t,$$
$$p_8(t,x)dx \text{ represents the probability of the system in state 8 and the repairman is dealing with component 1 with the elapsed time lying in } [x, x + dx) \text{ at time } t.$$

With the help of these functions, the dynamic behavior of the system is governed by the following partial differential equations

$$
\begin{align*}
\frac{dP_1(t)}{dt} & = -\lambda_1 P_1(t) + \int_0^\infty \eta(u) P_3(t,u)du, \\
\left(\frac{d}{dt} + \frac{\partial}{\partial x}\right) p_2(t,x) & = -(\lambda_2 + \mu_1(x)) p_2(t,x), \\
\left(\frac{d}{dt} + \frac{\partial}{\partial y}\right) p_3(t,y) & = -(\lambda_1 + \mu_2(y)) p_3(t,y) + \int_0^\infty \mu_1(x) p_4(t,x,y)dx, \\
\left(\frac{d}{dt} + \frac{\partial}{\partial x}\right) p_4(t,x,y) & = -\mu_1(x) p_4(t,x,y), \\
\left(\frac{d}{dt} + \frac{\partial}{\partial u}\right) p_5(t,u) & = -(\lambda_1 + \eta(u)) p_5(t,u), \\
\left(\frac{d}{dt} + \frac{\partial}{\partial u}\right) p_6(t,u) & = -(\lambda_2 + \eta(u)) p_6(t,u) + \lambda_3 p_3(t,u), \\
\left(\frac{d}{dt} + \frac{\partial}{\partial x}\right) p_7(t,u) & = -\eta(u) p_7(t,u) + \lambda_2 p_6(t,u), \\
\left(\frac{d}{dt} + \frac{\partial}{\partial y}\right) p_8(t,x) & = -\mu_1(x) p_8(t,x) + \lambda_2 p_2(t,x),
\end{align*}
$$

with the boundary conditions

$$
\begin{align*}
p_2(t,0) & = \lambda_1 P_1(t) + \int_0^\infty \eta(u) p_6(t,u)du, \\
p_3(t,0) & = \int_0^\infty \mu_1(x) p_4(t,x)dx, \\
p_4(t,0,y) & = \lambda_2 p_3(t,y), \\
p_5(t,0) & = \int_0^\infty \mu_1(x) p_2(t,x)dx + \int_0^\infty \mu_2(y) p_3(t,y)dy, \\
p_6(t,0) & = p_7(t,0) = 0, \\
p_8(t,0) & = \int_0^\infty \eta(u) p_7(t,u)du
\end{align*}
$$

and initial conditions

$$
P_1(0) = 1, \quad p_2(0,x) = p_3(0,y) = p_4(0,x,y) = 0, \quad p_5(0,u) = p_6(0,u) = p_7(0,u) = p_8(0,x) = 0.$$

3. The well-posedness of the system

In this section, we will discuss the well-posedness of the system described by (2.1) and (2.2). For simplicity, in the sequel, we use notation $\mathbb{R}^+$ to denote the real number set $[0, \infty)$ and by $L^1(\mathbb{R}^+)$ the usual Lebesgue integrable function space on $\mathbb{R}^+$.

To study the solvability of (2.1) and (2.2), we firstly formulate them into a Banach space. Based on the physical meaning of the problem, we take the space state

$$
X = \mathbb{R} \times (L^1(\mathbb{R}^+))^2 \times L^1(\mathbb{R}^+ \times \mathbb{R}^+) \times (L^1(\mathbb{R}^+))^4
$$
equipped with the norm
\[ \|P\| = |P_1| + \sum_{j=2}^{4} \|p_j\|_{L^1} + \|P_4\|_{L^1} \]
for each \( P = (P_1, p_2(x), p_3(x,y), p_4(u), p_5(u), p_6(x)) \in \mathcal{X} \). Obviously, \( \mathcal{X} \) is a Banach space.

Due to the practical background of the model, we can assume that the function \( \mu_1(x), \mu_2(y) \) and \( \eta(u) \) satisfies the following conditions: there exists \( M_j > 0, j = 1, 2, 3 \), such that
\[ 0 \leq \mu_1(x) \leq M_1, \quad 0 \leq \mu_2(y) \leq M_2, \quad 0 \leq \eta(u) \leq M_3, \]
\[ \int_0^\infty \mu_1(x)dx = \infty, \quad \int_0^\infty \mu_2(y)dy = \infty, \quad \int_0^\infty \eta(u)du = \infty. \]

These conditions are reasonable (see [16]). Furthermore without loss of generality we can assume that there exists \( c_j > 0, j = 1, 2, 3 \) such that
\[ \frac{1}{c} \int_0^t \mu_1(x)dt > c_1, \quad \frac{1}{c} \int_0^t \mu_2(y)dt > c_2, \quad \frac{1}{c} \int_0^t \eta(u)dt > c_3, \quad \forall x, y, u > 0. \]

Define an operator \( A \) in \( \mathcal{X} \) by
\[
A = \begin{pmatrix}
-\lambda_1 & 0 & 0 & 0 & 0 & 0 \\
-\lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda_1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\lambda_2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\lambda_1 & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda_2 & 0 \\
\end{pmatrix} + \begin{pmatrix}
\int_0^\infty \eta(u)du \\
\int_0^\infty \mu_1(x)dx \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]
with domain
\[
D(A) = \begin{cases}
p_i \text{is absolutely continuous, and } p_i, p_i' \in L^1(\mathbb{R}^+), \quad j = 2, 3, 5, 6, 7, 8, \\
p_2(x,y), p_2(x,y) \in L^1(\mathbb{R}^+ \times \mathbb{R}^+) \\
p_2(0) = \int_0^\infty \mu_1(x)p_2(x)dx, \quad p_2(0) = \int_0^\infty \mu_1(x)p_2(x)dx
\end{cases}
\]
\[
\begin{cases}
p_3(0) = \int_0^\infty \eta(u)p_3(u)du, \\
p_4(0) = \int_0^\infty \mu_1(x)p_2(x)dx + \int_0^\infty \mu_2(y)p_3(y)dy, \\
p_5(0) = \int_0^\infty \mu_1(x)p_2(x)dx + \int_0^\infty \mu_2(y)p_3(y)dy, \\
p_6(0) = p_7(0) = 0
\end{cases}
\]

Obviously, \( A \) is a linear operator. With the help of \( A \), we can write the system (2.1) and (2.2) as an abstract Cauchy problem in Banach space \( \mathcal{X} \):
\[
\begin{cases}
\frac{d}{dt}P(t) = AP(t), \quad t \geq 0, \\
P(0) = (1,0,\ldots,0),
\end{cases}
\]
where \( P(t) = (P_1(t), p_2(t,x), p_3(t,y), p_4(t,x,y), p_5(t,u), p_6(t,u), p_7(t,u), p_8(t,x)) \).

By [17, Theorem II 6.7 and Definition II 6.8], we know that the well-posedness of the system (2.1) and (2.2) is equivalent to the operator \( A \) generating a \( C_0 \) semigroup \( T(t) \) on \( \mathcal{X} \).

For \( A \), we can prove the following results.

**Theorem 3.1.** Let \( A \) be defined before. Then \( A \) is a closed and densely defined linear operator in \( \mathcal{X} \).

**Theorem 3.2.** Let \( A \) be defined before. Then \( A \) is a dissipative operator in \( \mathcal{X} \). Moreover, if for any \( \zeta \in \mathbb{R}^+ \), it holds that
\[
\sup_{t \geq 0} \int_0^\infty e^{-\zeta t} \mu_1(x)dx < \infty, \\
\sup_{t \geq 0} \int_0^\infty e^{-\zeta t} \mu_2(y)dy < \infty, \\
\sup_{t \geq 0} \int_0^\infty e^{-\zeta t} \eta(u)du < \infty,
\]
then \( T = \{ \gamma \in \mathbb{C} | \gamma \gamma > 0 \ \text{or} \ \gamma = \text{is} \} \setminus \rho(A) \).

**Theorem 3.3.** Let \( \mathcal{X} \) and \( A \) be defined as before. Then \( A \) generates a \( C_0 \)-semigroup \( T(t) \) of contractions on \( \mathcal{X} \). Hence the abstract Cauchy problem has a unique solution.
Theorem 4.4. Let $\mathcal{K}$ and $\mathcal{A}$ be defined as before and $T(t)_{t\geq 0}$ be the $C_0$ semigroup generated by $\mathcal{A}$. Then $T(t)_{t\geq 0}$ is positive semigroup and satisfies $\|T(t)\| = \|P\|$ for all $t \geq 0$ and any positive vector $P \in D(\mathcal{A})$.

Here we only list the main results, the proof will be postponed in Appendix A.

4. Asymptotic stability of the system

In this section we shall discuss the stability of the system (2.1). We begin with a technique lemma.

Lemma 4.1. Let $\lambda_2$ and $\lambda_1$ be described as model (2.1) and define

$$
\begin{align*}
&h_1 = \int_0^\infty \eta(u)e^{-\int_0^u (\lambda_1 + \eta(t))dt} du, \\
&h_2 = \frac{\lambda_1}{s_2 - s_1} \int_0^\infty \eta(u)(e^{-\int_0^u (\lambda_1 + \eta(t))dt} - e^{-\int_0^u (\lambda_2 + \mu_1(t))dt}) du, \\
&h_3 = 1 - \int_0^\infty \mu_1(x)e^{-\int_0^x (\lambda_2 + \mu_1(t))dt} dx, \\
&h_4 = \int_0^\infty \mu_1(x)e^{-\int_0^x (\lambda_2 + \mu_1(t))dt} dx, \\
&h_5 = 1 - \frac{\lambda_4}{s_2 - s_1} \int_0^\infty \eta(u)e^{-\int_0^u (\lambda_1 + \eta(t))dt} du + \frac{\lambda_1}{s_2 - s_1} \int_0^\infty \eta(u)e^{-\int_0^u (\lambda_2 + \eta(t)) dt} du.
\end{align*}
$$

Then $h_j$, $j = 1, \ldots, 5$ are positive numbers and satisfy $h_3 + h_4 = 1$, $h_1 + h_2 + h_5 = 1$.

Proof. In model (2.1), it holds that $\lambda_2 > \lambda_1$. A direct verification shows that $h_j > 0$ and $h_3 + h_4 = 1$ and $h_1 + h_2 + h_5 = 1$.  

Theorem 4.1. 0 is a simple eigenvalue of $\mathcal{A}$.

