Constructive solid analysis: a hierarchical, geometry-based meshless analysis procedure for integrated design and analysis

Devendra Natekar¹, Xuefeng Zhang, Ganesh Subbarayan*

School of Mechanical Engineering, Purdue University, 585 Purdue Mall, West Lafayette, IN 47907-2088, USA

Received 31 December 2002; received in revised form 3 June 2003; accepted 11 June 2003

Abstract

In this paper, we propose an analysis methodology that is procedurally analogous to Constructive Solid Geometry (CSG) integrating design and analysis, and thereby enabling efficient optimal design. The procedure, due to its analogous nature to CSG, is termed Constructive Solid Analysis (CSA). The analysis methodology is partitioned, hierarchical and is based on constructing the boundary value problem for a compound geometry through operations on the field quantities defined on the primitives. Although the CSA procedure will allow any basis for approximating the fields, Non-Uniform Rational B-Splines (NURBS), currently popular in the geometric modeling literature, are used to represent the geometry of the primitives as well as the analysis fields. The use of the same basis to represent geometry and analysis fields enables ‘representational’ integration, and further, the developed methodology may be classified as a partition of unity meshless analysis scheme. A more general null-space solution scheme and a somewhat restrictive range-space solution scheme are outlined to solve the discretized equations resulting from the use of NURBS. Several representative problems from the field of linear elasticity are solved to demonstrate the validity of the procedure and to evaluate its computational cost relative to the finite element method. The optimal orientation of an elliptical hole to applied tractions is determined to demonstrate the power of the proposed methodology for shape optimal design.

Keywords: Constructive solid geometry; Optimal design; Constructive solid analysis

1. Introduction

Optimal design is an iterative process often involving Computer Aided Geometric Design or a modeling phase and an analysis phase. Despite advances in both fields, the integration of the two is still an area of research. The basic hurdle to integrating the two phases is the transfer of the parametrized geometric information between the design and analysis phases. While the geometric modeling phase often uses sections or primitives that are manipulated and combined to construct a three-dimensional shape, during the analysis phase the modeling operations and the parametric design representation that is inherent in these operations are often ignored. For instance, to build a finite element mesh, the process typically begins with the final geometry. By using the construction procedure intrinsic to the final geometry during the analysis phase, one may be able to derive advantage in optimal design problems where the geometry is iteratively changed as per the analysis determined criteria. Any changes to the geometry currently necessitates the rebuilding of the final geometry from scratch and rediscretization of the geometry. Although significant advances in automatic mesh generation have mitigated the challenge of remeshing, the need for meshing remains a critical bottleneck to optimal design.

Various procedures have been adopted to automate the transition between the modeling and analysis phases. Much of the research work in the past has concentrated on automated discretization schemes. Tarrago et al. [1] developed an ‘integrated’ design approach based on the finite element method which included an automated meshing technique. The resulting procedure, while automated, does not address the computational expense associated with the analyses of an iteratively changing geometry. Iteratively changing geometry is typical of shape optimization problems and topology optimization schemes that seek to determine the optimal topology followed by shape refinement [2,3]. The shape refinement is often
accomplished by a ‘smoothening’ phase where the jagged boundary corresponding to the (finite element) discretized structure is smoothened by fitting B-splines or other entities that were not part of the initial geometric model definition [4]. In these schemes, a change in shape would have to be followed by a re-discretization of the new geometry prior to analysis.

Several researchers have also proposed the use of a common basis function for geometric modeling and for representing the fields, which as a solution to a boundary value problem represents the ‘analysis’. Kagan et al. [5] developed a B-spline based finite element scheme. Cassale [6] has utilized trimmed surface patches as boundary elements. Renken and Subbarayan [7] used NURBS to represent the shape of droplets as well as to determine the solid–liquid, liquid–vapor interaction energies at the surfaces. These latter quantities were obtained by integrating surface energy coefficients over the appropriate surfaces. Kulkarni [8] used NURBS based field representations on the boundary to develop a procedure analogous to the boundary element method. By using a common mathematical representation for both geometry modeling and the fields, these procedures are significant in advancing the integration of design and analysis. However, substantially greater integration than above can be achieved by an analysis procedure that emulates the hierarchical nature of the modern three-dimensional geometry construction techniques.