Proof.  Let us consider the equation $\mathcal{A}P = 0$, with $P \in D(\mathcal{A})$, i.e.,

$$
\begin{align*}
-\lambda_1 P_1 + \int_0^\infty \eta(u)p_3(u)du &= 0, \\
-\lambda_2 + \lambda_1 P_1 - p_2(x) &= 0, \\
-\lambda_1 + \lambda_2 + \mu_1(y) p_3(y) - p_4(y) + \int_0^\infty \mu_1(x)p_4(x,y)dx &= 0, \\
-\mu_1(x)p_4(x,y) - \frac{1}{s_2} p_4(x,y) &= 0, \\
-\lambda_1 + \eta(u) p_5(u) - p_5(u) &= 0, \\
\lambda_1 p_5(u) - (\lambda_2 + \eta(u)) p_6(u) - p_6(u) &= 0, \\
\lambda_2 p_6(u) - \eta(u) p_2(u) - p_2(u) &= 0, \\
\lambda_2 p_2(x) - \mu_1(x)p_8(x) - p_8(x) &= 0.
\end{align*}
$$

with boundary conditions

$$
\begin{align*}
p_3(0) &= \lambda_1 P_1 + \int_0^\infty \eta(u)p_6(u)du, \\
p_3(0) &= \int_0^\infty \mu_1(x)p_6(x)dx, \\
p_4(0,y) &= \lambda_1 p_3(y), \\
p_5(0) &= \int_0^\infty \mu_1(x)p_2(x)dx + \int_0^\infty \mu_2(y)p_3(y)dy, \\
p_6(0) &= p_2(0) = 0, \\
p_8(0) &= \int_0^\infty \eta(u)p_7(u)du.
\end{align*}
$$

Solving the differential equations in (4.2) yields

$$
\begin{align*}
p_3(x) &= p_3(0)e^{-\int_0^x (\lambda_2 + \mu_1(t))dt}, \\
p_3(y) &= p_3(0)e^{-\int_0^y \mu_2(t)dt}, \\
p_4(x,y) &= \lambda_1 p_3(y)e^{-\int_0^y \mu_1(t)dt}, \\
p_5(u) &= p_5(0)e^{-\int_0^u (\lambda_1 + \eta(t))dt}, \\
p_6(u) &= \int_0^u \lambda_1 p_5(s)e^{-\int_0^s (\lambda_2 + \eta(t)) dt} ds, \\
p_7(u) &= \int_0^u \lambda_2 p_6(s)e^{-\int_0^s \eta(t)dt} ds, \\
p_8(x) &= p_8(0)e^{-\int_0^x \mu_1(t)dt} + \int_0^x \lambda_2 p_8(s)e^{-\int_0^s \mu_1(t) dt} ds.
\end{align*}
$$
As a direct result of Lemma 4.1, we get
\[
\begin{align*}
A P &= 0, \\
-\lambda_1 P_1 + p_3(0) &= 0, \\
-h_2 P_1 + p_2(0) - h_3 p_5(0) &= 0, \\
-h_2 P_2(0) + p_2(0) - p_4(0) &= 0, \\
-h_3 P_2(0) - p_3(0) + p_5(0) &= 0, \\
-h_3 P_3(0) + p_4(0) &= 0,
\end{align*}
\]
where \(h_j, j = 1, \ldots, 5\) are defined in Lemma 4.1. Set
\[
D = \begin{pmatrix}
\lambda_1 & 0 & 0 & -h_1 & 0 \\
-\lambda_3 & 1 & 0 & -h_2 & 0 \\
0 & -h_3 & 1 & 0 & -1 \\
0 & -h_4 & -1 & 1 & 0 \\
0 & 0 & 0 & -h_5 & 1
\end{pmatrix},
\]
As a direct result of Lemma 4.1, we get \(|D| = 0\). This means that the equation \(AP = 0\) has a non-zero solution. Therefore, 0 is an eigenvalue of \(A\). A direct calculation shows that
\[
\begin{align*}
P_1 &= \frac{h_2}{h_1} p_3(0), \\
p_2(0) &= (h_1 + h_2) p_3(0), \\
p_3(0) &= (h_1 h_3 + h_2 h_3 + h_5) p_3(0), \\
p_8(0) &= h_5 p_3(0)
\end{align*}
\]
is a solution to the algebraic equations. Substituting the above into (4.4) we get
\[
\begin{align*}
P_1 &= \frac{h_1}{h_3} p_3(0), \\
p_2(x) &= (h_1 + h_2) p_3(0) e^{-\int_0^{s} (2 + \mu_1(t)) dt}, \\
p_3(y) &= (h_1 h_3 + h_2 h_3 + h_5) p_3(0) e^{-\int_0^{s} \mu_2(t) dt}, \\
p_4(x, y) &= \lambda_1 p_3(y) e^{-\int_0^{s} \mu_1(s) dt}, \\
p_5(u) &= p_5(0) e^{-\int_0^{s} (l_1 + \mu_1(t)) dt}, \\
p_6(u) &= \int_0^{s} \lambda_1 p_5(s) e^{-\int_0^{s} (l_1 + \mu_1(t)) dt} ds, \\
p_7(u) &= \int_0^{s} \lambda_2 p_6(s) e^{-\int_0^{s} \mu_1(t) dt} ds, \\
p_8(x) &= p_8(0) e^{-\int_0^{s} \mu_1(t) dt} + \int_0^{s} \lambda_2 p_2(s) e^{-\int_0^{s} \mu_1(t) dt} ds.
\end{align*}
\]
So 0 is an eigenvalue of \(A\) of geometric multiplicity one.

A direct computation shows that the dual operator \(A^*\) of \(A\) is of the form
\[
A^* Q = \begin{pmatrix}
-\lambda_1 q_1 + \lambda_4 q_2(0) \\
-(\lambda_2 + \mu_1(x)) q_2(x) + q_3(x) + \lambda_2 q_4(x) + \mu_1(x) q_3(0) \\
-(\lambda_1 + \mu_2(y)) q_3(y) + q_4(y) + \lambda_1 q_4(0, y) + \mu_2(y) q_3(0) \\
\lambda_1 q_3(y) - \lambda_1 q_4(x, y) + \frac{\partial}{\partial x} q_4(x, y) \\
\eta u q_4(x) - (\lambda_1 + \eta u) q_5(u) + q_6(u) + \lambda_3 q_6(u) \\
-(\lambda_1 + \eta u) q_6(0) + q_6(0) + \lambda_1 q_6(u) + \mu_1(x) q_6(0) \\
-\eta u q_6(u) + q_6(u) + \eta u q_6(0) \\
-\mu_1(x) q_6(x) + q_6(x) + \mu_1(x) q_6(0)
\end{pmatrix},
\]
with domain
\[
D(A^*) = \left\{ \begin{array}{l}
Q = (q_1, q_2, \ldots, q_8), \text{ for } j = 2, \ldots, 8, q_j \text{ is absolutely continuous and } \\
q_4(x, y), \frac{\partial q_4(x, y)}{\partial x} \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+) \\
q_j(0) \text{ is finite, } q_j(\infty), q_4(\infty, y) \in \mathbb{R}
\end{array} \right\}.
\]
Obviously, \(Q = (1, \ldots, 1) \in D(A^*)\) and \(A^* Q = 0\), so 0 is an eigenvalue of \(A^*\) and \(Q = (1, \ldots, 1)\) is a corresponding eigenvector. Using (4.6) we get
\[ (P, Q) = P_1 + \int_0^\infty p_2(x)dx + \int_0^\infty p_3(y)dy + \int_0^\infty \int_0^\infty p_4(x,y)dxdy + \sum_{j=5}^7 \int_0^\infty p_j(u)du + \int_0^\infty p_8(x)dx \neq 0 \]

Therefore, 0 is a simple eigenvalue of \( A \). \( \square \)

Summarizing the discussions above, by the stability theorem (e.g., see, [18]) we have the following assertion.

**Theorem 4.2.** Let \( \% \) and \( A \) be defined as before. Then the following statements are true

1. The operator \( A \) generates a positive \( C_0 \)-semigroup of contractions \( T(t) \).
2. For any initial value \( P(0) \), there is a unique dynamic solution \( P(t) \). In particular, \( P(t) \) is also positive when \( P(0) \) is positive.
3. The system that has a positive steady-state solution \( \tilde{P} = (p_1, p_2, \ldots, p_k) \) whose entries are defined by (4.6) and \( p_3(0) > 0 \) is chosen such that \( ||P|| = 1 \), which is the positive eigenvalue corresponding to the eigenvalue 0 of the operator \( A \). For any \( P \in \% \), we have

\[ \lim_{t \to -\infty} P(t) = \lim_{t \to -\infty} \gamma(t)P(0) = (P(0), Q)\tilde{P} \]

where \( Q = (1, \ldots, 1) \) and \( P(0) \) is the initial value. i.e., the dynamic solution \( P(t) \) converges to the nonnegative steady-state solution \( \tilde{P} \) in the sense of norm in \( \% \).

5. **Exponential stability of the system**

In the previous section we have shown the dynamic solution \( P(t) \) of (2.1) converges to its steady state \( \tilde{P} \) in the sense of norm in \( \% \). In this section we consider the rate of converging zero for \( P(t) - (P(0), Q)\tilde{P} \). We shall prove that under certain conditions, it decays exponentially to zero.

Firstly let us consider the spectrum distribution of \( A \).

**Lemma 5.1.** Let \( c_1, c_2 \) and \( c_3 \) be defined as (3.1) and the matrix \( B(\gamma) \) be defined as (A.5) in Appendix A. Set \( c = \min\{c_1, c_2, c_3\} \) and \( f(\gamma) = \det(B(\gamma)) \). Then \( f(\gamma) \) has at most finite number of zeros in the strip region \( \{ \gamma \in \mathbb{C} | \Re(\gamma) > -c \} \).

**Proof.** Let \( g_j, j = 1, \ldots, 17 \) be defined in Lemma A.2. Then they are analytic in the half-plane \( \{ \gamma \in \mathbb{C} | \Re(\gamma) > -c \} \). Hence the function \( f(\gamma) \) is analytic in the same half-plane. In addition, we have an expression for \( f(\gamma) \)

\[ f(\gamma) = (\gamma + \lambda_1) \left( 1 - \lambda_1 \int_0^\infty \mu_1(x)g_1(x)dx \int_0^\infty \eta(u)g_{10}(u)du - \lambda_1 \lambda_2 \int_0^\infty \mu_2(y)g_3(y)dy \int_0^\infty \eta(u)g_{13}(u)du \right) \]

\[ \quad - \lambda_1 \lambda_2 \int_0^\infty \mu_2(y)g_3(y)dy \int_0^\infty \mu_1(x)g_{16}(x)dx \int_0^\infty \eta(u)g_{10}(u)du \]

\[ \quad - \lambda_1 \lambda_2 \int_0^\infty \mu_1(x)g_{16}(x)dx + \lambda_2 \int_0^\infty \mu_1(x)g_{16}(x)dx \int_0^\infty \eta(u)g_{10}(u)du \]

where \( g_j \) are defined as Lemma A.2. Since for any \( \Re(\gamma) > -c \), \( \int_0^\infty \mu_2(y)g_3(y)dy < 1 \) and

\[ \left| \int_0^\infty \eta(u)g_{13}(u)du \right| \leq \frac{2}{\lambda_1(\lambda_2 - \lambda_1)} \]

and \( \int_0^\infty \eta(u)g_{10}(u)du, \int_0^\infty \mu_1(x)g_j(x)dx, j = 1, 15, 16, \) are bounded by a finite constant. The Riemann Lemma asserts that

\[ \lim_{\gamma \to -\infty, \Re(\gamma) > -c} \int_0^\infty \eta(u)g_j(u)du = 0 \]

\[ \lim_{\gamma \to -\infty, \Re(\gamma) > -c} \int_0^\infty \mu_1(x)g_j(x)dx = 0 \]

and

\[ \lim_{\gamma \to -\infty, \Re(\gamma) > -c} \int_0^\infty \mu_2(y)g_3(y)dy = 0 \]. So we have

\[ \lim_{\gamma \to -\infty, \Re(\gamma) > -c} \left( \frac{\gamma}{\lambda_1} + \gamma \right) = 1 \]

The desired result follows from above relation. \( \square \)

**Lemma 5.2.** There exists \( \delta > 0 \) such that \( \{ \gamma \in \mathbb{C} | \Re(\gamma) > -\delta, \gamma \neq 0 \} \subset \rho(A) \).

**Proof.** According to the proof of Theorem 3.2, we know that for \( \Re(\gamma) > -c, \gamma \in \rho(A) \) if and only if \( f(\gamma) \neq 0 \). According to Lemma 5.1, we can assume that the number zero of \( f(\gamma) \) in this strip is \( n + 1 \). Let \( \gamma_j, j = 0, 1, 2, \ldots, n \) be the zero of \( f(\gamma) = 0 \) in the region \( \Re(\gamma) > -c \). Clearly, \( \gamma_0 = 0 \) is an eigenvalue. We see from Step 2 of the proof of Theorem 3.2 that there is no eigenvalue on the imaginary axis besides 0. Therefore, we can take \( \delta = -\max_{1 \leq j \leq n} \{ \Re(\gamma_j) \} \). Then the result is as follows. \( \square \)
From the Lemma 5.2 and Theorem 4.1, we can get the following result.

**Corollary 5.1.** 0 is a strictly dominant eigenvalue of the operator A.

In order to study the exponential convergence, since

$$\sup_{s \geq 0} \int_0^\infty e^{-\int_s^\infty \eta(t) dt} dx = \sup_{s \geq 0} \int_0^\infty e^{-\int_0^s \eta(t) dt} dx,$$

we can define nonnegative real number $\hat{\mu}_1, \hat{\mu}_2$ and $\hat{\eta}$ by

$$\hat{\mu}_1 = \sup \left\{ \gamma \geq 0 \mid \sup_{s \geq 0} \int_0^\infty e^{-\int_s^\infty \eta(t) dt} dx < \infty \right\},$$

$$\hat{\mu}_2 = \sup \left\{ \gamma \geq 0 \mid \sup_{s \geq 0} \int_0^\infty e^{-\int_s^\infty \eta(t) dt} dy < \infty \right\},$$

$$\hat{\eta} = \sup \left\{ \gamma \geq 0 \mid \sup_{s \geq 0} \int_0^\infty e^{-\int_s^\infty \eta(t) dt} du < \infty \right\}.$$

Obviously, when $\gamma < \hat{\mu}_1$, it holds that

$$\sup_{s \geq 0} \int_0^\infty e^{-\int_s^\infty (\mu_1(t) - \gamma) dt} dx < \infty,$$

and for $\gamma > \hat{\mu}_1$, it holds that

$$\int_0^\infty e^{-\int_s^\infty (\mu_1(t) - \gamma) dt} dx = \infty.$$

Similar results hold for $\hat{\mu}_2$ and $\hat{\eta}$. Under the assumption (3.1) we have $\hat{\mu}_1 \geq c_1, \hat{\mu}_2 \geq c_2$ and $\hat{\eta} \geq c_3$. Now we define

$$\hat{\mu} = \min \{ \hat{\mu}_1, \hat{\mu}_2, \hat{\eta} \}.$$

The following theorem gives a more exact description of the spectrum $A$ using $\hat{\mu}$.

**Theorem 5.1.** Let $\lambda, \mu$ and $\hat{\mu}$ be defined as before. Then we have

1. $\{ \gamma \in \mathbb{C} \mid \Re \gamma + \mu < 0 \} \subset \sigma(A)$.
2. $\{ \gamma \in \mathbb{C} \mid \Re \gamma + \mu > 0, \det |B(\gamma)| \neq 0 \} \subset \rho(A)$, set $\{ \gamma \in \mathbb{C} \mid \Re \gamma + \mu > 0, \det |B(\gamma)| = 0 \}$ consists of eigenvalues of $A$.
3. For any $\delta > 0$, the number of eigenvalues of $A$ in the region $\{ \gamma \in \mathbb{C} \mid \Re \gamma + \mu \geq \delta \}$ is finite.
4. There exists a constant $\hat{\omega}_1 > 0$, which satisfies that there is only one eigenvalue $\gamma_0 = 0$ in the region $\{ \gamma \in \mathbb{C} \mid \Re \gamma \geq -\hat{\omega}_1 \}$ and $\gamma_0$ is strictly dominant.

**Proof.** Let $\gamma \in \mathbb{C}$. For any $F = (f_1, \cdots, f_6) \in \mathbb{C}$, we consider the resolvent equation $(\gamma I - A)P = F$, which is the same as (A.2), the formal solutions are the same as (A.3).

1. When $\Re \gamma + \mu > 0$, we have

$$\int_0^\infty \left| \int_0^\infty f_2(s) e^{-\int_s^\infty (\gamma + \mu + 1) dt} ds \right| dx \leq \int_0^\infty |f_2(s)| ds \int_0^\infty e^{-\int_0^\infty (\Re \gamma + \mu + 1) dt} dx \leq M_2(\Re \gamma) |f_2|_{L^2},$$

where $M_2(\Re \gamma) = \sup \int_0^\infty e^{-\int_s^\infty (\Re \gamma + \mu + 1) dt} dx < \infty$. So $p_2(x) \in L^1(\mathbb{R}^+)$. Similarly, $p_j, p'_j \in L^1(\mathbb{R}^+), j = 3, 5, 6, 7, 8$ and $p_2(x, y) = \omega_2 p_2(x, y) \in L^1(\mathbb{R}^+ \times \mathbb{R}^+)$. Therefore, $P \in D(A)$.

When $\Re \gamma + \mu < 0$, for

$$p_8(x) = p_8(0) e^{-\int_0^x (\gamma + \mu + 1) dt} + \int_0^x (\gamma + \mu + 1) dt,$$

there is a $f_8 \in L^1(\mathbb{R}^+)$ such that $\int_0^\infty |f_8(x)| e^{-\int_0^x (\gamma + \mu + 1) dt} dx$ is unbounded, so $p_8(x)$ is not in $L^1(\mathbb{R}^+)$. Therefore, $\{ \gamma \in \mathbb{C} \mid \Re \gamma + \mu < 0 \} \subset \sigma(A)$.

2. Let $\Re \gamma + \mu > 0$. Substitute the formal solution into the boundary conditions and get an algebraic equation about $(P_1, p_2(0), p_3(0), p_5(0), p_9(0))$. Then the coefficient matrix is $B(\gamma)$ which is the same as (A.5).

Note that the algebraic equation is solvable if and only if $\det |B(\gamma)| \neq 0$. When $\det |B| = 0$, the Eq. (A.4) has a unique solution $(P_1, p_2(0), p_3(0), p_5(0), p_9(0))$ Then

$$P = (P_1, p_2(x), p_3(y), \cdots, p_8(x)),$$

where $p_j$ are defined as (A.3) is in $D(A)$. So $\gamma \in \rho(A)$. Therefore, $\{ \gamma \in \mathbb{C} \mid \Re \gamma + \mu > 0, \det |B| \neq 0 \} \subset \rho(A)$. 

When det |B(γ)| = 0, the eigenvalue problem has a non-zero solution (p₁, p₂(x), p₃(y), \cdots, p₈(x)). So, γ is an eigenvalue of A.

(III) For any δ > 0, let γ ∈ C satisfying −μ + δ ≤ Re{γ} < 0. Since det |B(γ)| is analytic and \( \lim_{|γ| \to \infty} \frac{\text{det } B(γ)}{|γ|^p} = 1 \). Then det |B(γ)| has at most finite number of zeros in this strip. So the result is true.

(IV) In Lemma 5.1, we have proven that \( f(γ) = \text{det } B(γ) \) has at most a finite number of zero in the region \( \{ γ ∈ C : −c < Re{γ} < 0 \} \). Note that \( μ > c \). So, −μ ≤ −c. From Lemma 5.2 we see that one can choose 0 < \( \hat{w}_1 < δ \). For such a choice, the assertion is true. □

Corollary 5.2. Let \( \hat{w}_1 \) and δ be given as in Theorem 5.1 and Lemma 5.2, respectively. Then the resolvent \( R(γ, A) \) is uniformly bounded on the region \( \{ γ ∈ C : |Re{γ}| ≥ −\hat{w}_1, |γ| ≥ δ \} \).

Proof. Let \( \hat{w}_1 \) and δ be given as in Theorem 5.1 and Lemma 5.2, respectively. For any fixed \( F ∈ X \), the solution to equation \( (γI − A)\hat{F} = F \) is given by \( R(γ, A)\hat{F} \) that is of the form

\[ R(γ, A)\hat{F} = (p₁, p₂, p₃, p₄, p₅, p₆, p₇, p₈), \]

where the functions \( p₁, p₂(x), p₃(y), p₄(x, y), p₅(υ), p₆(υ), p₇(υ), p₈(x) \) are given as (A.3), in which the coefficients are determined by the algebraic Eq. (A.4). Note that the limit

\[ \lim_{|γ| \to \infty, |Re{γ}| ≥ −\hat{w}_1} \frac{\text{det } B(γ)}{1 + |γ|} = 1 \]

converges uniformly. On the other hand, by the definition of functions \( gᵢ(u) \), it holds that \( \int_0^∞ |pᵢ(x)|dx \) is uniformly bounded with respect to \( |Re{γ}| ≥ −\hat{w}_1 \). Therefore, we have

\[ \sup_{|Re{γ}| ≥ −\hat{w}_1, |γ| ≥ δ} ||R(γ, A)\hat{F}|| < ∞. \]

The uniform bounded principle asserts that \( \sup_{|Re{γ}| ≥ −\hat{w}_1, |γ| ≥ δ} ||R(γ, A)|| < ∞. \) The desired result follows. □

Based on the result of Theorem 5.1, we have the following result about the exponential decay.

Theorem 5.2. Let \( X \) and \( A \) be defined as before and \( T(t)\) for \( t ≥ 0 \) be the \( C₀ \) semigroup generated by \( A \). Let \( \hat{w}_1 \) be given as Theorem 5.1. Then for any initial \( P(0) \) and \( t ≥ 0 \), we have \( ||P(t) − (P(0), Q)\hat{P}_0|| ≤ 2e^{-\hat{w}_1t}||P(0)|| \), where \( P(t) = T(t)P(0) \) and \( Q = \{1, \ldots, 1\} \).

Proof. Since \( γ₀ = 0 \) is a strictly dominant eigenvalue of \( A \), the Riesz spectral project corresponding to \( γ₀ \) is

\[ E(γ₀, A)\hat{F} = \frac{1}{2π𝑖} \int_{d = ϵ} R(z, A)\hat{F}dz = (F, Q)\hat{P}_0, \quad ∀F ∈ X \]

and \( \text{∥}E(γ₀, A)\text{∥} = ||Q||||\hat{P}_0|| = 1 \).

Note that \( E(γ₀, A)T(t) = T(t)E(γ₀, A) \) and \( X = \text{E} (γ₀, A)X + (I − E(γ₀, A))X \). The semigroup \( T(t) \) is dissipative on space \( X \), so it is in the subspace \( (I − E(γ₀, A))X \). Using the expression of semigroup \( T(t) \) (see [19, Corollary 7.5, pp. 29]),

\[ T(t)\hat{F} = \frac{1}{2π𝑖} \int_{δ − 𝑖∞}^{δ + 𝑖∞} e^{s}R(γ, A)\hat{F}ds \]

by shifting the integral path, we get

\[ T(t)\hat{F} = (F, Q)\hat{P}_0 + \lim_{R → ∞} \frac{1}{2π𝑖} \int_{−\hat{w}_1 + 𝑖R}^{−\hat{w}_1 + 𝑖R} e^{s}R(γ, A)\hat{F}ds = \lim_{R → ∞} \frac{1}{2π𝑖} \int_{−\hat{w}_1}^{−\hat{w}_1} e^{s}R(γ, A)\hat{F}ds \]

Therefore, for \( T(t)(I − E(γ₀, A))\hat{F} = \frac{1}{2π𝑖} \int_{−\hat{w}_1 + 𝑖∞}^{−\hat{w}_1 + 𝑖∞} e^{s}R(γ, A)\hat{F}ds \). So, for any initial \( P(0) \) we have

\[ \|P(t) − (P(0), Q)\hat{P}_0\| = \|T(t)(P(0) − E(γ₀, A)T(t)P(0))\| = \|T(t)(I − E(γ₀, A))P(0)\| ≤ e^{-\hat{w}_1t}\|I − E(γ₀, A)\|P(0)\| \]

≤ 2e^{-\hat{w}_1t}||P(0)||.