The ‘Domain Composition’ method proposed by Cox et al. [9–11] aims to integrate the CSG procedure for geometric modeling with the analysis phase by meshing the primitives instead of the final geometry. Although meshing of the primitives alleviated the problems associated with meshing the final geometry, the procedure required the removal of overlapping elements belonging to the primitives. This was since the underlying boundary value problem solution was still on fields defined over the final geometry rather than using the fields defined on the primitives. Also, the removal of the overlapping elements results in an approximate scheme since there still exists a partially overlapping region along primitive boundaries which causes an increase in the stiffness. Thus, the solution will only approximate the true numerical solution when this small overlapping zone is present, and will approach the true solution as the overlap is reduced. Furthermore, the implementation requires a routine for recognizing the overlapping elements and results in the creation of new ‘boundaries’ along the regions of elemental overlap. These coincide with the elemental sides.

Fundamental to the Constructive Solid Analysis (CSA) procedure that is developed in the present paper is the formulation and solution of boundary value problems corresponding to the resultant structure through Boolean operations using the fields defined on the primitives. A NURBS representation is used to define both the primitive geometry as well as the analysis fields defined on the primitives. To facilitate this integration, the NURBS approximation is formulated as a partition of unity based meshless analysis method. The use of NURBS enables adaptivity in analysis through degree elevation (analogous to p-refinement) or through knot insertion (analogous to h-refinement), for which the algorithms are well established. Established problems from the field of linear elasticity theory that demonstrate the subtraction, intersection and union operations on the primitives are solved to validate the procedure. In Section 2, the developed methodology is described in detail. The validation examples and the analysis results are detailed following the methodology description. The article is concluded with remarks on the developed procedure and potential future applications.

2. Methodology

As mentioned earlier, Renken and Subbarayan [7] developed a droplet shape prediction procedure using the NURBS definition for droplet surface representation and integrating over the surface entities to calculate the surface energies. While using the same mathematical basis for geometry and analysis fields representation enables tighter integration, here, we achieve even greater integration by emulating (during analysis) the hierarchical procedure used in the Constructive Solid Geometry (CSG) technique for geometric modeling. At the present time, the CSG procedure is a well established one in the field of geometric modeling [12–15]. While this may be obvious to most readers, for the sake of completeness, we show the Boolean operations at the core of CSG in Fig. 1. Prior to describing the algorithmic details of the analysis procedure, we state the boundary value problem or the ‘analysis’ problem below.

2.1. Problem definition

Given:

1. Design domain \( \Omega \)
2. Design boundary \( \Gamma \)
3. Boundaries \( \Gamma_i \) and \( \Gamma_u \) such that \( \Gamma_i \cap \Gamma_u = \{ \phi \} \) and \( \Gamma_i \cup \Gamma_u = \Gamma \)
4. Boundary conditions on fields \( u \) and \( t \) as \( u_{\Gamma_i} = \tilde{u} \) and \( t_{\Gamma_i} = \tilde{t} \)

Find: field variable \( \tilde{u}_\Omega \) over the domain \( \Omega \).

Such that: \( u = \{ u_1, u_2, \cdots, u_n \} \), where \( u_i \), \( i = 1, \cdots, n \) are fields defined over the primitives \( \Omega_i \) constituting the final geometry and \( \{ \} \) refers to the set operations on all \( u_i \) constituting \( u \). The domains \( \Omega_i \) satisfy the property \( \Omega_1 \ast \Omega_2 \ast \cdots \ast \Omega_n = \Omega \), where \( \ast \) represents a regularized Boolean operation (\( \cup^*, \cap^*, \ast^* \)).

Subject to: field variable (displacement \( u \) or traction \( t \)) constraints across primitive interfaces.
2.2. Algorithm

We first present an outline of the algorithmic steps involved in the analysis procedure. The detailed descriptions of the steps follow in subsequent subsections. The items below contain references to the sections where the step is described in detail.

1. From the geometry construction step, determine \( n \) primitives \( V_1, V_2, \ldots, V_n \) such that \( V_1 \ast V_2 \ast \cdots \ast V_n = V \), where \( \ast \) represents a regularized Boolean operation \((\cup^*, \cap^*, -^*)\) [15].

2. Classify \( G_I \) as \( G_I^{in}, G_I^{out}, G_I^{sh}, G_I^{anti-shared} \) (Section 2.3) where \( I = 1, \ldots, n \).

3. Given \( \Omega_1, \Omega_2, \ldots, \Omega_n \) and Boolean operations determine \( \Gamma = f(G_I^{in}, G_I^{out}, G_I^{sh}, G_I^{anti-shared}) \) (Section 2.3).

4. Formulate the mechanics problem (Section 2.4)
   (a) Form individual mechanics problem for each Boolean operation (Sections 2.4.2 and 2.4.3).
   (b) Combine to formulate the system level problem

5. For each \( \Omega_I \) where \( I = 1, \ldots, n \), approximate field \( u = \sum_j N_j(\xi, \eta)u_j \) where \( u_j \) are values at points of approximation (Section 2.5) and \( N_j \) are the basis functions (Section 2.5) [16].