The desired result follows. □
6. Numerical simulation

In this section we discuss some indices of reliability of the system such as steady-state availability and the steady-state failure frequency of the system and analyze the profit of the system. Further we discuss some changes of the reliability indices with parameters changing by numerical simulation.

6.1. Indices of reliability of the systems and their profits

Firstly we need to get the positive eigenvector $\hat{P}$ corresponding to the eigenvalue 0 and $\|\hat{P}\| = 1$. In fact, if we take $p_3(0) = 1$ in (4.4), then the following functions

$$
\begin{align*}
\phi_1(x) &= \frac{h_1}{h_1}, \\
\phi_2(x) &= (h_1 + h_2)e^{-\int_0^x (h_1 + \mu_1(y))dy}, \\
\phi_3(y) &= (h_1 h_3 + h_2 h_3 + h_5)e^{-\int_0^y \mu_1(y)dy}, \\
\phi_4(x, y) &= \lambda_1 \phi_3(y)e^{-\int_0^y \mu_1(y)dy}, \\
\phi_5(u) &= e^{-\int_0^u (\lambda_1 + \eta(y))dy}, \\
\phi_6(u) &= \frac{\lambda_1}{\lambda_2 - \lambda_1} \left( e^{-\int_0^u (\lambda_1 + \eta(y))dy} - e^{-\int_0^u (\lambda_2 + \eta(y))dy} \right), \\
\phi_7(u) &= \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\int_0^u (\lambda_1 + \eta(y))dy} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\int_0^u (\lambda_2 + \eta(y))dy} + e^{-\int_0^u \eta(y)dy}, \\
\phi_8(x) &= e^{-\int_0^x \mu_1(y)dy} - (h_1 + h_2)e^{-\int_0^x \mu_1(y)dy}
\end{align*}
$$

are positive functions and $\Phi = (\phi_1, \phi_2, \ldots, \phi_8)$ is an eigenvector of $A$ corresponding to eigenvalue 0. Then $\hat{P} = \|\Phi\|^{-1}\Phi$ is desired.

By definition of the steady-state availability and the steady-state failure frequency (see, [21]), we have the following statements.

**Theorem 6.1.** The steady-state availability of the system is

$$
Av = \frac{1}{\|\Phi\|} \left( \int_0^\infty \phi_1(x)dx + \int_0^\infty \phi_3(y)dy + \int_0^\infty \phi_5(u)du + \int_0^\infty \phi_6(u)du \right)
$$

and the frequency of the steady-state failure is

$$
Mv = \frac{1}{\|\Phi\|} \left( \lambda_1 \int_0^\infty \phi_1(y)dy + \int_0^\infty \phi_5(u)du \right),
$$

where $\|\Phi\| = |\phi_1| + \sum_{j=2}^7 \int_0^\infty \phi_j(x)dx + \int_0^\infty \int_0^\infty \phi_4(x, y)dxdy$. Then the steady-state probability of the repairman’s vacation, i.e.

$$
Pv = \frac{1}{\|\Phi\|} \sum_{j=2}^7 \int_0^\infty \phi_j(u)du.
$$

Next, we discuss the revenue of the system. Assume that the profit of the system per unit time is $f_1$ if the system works, the average loss for each failure is $f_2$, while the repairman can save $f_3$ in unit time for every vacation. Then the total profit of the system per unit time is $Bv = Av \cdot f_1 + Pv \cdot f_3 - Mv \cdot f_2$.

If the repairman gives up his vacation, the model of the system becomes

$$
\begin{align*}
\frac{dP_1(t)}{dt} &= -\lambda_1 P_1(t) + \int_0^\infty \mu_1(x)p_2(t, x)dx + \int_0^\infty \mu_2(y)p_3(t, y)dy, \\
\frac{dP_2(t, x)}{dt} &= -\lambda_2 + \mu_1(x)p_2(t, x), \\
\frac{dP_3(t, y)}{dt} &= -(\lambda_1 + \mu_2(y))p_3(t, y) + \int_0^\infty \mu_1(x)p_4(t, x, y)dx, \\
\frac{dP_4(t, x, y)}{dt} &= -\mu_1(x)p_4(t, x, y), \\
\frac{dP_5(t, x)}{dt} &= -\mu_1(x)p_5(t, x) + \lambda_2 p_2(t, x), \\
\frac{dP_6(t, x)}{dt} &= -\mu_1(x)p_6(t, x) + \lambda_2 p_2(t, x)
\end{align*}
$$

with the boundary conditions

$$
\begin{align*}
p_2(t, 0) &= \lambda_1 P_1(t), \\
p_3(t, 0) &= \int_0^\infty \mu_1(x)p_8(t, x)dx, \\
p_4(t, 0, y) &= \lambda_1 p_3(t, y), \\
p_5(t, 0) &= 0, \\
p_6(t, 0) &= 0, \\
p_7(t, 0) &= p_8(t, 0) = p_9(0, x) = p_8(0, x) = 0.
\end{align*}
$$
By solving the steady state equation, we can get

\[
\begin{aligned}
P_1 &= \frac{1}{k_1} p_2(0), \\
p_2(x) &= p_2(0) e^{-\int_0^x (\lambda_2 + \mu_1(t)) dt}, \\
p_3(y) &= \lambda_2 p_2(0) e^{-\int_0^y \mu_2(t) dt} e^{-\int_0^x (\lambda_2 + \mu_1(t)) dt} dx, \\
p_4(x, y) &= \lambda_1 p_3(y) e^{-\int_0^x \mu_1(t) dt}, \\
p_5(x) &= p_2(0) \left[ e^{-\int_0^x \mu_1(t) dt} - e^{-\int_0^y (\lambda_2 + \mu_1(t)) dt} \right].
\end{aligned}
\]

Let \( p_2(0) = 1 \) and \( \bar{P} = (P_1, p_2(x), p_3(y), p_4(x, y), p_5(x)) \). Then \( \bar{P} \) is a positive steady state of the system \((6.2)\). Therefore, we have the following results.

**Theorem 6.2.** The steady-state availability of the new system \((6.2)\) is

\[
Aw = \frac{P_1 + \int_0^\infty p_2(x) dx + \int_0^\infty p_3(y) dy}{\|P\|}
\]

and the steady-state failure frequency is

\[
Mw = \frac{\lambda_1 P_1 + \lambda_1 \int_0^\infty p_3(y) dy + \lambda_2 \int_0^\infty p_2(x) dx}{\|P\|}.
\]

Then, the total profit of the new system is

\[
Bw = Aw \cdot f_1 - Mw \cdot f_2.
\]

6.2. Analysis

As we know, if we want to maximize greater profits, it must be \( Bv > Bw \). Obviously, when \( Bv > Bw \), the repairman should take his vacation, but when \( Bv < Bw \), the repairman should give up his vacation. In what follows, we shall show that if the system has a strict requirement on the availability, the repairman should not take a vacation anyway.

Firstly, we discuss the impacts of \( \lambda_1 \) on the availability and the profits of the two systems. Suppose that \( \lambda_2 = 0.3, \mu_1(x) = 0.7, \mu_2(y) = 0.55, \eta(u) = 0.3 \). Let \( \lambda_1 \) change from 0.01 to 0.29.

**Figs. 1–4** describe the variety of these indices with \( \lambda_1 \) change. The notation dashed line – represents the change of indices of the system \((2.1)\) while * line represents the system \((6.2)\).

From **Figs. 1–4**, we can see:
From Fig. 1, we can understand that it always has $A_v < A_w$, the steady-state availability of the system decreases as $\lambda_1$ increases.

From Fig. 2 says that the failure frequency also has $M_w > M_v$, it increases when $\lambda_1$ increases.

The steady-state probability of the repairman’s vacation shows an increasing trend as $\lambda_1$ increases. This is not the index when the repairman does not take his vacation.

Suppose that $f_1 = 40$, $f_2 = 50$, $f_3 = 9$. From Fig. 4, we have the following conclusions: when $0 < \lambda_1 < 0.1414$, $B_v > B_w$, in this case, the repairman can take his vacation; when $\lambda_1 = 0.1414$, $B_v = B_w$; when $0.1414 < \lambda_1 \leq 0.29$, $B_v < B_w$, in this case, the repairman should not take his vacation.
Next, we discuss the impact of the probability of the repairman's vacation on these indices. Suppose that 
\( \lambda_1 = 0.15, \lambda_2 = 0.25, \mu_1(x) = 0.8, \mu_2(y) = 0.6, \) and let \( \eta(u) \) change from 0 to 0.5.

Figs. 5–8 show us the impact of \( \eta(u) \) on these indices. The dashed line – represents the change of indices of the system (2.1) while * line represents the system (6.2).

Figs. 5–8 show that the following facts are true:
Aw is constant while Av increases as $\eta(u)$ increases. Aw is far larger than Av, see Fig. 1.

Mv has a trend similar to Av while Mw is constant. When $0 < \eta(u) < 0.15$, $Mv$ the growth is fast; The change becomes small when $0.15 < \eta(u) < 0.5$, see Fig. 2.

Pv has a decreasing trend as $\eta(u)$ increases if the repairman takes his vacation. When $\eta(u)$ is closer to zero, $Pv$ is larger than 0.9; while $\eta(u)$ is closer to 0.5, $0.2 < Pv < 0.3$. 

(1) $Aw$ is constant while $Av$ increases as $\eta(u)$ increases. $Aw$ is far larger than $Av$, see Fig. 1.
(2) $Mv$ has a trend similar to $Av$ while $Mw$ is constant. When $0 < \eta(u) < 0.15$, $Mv$ the growth is fast; The change becomes small when $0.15 < \eta(u) < 0.5$, see Fig. 2.
(3) $Pv$ has a decreasing trend as $\eta(u)$ increases if the repairman takes his vacation. When $\eta(u)$ is closer to zero, $Pv$ is larger than 0.9; while $\eta(u)$ is closer to 0.5, $0.2 < Pv < 0.3$. 

Fig. 6. $Mv$ and $Mw$ change when $\eta(u)$ changes.

Fig. 7. $Pv$ change when $\eta(u)$ changes.
(4) $B_v$ increases as $\eta(u)$ increases. When $0 < \eta(u) < 0.2609$, we have $B_v < B_w$. This means that the smaller rate of vacation does not enhance the profit of the system; when $\eta(u) = 0.2609$, we have $B_v = B_w$; when $0.2609 < \eta(u) < 0.5$, we have $B_v > B_w$. This means that the larger rate of vacation can enhance the profit of the system.

Here we cannot list all circumstances. In our daily work, we have to make a decision in accordance with the actual situation. Numerical Simulation provides us a method to judge whether the repairman should take a vacation or not.

7. Conclusion

In the present paper we studied the reliability of a cold standby system with a repairman. Herein the main component priority and repairman vacation are considered. Under certain conditions, we proved that the dynamic solution of the system converges exponentially in its steady state. Further we studied the reliability indices of the system and analyzed the profit and reliability indices of the system when the parameter changes, and compared these indices with those of a system without vacation. The result shows that the steady-state availability of the system with a single vacation of the repairman is less than that of the system without a vacation of the repairman. But, the change of $\lambda_1$ and $\eta(u)$ can impact the profit of the system. The numerical simulations show that if the repairman takes his vacation, he should have a larger vacation rate for example $g(u) > 0.3$ when $\lambda_1$ is smaller. When $\lambda_1 > 0.15$ or $\eta(u) < 0.3$, the repairman’s vacation does not increase the profit of the system, so it should cancel or reduce the repairman’s vacation.

Appendix A. The proof of results in Section 3

The Proof of Theorem 3.1. We divide the proof into the following several steps.

Step 1. $D(A)$ is dense in $\mathbb{X}$. We should prove that $\forall \varepsilon > 0, \forall F \in \mathbb{X}$, we can find a $P \in D(A)$ such that $\|F - P\| < \varepsilon$. For any $F = (f_1, \ldots, f_8) \in \mathbb{X}$ fixed, let us condition a $P \in \mathbb{X}$,

$$
\|F - P\|_x = |f_1 - p_1| + \sum_{j=2}^{8} \int_0^\infty |f_j(x) - p_j(x)|\,dx + \int_0^\infty |f_3(y) - p_3(y)|\,dy + \int_0^\infty \int_0^\infty |f_4(x, y) - p_4(x, y)|\,dxdy
$$

$$
+ \sum_{j=5}^{8} \int_0^\infty |f_j(u) - p_j(u)|\,du
$$

We can take $p_1 = f_1$. Since $F \in \mathbb{X}$, then for any $\forall \varepsilon > 0$, there exist constants $G_j > 0$ ($j = 2, 3, \ldots, 8$) such that $\int_0^\infty |f_j(t)|\,dt < \frac{\varepsilon}{G_j}$, ($j = 2, 3, 5, 6, 7, 8$) and $\int_0^\infty \int_0^\infty |f_4(x, y)|\,dxdy < \frac{\varepsilon}{G_4}$.
By the absolute continuity of the integral, we can choose a positive constant \( \delta_6 < \frac{\varepsilon}{21} \) such that \( \int_0^{\delta_6} |f_6(u)|\,du < \frac{\varepsilon}{21} \). We define a function by

\[
p_6(u) = \begin{cases} 
0, & 0 \leq u \leq \delta_6, \\
g_6(u), & \delta_6 \leq u \leq G_6, \\
0, & u \geq G_6,
\end{cases}
\]

where \( g_6(G_6) = 0, \, g_6(\delta_6) = 0, g_6(\delta_6) \) is continuously differentiable and \( g_6 \) satisfies \( \int_0^{\delta_6} |f_6(u) - g_6(u)|\,du < \frac{\varepsilon}{21} \). Obviously, \( p_6(u), \, p_6' \in L^1(\mathbb{R}^+) \) and

\[
\int_0^\infty |p_6(u) - f_6(u)|\,du = \int_0^\infty |f_6(u) - g_6(u)|\,du + \int_0^{\delta_6} |f_6(u) - g_6(u)|\,du + \int_{\delta_6}^{G_6} |f_6(u) - g_6(u)|\,du \leq \frac{2\varepsilon}{21}.
\]

Similarly, we can find a \( \delta_7 > 0 \) and define a function \( p_7(u) \) such that \( p_7(u), \, p_7' \in L^1(\mathbb{R}^+) \) and

\[
\int_0^\infty |p_7(u) - f_6(u)|\,du < \frac{2\varepsilon}{21}.
\]

Now we take a positive constant \( \delta_2 < \frac{\varepsilon}{42^3 + |\lambda_1 P_1| + \int \eta(u)p_6(u)\,du} \) such that \( \int_0^{\delta_2} |f_2(x)|\,dx < \frac{\varepsilon}{21} \). We define a function by

\[
p_2(x) = \begin{cases} 
\lambda_1 P_1 + \int_0^{\delta_2} \eta(u)p_6(u)\,du, & 0 \leq x \leq \delta_2, \\
g_2(x), & \delta_2 \leq x \leq G_2, \\
0, & x \geq G_2,
\end{cases}
\]

where \( g_2(G_2) = 0, \, g_2(\delta_2) \) is continuously differentiable and satisfies \( \int_0^{\delta_2} |f_2(x) - g_2(x)|\,dx < \frac{\varepsilon}{21} \). To ensure the condition \( g_2(\delta_2) = \lambda_1 P_1 + \int_0^{\delta_2} \eta(u)p_6(u)\,du \), we calculate

\[
\int_0^\infty \eta(u)p_6(u)\,du = \int_0^{\delta_2} \eta(u)\,du \int_0^{\delta_2} \eta(u)p_6(u)\,du + \int_{\delta_2}^{G_2} \eta(u)g_6(u)\,du = \lambda_1 P_1 \int_0^{\delta_2} \eta(u)\,du + \int_{\delta_2}^{G_2} \eta(u)g_6(u)\,du.
\]

We can take \( \delta_2 \) such that \( 1 - \int_0^{\delta_2} \eta(u)\,du \neq 0 \) and hence

\[
\int_0^\infty \eta(u)p_6(u)\,du = \frac{1}{1 - \int_0^{\delta_2} \eta(u)\,du} \left[ \lambda_1 P_1 \int_0^{\delta_2} \eta(u)\,du + \int_{\delta_2}^{G_2} \eta(u)g_6(u)\,du \right].
\]

With such a choice, we have \( p_2(x), \, p_2' \in L^1(\mathbb{R}^+) \) and

\[
\int_0^\infty |f_2(x) - p_2(x)|\,dx = \int_0^{\delta_2} |f_2(x) - \lambda_1 P_1| + \int_{\delta_2}^{G_2} |f_2(x) - g_2(x)|\,dx + \int_0^\infty |f_2(x)|\,dx < \frac{2\varepsilon}{21}.
\]

Similar to the construction of \( p_2 \), we can construct functions \( p_3(u), \, p_3' \) and \( p_4 \) and take \( \delta_3 = \frac{\varepsilon}{42^4 + |\lambda_1 P_2| + \int \eta(u)p_2(\delta_2)\,du} \) and \( \delta_3 = \frac{\varepsilon}{42^2 + |\lambda_1 P_3| + \int \eta(u)p_3(\delta_2)\,du} \) such that \( \int_0^{\delta_3} |f_3(x)|\,dx < \frac{\varepsilon}{21} \), the functions \( g_3, \, g_3, \, g_5 \) satisfy \( \int_0^{\delta_3} |f_3(x) - p_3(x)|\,dx < \frac{\varepsilon}{21} \) and \( \int_0^{1} |p_3(x)|\,dx < \frac{\varepsilon}{21} \), where \( j = 3, \, 5, \, 8 \). Thus we have \( p_j, \, p_j' \in L^1(\mathbb{R}^+) \), \( j = 3, \, 5, \, 8 \) and

\[
\int_0^\infty |f_j(t) - p_j(t)|\,dt < \frac{2\varepsilon}{21}, \quad j = 3, \, 5, \, 8.
\]

Set \( 0 < \delta_4 < \frac{\varepsilon}{42^4 + |\lambda_1 P_4|} \) satisfy \( \int_0^{\delta_4} \int_0^{\delta_4} |f_4(x, y)|\,dxdy < \frac{\varepsilon}{21} \). We define a function \( p_4(x, y) \) by

\[
p_4(x, y) = \begin{cases} 
\lambda_1 P_3(y), & 0 \leq y \leq \delta_4, \\
g_4(x, y), & \delta_4 \leq y \leq G_4, \\
0, & x \geq G_4,
\end{cases}
\]

where \( g_4(G_4, y) = 0, \, g_4(\delta_4, y) = \lambda_1 P_3(y), \, g_4(x, y) \) is continuously differentiable and satisfies \( \int_0^{\delta_4} \int_0^{\delta_4} |f_4(x, y) - g_4(x, y)|\,dxdy < \frac{\varepsilon}{21} \). Obviously, \( p_4(x, y), \, p_4' \in L^1(\mathbb{R}^+ \times \mathbb{R}^+) \) and

\[
\int_0^\infty \int_0^{\delta_4} |f_4(x, y) - p_4(x, y)|\,dxdy = \int_0^{\delta_4} \int_0^{\delta_4} |f_4(x, y) - \lambda_1 P_3(y)|\,dxdy + \int_0^{\delta_4} \int_{\delta_4}^{G_4} |f_4(x, y) - g_4(x, y)|\,dxdy
\]

\[
+ \int_0^{\delta_4} \int_{\delta_4}^{G_4} |g_4(x, y) - f_4(x, y)|\,dxdy + \int_0^{\delta_4} \int_{\delta_4}^{G_4} |f_4(x, y)|\,dxdy
\]

\[
< \frac{\varepsilon}{42^2} + \frac{\varepsilon}{42^2} + \frac{\varepsilon}{42^2} + \frac{\varepsilon}{21}.
\]
Let \( p_j \) be defined as above, and let
\[
P = (P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8).
\]
By the construction of functions, we have \( P \in D(A) \) and
\[
\|F - P\| = \int_0^\infty |f_2(x) - p_2(x)|dx + \int_0^\infty |f_3(y) - p_3(y)|dy + \int_0^\infty \int_0^\infty |f_4(x, y) - p_4(x, y)|dxdy
\]
\[
+ \sum_{j=0}^7 \int_0^\infty |f_j(u) - p_j(u)|du + \int_0^\infty |f_8(x) - p_8(x)|dx \leq 7 \frac{2\varepsilon}{21} < \varepsilon.
\]
Therefore \( \overline{D(A)} = \mathbb{K} \).

Step 2. \( A \) is a closed linear operator.

Obviously, \( A \) is a linear operator in \( \mathbb{K} \). We only need to prove \( A \) is a closed operator. By the definition of closed operator, \( A \) is closed if and only if when
\[
P^{(m)} \in D(A), \ P^{(m)} \to P^{(0)}, \ AP^{(m)} \to F,
\]
then \( P^{(0)} \in D(A) \) and \( AP^{(0)} = F \).

Now we suppose that there is a sequence \( P^{(m)} \in D(A) \) satisfying \( P^{(m)} \to P^0 \) and \( AP^{(m)} \to F \). Let
\[
P^{(m)} = (P_1^{(m)}, P_2^{(m)}(x), \ldots, P_8^{(m)}(x)), \quad P^{(0)} = (P_1^{(0)}, P_2^{(0)}(x), \ldots, P_8^{(0)}(x))
\]
and \( F = (f_1, f_2(x), \ldots, f_8(x)) \). Thus we have when \( m \to \infty \)
\[
|P_1^{(m)} - P_1^{(0)}| \to 0,
\]
\[
\int_0^\infty |p_j^{(m)}(t) - p_j^{(0)}(t)|dt \to 0, \quad j = 2, 3, 5, 6, 7, 8,
\]
\[
\int_0^\infty \int_0^\infty |p_4^{(m)}(x, y) - p_4^{(0)}(x, y)|dxdy \to 0,
\]
and
\[
|\lambda_1 P_1^{(m)} + \int_0^\infty \eta(u)p_2^{(m)}(u)du - f_1| \to 0,
\]
\[
\int_0^\infty \left| \left( \lambda_2 + \mu_1(x) + \frac{d}{dx} p_2^{(m)}(x) - f_2(x) \right) dx \right| \to 0,
\]
\[
\int_0^\infty \left| \left( \lambda_1 + \mu_2(x) + \frac{d}{dy} p_3^{(m)}(y) + \int_0^\infty \mu_1(x)p_4^{(m)}(x, y)dx - f_3(y) \right) dy \right| \to 0,
\]
\[
\int_0^\infty \int_0^\infty \left| \left( \mu_1(x) + \frac{d}{dx} p_4^{(m)}(x, y) - f_4(x, y) \right) dxdy \right| \to 0,
\]
\[
\int_0^\infty \left| \left( \lambda_1 + \eta(u) + \frac{d}{du} p_5^{(m)}(u) - f_5(u) \right) du \right| \to 0,
\]
\[
\int_0^\infty \left| \left( \lambda_2 + \eta(x) + \frac{d}{du} p_6^{(m)}(u) + \lambda_1 p_5^{(m)}(u) - f_6(u) \right) du \right| \to 0,
\]
\[
\int_0^\infty \left| \left( B(\eta(u) + \frac{d}{du} p_7^{(m)}(u) + \lambda_1 p_5^{(m)}(u)) - f_7(u) \right) du \right| \to 0,
\]
\[
\int_0^\infty \left| \left( \mu_1(x) + \frac{d}{dx} p_8^{(m)}(x) + \lambda_2 p_2^{(m)}(x) - f_8(x) \right) dx \right| \to 0.
\]

Note that \( \eta(x) \in L^\infty(\mathbb{R}^+) \) and \( p_5^{(m)} \to p_5^{(0)} \) in \( L^1(\mathbb{R}^+) \). From the first formula, we can get
\[
f_1 = -\lambda_1 P_1 + \int_0^\infty \eta p_5^{(0)}(u)du,
\]
where we have used \( \eta \) is a bounded linear functional on \( L^1(\mathbb{R}^+) \).

For any \( t \in \mathbb{R}^+ \), the function \( \chi_{[0, t]} \in L^\infty(\mathbb{R}^+) \), and is also a bounded and linear functional of \( L^1(\mathbb{R}^+) \). Then for each \( j = 2, 3, 4, 5, 6, 7, 8 \), \( \chi_{[0, t]}(AP^{(m)}) \) has meaning and \( \lim_{m \to \infty} \chi_{[0, t]}(AP^{(m)}) = \chi_{[0, t]}(f_j) \) due to the continuity. Thus we get
\[
\lim_{m \to \infty} \int_0^t \int_0^{f_2(x)} \left( \lambda_2 + \mu_1(x) + \frac{d}{dx} p_2^{(m)}(x) \right) dx = \int_0^t f_2(x)dx,
\]
\[
\lim_{m \to \infty} \int_0^t \int_0^{f_3(y)} \left( \lambda_1 + \mu_2(x) + \frac{d}{dy} p_3^{(m)}(y) + \int_0^\infty \mu_1(x)p_4^{(m)}(x, y)dx \right) dy = \int_0^t f_3(y)dy.
\]
\[
\begin{align*}
\lim_{m \to \infty} \int_0^t \int_0^t & \left[ -\left( \mu_1(x) + \frac{d}{dx} p_4^{(m)}(x,y) \right) \right] dx \, dy = \int_0^t \int_0^t f_4(x,y) \, dx \, dy, \\
\lim_{m \to \infty} \int_0^t & \left[ -\left( \lambda_1 + \eta(u) + \frac{d}{du} p_5^{(m)}(u) \right) \right] du = \int_0^t f_5(u) \, du, \\
\lim_{m \to \infty} \int_0^t & \left[ -\left( \lambda_2 + \eta(x) + \frac{d}{du} p_6^{(m)}(u) + \lambda_1 p_5^{(m)}(u) \right) \right] du = \int_0^t f_6(u) \, du, \\
\lim_{m \to \infty} \int_0^t & \left[ -\left( \eta(u) + \frac{d}{du} p_7^{(m)}(u) + \lambda_1 p_6^{(m)}(u) \right) \right] du = \int_0^t f_7(u) \, du, \\
\lim_{m \to \infty} \int_0^t & \left[ -B \left( \mu_1(x) + \frac{d}{dx} p_8^{(m)}(x) + \lambda_2 p_2^{(m)}(x) \right) \right] dx = \int_0^t f_8(x) \, dx.
\end{align*}
\]

From the above we can get

\[
\begin{align*}
- \int_0^t \left[ \lambda_2 + \mu_1(x) \right] p_2^{(0)}(x) \, dx - p_2^{(0)}(t) + p_2^{(0)}(0) &= \int_0^t f_2(x) \, dx, \\
- \int_0^t \left[ \lambda_1 + \mu_2(x) \right] p_3^{(0)}(x) - p_3^{(0)}(t) + p_3^{(0)}(0) + \int_0^\infty \mu_4(x) \, dx \int_0^t p_4^{(0)}(x,y) \, dy &= \int_0^t f_3(y) \, dy, \\
- \int_0^t \int_0^t \mu_1(x)p_4^{(0)}(x,y) \, dx \, dy - \int_0^t p_4^{(0)}(t,y) \, dy + \int_0^t p_4^{(0)}(0,y) \, dy &= \int_0^t \int_0^t f_4(x,y) \, dx \, dy, \\
- \int_0^t \left[ \lambda_1 + \eta(u) \right] p_5^{(0)}(u) \, du - p_5^{(0)}(t) + p_5^{(0)}(0) &= \int_0^t f_5(u) \, du, \\
- \int_0^t \left[ \lambda_2 + \eta(u) \right] p_6^{(0)}(u) \, du - p_6^{(0)}(t) + p_6^{(0)}(0) + \lambda_1 \int_0^t p_5^{(0)}(u) \, du &= \int_0^t f_6(u) \, du, \\
- \int_0^t \eta(u)p_7^{(0)}(u) - p_7^{(0)}(t) + p_7^{(0)}(0) + \lambda_1 \int_0^t p_6^{(0)}(u) \, du &= \int_0^t f_7(u) \, du, \\
- \int_0^t \mu_1(x)p_8^{(0)}(x) - p_8^{(0)}(t) - p_8^{(0)}(0) + \lambda_2 \int_0^t p_2^{(0)}(x) \, dx &= \int_0^t f_8(x) \, dx.
\end{align*}
\]

The above formulas show that \( p_j^{(0)} , j = 2, 3, 4, 5, 6, 7, 8 \) are absolute continuous and

\[
\begin{align*}
- \lambda_2 + \mu_1(x) p_2^{(0)}(x) - p_2^{(0)}(t) + p_2^{(0)}(0) &= f_2(x), \\
- \lambda_1 + \mu_2(x) p_3^{(0)}(x) - p_3^{(0)}(t) + p_3^{(0)}(0) &= f_3(x), \\
- \mu_1(x)p_4^{(0)}(x,y) - \frac{\partial p_4^{(0)}(x,y)}{\partial x} &= f_4(x,y), \\
- \lambda_1 + \eta(u) p_5^{(0)}(u) - p_5^{(0)}(t) + p_5^{(0)}(0) &= f_5(u), \\
- \lambda_2 + \eta(u) p_6^{(0)}(u) + p_6^{(0)}(t) + \lambda_1 p_5^{(0)}(u) &= f_6(u), \\
- \eta(u)p_7^{(0)}(u) - (p_7^{(0)})'(t) + \lambda_1 p_6^{(0)}(u) &= f_7(u), \\
- \mu_1(x)p_8^{(0)}(x) - (p_8^{(0)})'(t) + \lambda_2 p_2^{(0)}(x) &= f_8(x),
\end{align*}
\]

which imply that \( p_j^{(0)} \in L^1(\mathbb{R}^+) \), \( j = 2, 3, 4, 5, 6, 7, 8 \) and \( \frac{\partial p_j^{(0)}(x,y)}{\partial x} \in L^1(\mathbb{R}^+) \).

Since \( P^{(m)} \in D(A) \), we have

\[
\begin{align*}
p_2^{(m)}(0) &= \lambda_1 p_1^{(m)} + \int_0^\infty \eta(u)p_6^{(m)}(u) \, du, \\
p_3^{(m)}(0) &= \int_0^\infty \mu_1(x)p_6^{(m)}(x) \, dx, \\
p_4^{(m)}(0,y) &= \lambda_1 p_4^{(m)}(y), \\
p_5^{(m)}(0) &= \int_0^\infty \mu_1(x)p_5^{(m)}(x) \, dx + \int_0^\infty \mu_2(y)p_4^{(m)}(y) \, dy, \\
p_6^{(m)}(0) &= p_7^{(m)}(0) = 0, \\
p_8^{(m)}(0) &= \int_0^\infty \eta(u)p_8^{(m)}(u) \, du.
\end{align*}
\]

Using the fact that \( \mu_1, \mu_2, \eta \in L^\infty(\mathbb{R}^+) \), we get
Lemma A.1. Let $\mu_1(x), \mu_2(y)$ and $\eta(u)$ be defined as before. Then if the following inequalities hold, they satisfy
\[
\begin{align*}
|\int_0^\infty \mu_1(x) e^{-\int_0^y \gamma dt} dx| &< 1, \\
|\int_0^\infty \mu_2(y) e^{-\int_0^x \gamma dt} dy| &< 1, \\
|\int_0^\infty \eta(u) e^{-\int_0^u \gamma dt} du| &< 1.
\end{align*}
\]  
(A.1)

where $\gamma \in \mathbb{C}, \Re \gamma > 0$ or $\gamma = is, s \neq 0$.

The proof is a direct verification, we omit the detail.

Based on the above lemma, we obtain the following corollary.

Corollary A.1. Set $d_1 = \int_0^\infty \mu_1(x) e^{-\int_0^x \gamma dt} dx$ and $d_2 = \int_0^\infty \eta(u) e^{-\int_0^u \gamma dt} du$, where $\gamma \in \mathbb{C}$ with $\Re \gamma > 0$ or $\gamma = is, s \neq 0$. Then, $|d_1| < 1$ and $|d_2| < 1$.

Lemma A.2. Let $F = (f_1, \ldots, f_8) \in \mathbb{X}$ and $\Re \gamma > 0$ or $\Re \gamma = 0, \Im \gamma \neq 0$, and let $d_1, d_2$ be defined by Corollary A.1. Define functions by
\[
\begin{align*}
g_1(x) &= e^{-\int_0^x \gamma dt}, \\
g_2(x) &= \int_0^x f_2(s) e^{-\int_0^s \gamma dt} ds, \\
g_3(y) &= e^{-\int_0^y \gamma dt}, \\
g_4(y) &= \int_0^y f_3(s) e^{-\int_0^s \gamma dt} ds, \\
g_5(y) &= \int_0^y e^{-\int_0^s \gamma dt} ds \int_0^\infty \mu_1(x) e^{-\int_0^x \gamma dt} dx ds dy, \\
g_6(x) &= e^{-\int_0^x \gamma dt}, \\
g_7(x,y) &= \int_0^x e^{-\int_0^s \gamma dt} ds f_4(s,y) dy, \\
g_8(u) &= e^{-\int_0^u \gamma dt}, \\
g_9(u) &= \int_0^u f_5(s) e^{-\int_0^s \gamma dt} ds, \\
g_{10}(u) &= \int_0^u e^{-\int_0^s \gamma dt} ds f_5(s) e^{-\int_0^s \gamma dt} ds dy, \\
g_{11}(u) &= \int_0^u \int_0^3 f_6(\tau) e^{-\int_0^\tau \gamma dt} d\tau ds, \\
g_{12}(u) &= \int_0^u f_6(s) e^{-\int_0^s \gamma dt} ds, \\
g_{13}(u) &= \int_0^u \int_0^x e^{-\int_0^s \gamma dt} ds f_5(s) e^{-\int_0^s \gamma dt} ds dy, \\
g_{14}(u) &= \int_0^u \int_0^x e^{-\int_0^s \gamma dt} ds f_5(s) e^{-\int_0^s \gamma dt} ds dy.
\end{align*}
\] Therefore, we have $P^{(0)} \in D(A)$ and $AP^{(0)} = F$. So, $A$ is a closed and dense defined linear operator in $\mathbb{X}$. \qed
\[ g_{15}(x) = e^{\int_0^x (\gamma + \mu_1(t)) \, dt}, \]
\[ g_{16}(x) = \int_0^x g_1(s) e^{\int_s^x (\gamma + \mu_1(t)) \, dt} \, ds, \]
\[ g_{17}(x) = \int_0^x (\dot{z}g_2(s) + f_8(s)) e^{\int_s^x (\gamma + \mu_1(t)) \, dt} \, ds. \]

Then these functions are absolutely continuous and absolutely integrable.

**Proof.** By definition these functions are absolutely continuous. Obviously, \( g_j \in L^1(\mathbb{R}^+) \), \( j = 1, 3, 6, 8, 15 \). Let us consider \( g_2 \).

\[
\int_0^\infty |g_2(x)| \, dx = \int_0^\infty \left| \int_0^x f_2(s) e^{-\int_s^x (\gamma + \mu_1(t)) \, dt} \, ds \right| \, dx \leq \int_0^\infty \int_0^\infty e^{-(\gamma + \mu_1(t)) (x-s)} |f_2(s)| \, ds \, dx = \int_0^\infty ds \int_0^\infty e^{-(\gamma + \mu_1(t)) (x-s)} |f_2(s)| \, dx
\]
\[
= \frac{1}{\gamma \gamma + \lambda_2} \int_0^\infty |f_2(s)| \, ds < \infty.
\]
So \( g_2 \in L^1(\mathbb{R}^+) \). Similarly we can prove that \( g_j \in L^1(\mathbb{R}^+) \), \( j = 4, 7, 9, 12 \).

Since
\[
g_{10}(u) = \int_0^u e^{\int_0^s (\gamma + \lambda_1 + \eta(t)) \, dt} e^{-\int_s^u (\gamma + \lambda_1 + \eta(t)) \, dt} \, ds = e^{-\int_0^u (\gamma + \lambda_1 + \eta(t)) \, dt} \int_0^u e^{\int_s^u (\gamma + \lambda_1 + \eta(t)) \, dt} \, ds
\]
\[
= \frac{1}{\lambda_2 - \lambda_1} \left( e^{-\int_0^u (\gamma + \lambda_1 + \eta(t)) \, dt} - e^{-\int_0^u (\gamma + \lambda_1 + \eta(t)) \, dt} \right),
\]
we have \( g_{10}(u) \in L^1(\mathbb{R}^+) \).

Now we consider \( g_5 \). Since \( f_4 \in L^1(\mathbb{R}^+ \times \mathbb{R}^+) \) and
\[
\int_0^\infty |g_5(y)| \, dy = \int_0^\infty \int_0^\infty e^{-(\gamma + \lambda_1 + \eta(t)) (s-t)} \, ds \, dy \leq \frac{1}{\gamma \gamma + \lambda_2 \lambda_4} \int_0^\infty \int_0^\infty e^{-\int_0^x (\gamma + \lambda_1 + \eta(t)) \, dt} \, dx \, dy \int_0^\infty \int_0^\infty e^{-\int_0^x (\gamma + \lambda_1 + \eta(t)) \, dt} \, dx \, dy
\]
\[
= \frac{1}{\gamma \gamma + \lambda_2 \lambda_4} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\gamma + \lambda_1 + \eta(t)) (s-t)} \, ds \, dy \int_0^\infty \int_0^\infty e^{-\int_0^x (\gamma + \lambda_1 + \eta(t)) \, dt} \, dx \, dy \int_0^\infty \int_0^\infty e^{-\int_0^x (\gamma + \lambda_1 + \eta(t)) \, dt} \, dx \, dy \int_0^\infty \int_0^\infty e^{-\int_0^x (\gamma + \lambda_1 + \eta(t)) \, dt} \, dx \, dy
\]
\[
= \frac{1}{\gamma \gamma + \lambda_2 \lambda_4} \int_0^\infty \int_0^\infty |f_4(s, t)| \, ds \, dt < \infty.
\]
So \( g_5(y) \in L^1(\mathbb{R}^+) \).

For \( g_{11} \) we have
\[
g_{11}(u) = \int_0^u f_2(\tau) e^{\int_0^\tau (\gamma + \lambda_1 + \eta(t)) \, dt} e^{-\int_0^\tau (\gamma + \lambda_1 + \eta(t)) \, dt} \, d\tau ds = e^{\int_0^u (\gamma + \lambda_1 + \eta(t)) \, dt} \int_0^u f_2(\tau) e^{\int_0^\tau (\gamma + \lambda_1 + \eta(t)) \, dt} e^{-\int_0^\tau (\gamma + \lambda_1 + \eta(t)) \, dt} \, d\tau ds
\]
\[
= e^{\int_0^u (\gamma + \lambda_1 + \eta(t)) \, dt} \int_0^u f_2(\tau) e^{\int_0^\tau (\gamma + \lambda_1 + \eta(t)) \, dt} e^{-\int_0^\tau (\gamma + \lambda_1 + \eta(t)) \, dt} \, d\tau ds
\]
\[
= \frac{1}{\lambda_2 - \lambda_1} \int_0^u f_2(\tau) e^{\int_0^\tau (\gamma + \lambda_1 + \eta(t)) \, dt} \, d\tau ds - \frac{1}{\lambda_2 - \lambda_1} \int_0^u f_2(\tau) e^{\int_0^\tau (\gamma + \lambda_1 + \eta(t)) \, dt} \, d\tau ds
\]
and hence \( g_{11} \in L^1(\mathbb{R}^+) \). Similarly we can get \( g_{13} \in L^1(\mathbb{R}^+) \). Since \( g_1 \in L^1(\mathbb{R}^+) \), \( j = 11, 12, 1, 2 \), we also have \( g_{14}, g_{16}, g_{17} \in L^1(\mathbb{R}^+) \). Therefore, these functions are absolutely continuous and absolutely integrable. \( \Box \)

**The proof of Theorem 3.2** By the direct verification, the dual space of \( X \) is
\[ X^* = \mathbb{R} \times (L^\infty(\mathbb{R}^+))^2 \times L^\infty(\mathbb{R}^+ \times \mathbb{R}^+) \times (L^\infty(\mathbb{R}^+))^4 \]
with the norm for \( F \in X^* \)
\[ ||F|| = \max \{|f_1|, |f_2|, \ldots, |f_6|\}. \]

Step 1. \( A \) is dissipative operator in \( X \).
For any \( P \in D(A) \), we define \( Q = (q_1, \ldots, q_8) \) where \( q_1 = ||P|| \text{sign}(P), q_j = ||P|| \text{sign}(p_j), j = 2, \ldots, 8 \). Obviously, \( Q \in X^* \) and \( Q \in E(P) = Q \in X^*(P, Q) = ||P||^2 = ||Q||^2 \). Moreover,
\[
\left(\frac{\partial P}{\partial t}\right) = -\lambda_1 P_1 + \int_0^\infty \eta(u)p_5(u) \text{sign}(P_1) \text{d}u + \int_0^\infty [-\lambda_2 - \left(\lambda_1 + \mu(u)\right) p_2(x)] \text{d}x - p_3'(y) \text{sign}(p_3(y))
\]
\[
+ \int_0^\infty \lambda_2 p_2(x) \text{sign}(p_3(x)) \text{d}x - \int_0^\infty \lambda_1 p_2(x) \text{sign}(p_3(x)) \text{d}x - \mu_1(x)p_4(x,y) \text{sign}(p_3(x)) \text{d}y + \int_0^\infty \int_0^\infty [-\lambda_1(x)p_4(x,y)] \text{d}y \text{d}x
\]
\[
- \frac{\partial}{\partial x} p_4(x,y) \text{sign}(p_4(x,y)) \text{d}x \text{d}y + \int_0^\infty \int_0^\infty [-\lambda_1(x)p_4(x,y)] \text{d}y \text{d}x + \int_0^\infty \lambda_2 p_5(x) \text{sign}(p_6(u)) \text{d}u + \int_0^\infty \lambda_2 p_5(x) \text{sign}(p_6(u)) \text{d}u + \int_0^\infty \lambda_1(x)p_6(x) \text{sign}(p_7(u)) \text{d}u - \eta(u)p_7(u) - p_7(u) \text{sign}(p_7(u)) \text{d}u
\]
\[
+ \int_0^\infty \lambda_1(x)p_6(x) \text{sign}(p_6(u)) - \mu_1(x)p_6(x) - p_6'(y) \text{sign}(p_6(x)) \text{d}y - \mu_1(x)p_6(x) \text{sign}(p_6(u)) \text{d}u - \mu_1(x)p_6(x) \text{sign}(p_6(u)) \text{d}u - \mu_1(x)p_6(x) \text{sign}(p_6(u)) \text{d}u - \mu_1(x)p_6(x) \text{sign}(p_6(u)) \text{d}u - \mu_1(x)p_6(x) \text{sign}(p_7(u)) \text{d}u
\]
\[
+ \int_0^\infty \lambda_1(x)p_6(x) \text{sign}(p_6(u)) - \mu_1(x)p_6(x) - p_6'(y) \text{sign}(p_6(x)) \text{d}y - \mu_1(x)p_6(x) \text{sign}(p_6(u)) \text{d}u - \mu_1(x)p_6(x) \text{sign}(p_6(u)) \text{d}u - \mu_1(x)p_6(x) \text{sign}(p_7(u)) \text{d}u
\]
\[
\text{Therefore, } A \text{ is dissipative.}
\]

Step 2. Set \( \gamma \in C \cap \gamma \geq 0, \gamma \neq 0 \), then \( T \subset \rho(A) \).

For any \( F \in \mathbb{X} \) and \( \gamma \in T \), we consider the resolvent equation \((\gamma I - A)P = F \), that is
\[
\left\{
\begin{align*}
(\gamma + \lambda_1)P_1 - \int_0^\infty \eta(u)p_3(u) \text{d}u &= f_1, \\
(\gamma + \lambda_2 + \mu(u))p_2(x) + p_5'(y) &= f_2(x), \\
(\gamma + \lambda_2 + \mu(u))p_2(x) + p_5'(y) - \int_0^\infty \mu_1(x)p_4(x,y) \text{d}x &= f_3(y), \\
(\gamma + \mu_1(x))p_4(x,y) + \frac{\partial}{\partial x} p_4(x,y) &= f_4(x,y), \\
(\gamma + \lambda_1 + \eta(u))p_5(u) + p_5'(u) &= f_5(u), \\
(\gamma + \lambda_2 + \eta(u))p_6(u) + p_6'(u) - \lambda_2 p_6(u) &= f_6(u), \\
(\gamma + \eta(u))p_7(u) + p_7'(u) - \lambda_2 p_6(u) &= f_7(u), \\
(\gamma + \mu_1(x))p_6(x) + p_6'(x) - \lambda_2 p_6(x) &= f_8(x).
\end{align*}
\right.
\]
(A.2)

with boundary conditions
\[
p_2(0) = \lambda_1 P_1 + \int_0^\infty \eta(u)p_5(u) \text{d}u, \\
p_5(0) = \int_0^\infty \mu_1(x)p_5(x) \text{d}x, \\
p_4(0,y) = \lambda_1 p_3(y).
\]
\[ p_5(0) = \int_0^\infty \mu_1(x)p_2(x)dx + \int_0^\infty \mu_2(y)p_3(y)dy, \]
\[ p_6(0) = p_7(0) = 0, \]
\[ p_8(0) = \int_0^\infty \eta(u)p_2(u)du. \]

At first, we solve \( p_5 \) from Eq. (A.2) and get
\[ p_5(u) = p_5(0)e^{-\int_0^u (\gamma + \lambda_1)dt} + \int_0^u f_5(s)e^{-\int_s^u (\gamma + \lambda_1)dt} ds. \]

When \( \Re \gamma > 0 \)
\[ \int_0^\infty |f_5(s)|ds|du \leq \int_0^\infty |f_5(s)|ds \int_s^\infty \frac{1}{\Re \gamma + \lambda_1} \leq \frac{1}{\Re \gamma} \frac{1}{\Re \gamma}. \]

When \( \gamma = \Re, \ s \neq 0 \)
\[ \int_0^\infty |p_5(u)|du \leq |p_5(0)| \int_0^\infty e^{-\int_0^u (\gamma + \lambda_1)dt} du + \int_0^\infty |f_5(s)|ds \int_s^\infty e^{-\int_s^u (\gamma + \lambda_1)dt} du. \]

By the assumption (3.2), we have
\[ \int_0^\infty |f_5(s)|ds \int_0^\infty e^{-\int_0^u (\gamma + \lambda_1)dt} du < \infty. \]

Therefore, \( p_5(u) \in L^1(\Re^+) \), and
\[ \int_0^\infty \eta(u)p_5(u)du = \int_0^\infty f_5(u)du + p_2(0) - (\gamma + \lambda_1) \int_0^\infty p_5(u)du \]

is finite. So we can prove \( p_1 \in \Re \).

Similarly, we can prove that \( p_2(x), p_3(x) \in L^1(\Re^+) \) and \( p_4(x,y), \frac{\partial}{\partial y} p_4(x,y) \in L^1(\Re^+ \times \Re^+) \). Using \( p_2, p_4, \) and \( p_3 \) we can prove \( p_5(y), p_6(y), p_7(u), p_8(u), p_9(u), p_8(p) \in L^1(\Re^+) \). Finally, we get \( p_7(u), p_8(u) \in L^1(\Re^+) \).

A direct calculation gives
\[
\begin{align*}
P_1 = \frac{1}{\gamma + \lambda_1} \left[ f_1 + \int_0^\infty \eta(u)p_5(u)du \right], \\
P_2(x) = p_2(0)g_1(x) + g_2(x), \\
P_3(y) = p_3(0)g_3(y) + g_4(y) + g_5(y), \\
P_4(x,y) = \lambda_1 p_2(x)g_3(x) + g_7(x), \\
P_5(u) = p_5(0)g_1(u) + g_9(u), \\
P_6(u) = \lambda_1 p_2(0)g_{10}(u) + g_{11}(u) + g_{12}(u), \\
P_7(u) = \lambda_1 \lambda_2 p_2(0)g_{13}(u) + g_{14}(u), \\
P_8(x) = p_8(0)g_{15}(x) + \lambda_2 p_2(0)g_{16}(x) + g_{17}(x),
\end{align*}
\]

where \( g_j, j = 1, \ldots, 17 \), are defined as in Lemma A.2.

Next, we substitute the expression in (A.3) into the boundary conditions and get an algebraic equation about \( (P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8) \), i.e.,
\[
\begin{align*}
-\lambda_1 P_1 + P_2(0) - \lambda_1 \int_0^\infty \eta(u)g_{10}(u)du p_5(0) = \int_0^\infty \eta(u)g_{11}(u) + g_{12}(u)du, \\
-\lambda_2 \int_0^\infty \mu_1(x)g_{16}(x)dx p_2(0) + p_3(0) - \int_0^\infty \mu_1(x)g_{15}(x)dx p_6(0) = \int_0^\infty \mu_1(x)g_{17}(x)dx, \\
-\int_0^\infty \mu_2(x)g_1(x)dx p_2(0) + p_3(0) - \int_0^\infty \mu_2(y)g_3(y)dy p_6(0) = m, \\
-\lambda_2 \int_0^\infty \eta(u)g_{13}(u)du p_5(0) + p_6(0) = \int_0^\infty \eta(u)g_{14}(u)du, \\
(\gamma + \lambda_1) P_1 - p_5(0) = \int_0^\infty \eta(u)f_5(u)du + f_1 - (\gamma + \lambda_1) \int_0^\infty f_5(s)dx e^{-\int_0^u (\gamma + \lambda_1 + \eta(u))dt} dsdu,
\end{align*}
\]

where \( m = \int_0^\infty \mu_1(x)g_2(x)dx + \int_0^\infty \mu_2(y)g_4(y) + g_5(y))dy \). Set \( B(\gamma) \) denotes matrix
\[
\begin{pmatrix}
\gamma + \lambda_1 & 0 & 0 & 0 & -d_2 & 0 \\
-\lambda_1 & 1 & 0 & 0 & -\lambda_1 \int_0^\infty \eta(u)g_{10}(u)du & 0 \\
0 & -\lambda_2 \int_0^\infty \mu_1(x)g_{16}(x)dx & 1 & 0 & 0 & -\int_0^\infty \mu_1(x)g_{15}(x)dx \\
0 & -\int_0^\infty \mu_2(x)g_1(x)dx & -\int_0^\infty \mu_2(y)g_3(y)dy & 1 & 0 & 0 \\
0 & 0 & 0 & -\lambda_2 \int_0^\infty \eta(u)g_{13}(u)du & 1
\end{pmatrix}.
\]
Since we have the following inequalities
\[ -\lambda_1 < |\gamma + \lambda_1|, \]
\[ -\lambda_2 \int_0^\infty \mu_1(x)g_{10}(x)dx - \int_0^\infty \int_0^\infty \mu_1(x)e^{\int_0^t (\gamma + \mu_1(t))dt}dx < 1, \]
\[ -\int_0^\infty \mu_2(y)g_{12}(y)dy < 1, \]
\[ -\lambda_1 \int_0^\infty \eta(u)g_{10}(u)du - \lambda_2 \int_0^\infty \eta(u)g_{13}(u)du < d_2; \]
\[ -\int_0^\infty \int_0^\infty \eta(u)e^{\int_0^t (\gamma + \eta(t))dt}du < 1, \]
so \( B(\gamma) \) is a column that is strictly diagonal dominant. So linear algebraic Eq. (A.4) has a unique solution \((P_1, P_2(0), P_3(0), P_5(0), P_6(0))\). Therefore, for each \( F \in X \) there is a unique solution \( P = (P_1, \ldots, P_8) \in D(A) \) such that \((\gamma I - A)P = F\). The closed operator theorem asserts that \((\gamma I - A)^{-1}\) exists and is bounded. Hence, \( T \subseteq \rho(A) \).

As a direct consequence, we have the following corollary.

**Corollary A.2.** There is no spectrum of \( A \) besides zero on the imaginary axis. Moreover, the spectra \( \sigma(A) \) are located in the left-half of the plane.

**The proof of Theorem 3.3.** Since \( A \) is a dissipative operator and \((0, \infty) \subset \rho(A) \) (see corollary.), the Lumer–Phillips Theorem (see [19]) asserts that \( A \) generates a \( C_0 \)-semigroup \( T(t) \) of contractions on \( X \). Hence the abstract Cauchy problem has a unique solution.

**The proof of Theorem 3.4.** Let \( X \) and \( A \) be defined as before, and \( T(t) \) be the \( C_0 \) semigroup generated by \( A \). We regard \( X \) as a real Banach space, and the positive cone defined by
\[ X_+ = \{ F = (f_1, f_2, \ldots, f_8) \in X | f_j \geq 0 \} \]
It is a closed convex cone, also called positive cone. A bounded linear operator \( T \) is called a positive operator if \( TX_+ \subset X_+ \). From what follows, we shall show \( T(t) \) is positive semigroup and satisfies \( \| T(t)P \| = |P| \) for any positive vector \( P \in D(A) \).

**Step 1.** \( T(t) \) is positive semigroup.

According to the positive semigroup theory (see [20]), \( A \) generates a positive \( C_0 \)-semigroup of contractions if and only if \( A \) is a dissipative and \( \Re(I - A) = \infty \). Since Theorem 3.2 has asserted that \( \Re(I - A) = \infty \), we only need to prove \( A \) is a dissipative operator.

**Step 2.** For any \( P \in D(A) \), we take \( Q = \|P\|(q_1, \ldots, q_8) \) where \( q_1 = \text{sign}_+(P_1), q_j = \text{sign}_+(P_j), j = 2, \ldots, 8 \) where
\[ \text{sign}_+(P_j) = \begin{cases} 1 & p_j > 0, \\ 0 & p_j \leq 0. \end{cases} \]

Similar to calculation in the proof of Theorem 3.2 can show that \( (AP, Q) \leq 0 \), so \( A \) is dissipative operator. Therefore, \( T(t) \) is a positive operator for all \( t \geq 0 \).

**Step 3.** For any \( F \in X_+ \), it holds that \( \| T(t)F \| = \| F \| \). Due to \( D(A) = X \), we only need to prove \( \| T(t)P \| = |P| \) for any positive vector \( P \in D(A) \). Let \( P \in D(A) \cap X_+ \). For \( t \geq 0 \), we get \( T(t)P \in D(A) \cap X_+ \) since \( T(t) \) is positive. Let
\[ P(t) = (P_1(t), P_2(t,x), P_3(t,y), P_4(t,x,y), P_5(t,u), P_6(t,u), P_7(t,u), P_8(t,x)) = T(t)P. \]

Then it satisfies the equation \( \frac{d}{dt} T(t)P = AP \), or equivalently the differential equation (2.1). Due to \( T(t)P \geq 0 \) we have
\[ \| P(t) \| = \| P_1(t) \| + \int_0^\infty p_2(t,x)dx + \int_0^\infty p_3(t,y)dy + \int_0^\infty \int_0^\infty p_4(t,x,y)dxdy + \sum_{j=5}^8 \int_0^\infty p_j(t,u)du + \int_0^\infty p_8(t,x)dx \]
and
\[ \frac{d}{dt} \| P(t) \| = \frac{d}{dt} \| P_1(t) \| + \frac{d}{dt} \int_0^\infty p_2(t,x)dx + \frac{d}{dt} \int_0^\infty p_3(t,y)dy + \frac{d}{dt} \int_0^\infty \int_0^\infty p_4(t,x,y)dxdy + \sum_{j=5}^8 \frac{d}{dt} \int_0^\infty p_j(t,u)du = 0 \]
so \( \| P(t) \| \) is a constant. By the continuity of \( T(t)P \), we get \( \| T(t)P \| = |P| \).

**References**