6. Integration (Section 2.6) The current methodology has been implemented with constant background integration net.
   (a) Loop over integration points (IP).
   (b) For each integration point IP, determine if IP \( \in \Omega \) or IP \( \notin \Omega \).
   (c) If IP \( \in \Omega \) then integrate else eliminate.

7. Apply boundary conditions (Section 2.7) Detect control points influencing a boundary point or the boundary of the intersection region and apply the appropriate conditions or constraints.

8. Form solution system (Section 2.8).

9. Solve

2.3. Boundary definition

The solution to the boundary value problem involves constraint conditions which are defined on the boundaries of...
the resultant geometry. Since the analysis is carried out on the primitives rather than the final geometry, the boundary conditions applied on the resultant geometry need to be described in terms of those on the primitive boundary. Furthermore, primitive based analysis requires the imposition of additional conditions on the boundaries of the intersecting regions of the primitives (Sections 2.4.2 and 2.4.3). Thus, a definition of the boundaries of the objects resulting from Boolean operations on the primitives is essential for constructive analysis. Here, we use the convention adopted in Ref. [14]. Similar conventions can also be found in Refs. [13,15].

We start by defining the intersection problem. For the convenience of representation, the union and the subtraction problem are defined using the intersection problem and primitive definitions. The relationship between the formulation stated here and the one in Ref. [14] is straightforward to show.

We define (as shown in Fig. 2):

1. \( \Omega \): domain under consideration. e.g. \( \Omega_P \) is the region occupied by BCEF
2. \( \Gamma \): boundary of domain. \( \Gamma_P = CDEFIJB \)
3. \( n_P \): normal at point ‘i’ on \( \Gamma_P \), +ve direction points towards \( \Omega \not\in \Omega_P \)
4. \( n_Q \): normal at point ‘j’ on \( \Gamma_Q \), +ve direction points towards \( \Omega \not\in \Omega_Q \)
5. \( \text{Pin}_Q : \Gamma_P \in \Omega_Q; \text{seg}(JB) \) is \( \text{Pin} Q \)
6. \( \text{Qin}_P : \Gamma_P \in \Omega_P; \text{seg}(JD) \) is \( \text{Qin} P \)
7. \( \text{Pout}_Q : \Gamma_P \not\in \Omega_Q; \) segments DE, EF, FI, IJ constitute PoutQ
8. \( \text{Qout}_P : \Gamma_Q \not\in \Omega_P; \) segments JK, KA, AB constitute QoutP

For the intersection problem in Fig. 1:

\[ \text{PQ}_\text{shared} = \{ \phi \} \]
\[ \text{Pin}_Q = 1-3 \]
\[ \text{Qin}_P = \text{arc}(1-4-3) \]

The union problem is defined as:

\[ \Omega_{\text{p}\cup\text{q}} = \Omega_P + \Omega_Q \]
\[ \Gamma_{\text{p}\cup\text{q}} = \Gamma_P + \Gamma_Q - 2 + \text{PQ}_\text{anti-shared} \]

For the union problem in Fig. 1:

\[ \Gamma_{\text{p}\cup\text{q}} = \text{Quad}(ABCD) + \text{Circle}(1234) - \text{arc}(143) + \text{seg}(1-3) \] - \{ \phi \} \]

The subtraction problem can be defined as:

\[ \Omega_{\text{p}-\text{q}} = \Omega_P - \Omega_Q \]
\[ \Gamma_{\text{p}-\text{q}} = \Gamma_P - \Gamma_Q \]

For the subtraction problem in Fig. 1:

\[ \Gamma_{\text{p}-\text{q}} = \text{Quad}(ABCD) - \text{Semi Circle}(1-3-4-1) \]

In this case the negative sign associated with \( \Gamma_{\text{p}\cap\text{q}} \) is associated with the normals of the boundaries forming \( \Gamma_{\text{p}\cup\text{q}} \) and refers to reversal in ‘material’ region. In this example, prior to the subtraction operation the boundary \( \Gamma_Q = 1-2-3-4 \) enclosed material inside it. After the operation, as indicated by the negative sign, the material region lies outside the circle and bounded by \( \Gamma_P \) on the outside and \( \Gamma_Q \) on the inside. Thus the reversal in material direction is implicit in this mathematical formulation.

The union problem is defined as:

\[ \Omega_{\text{p}\cup\text{q}} = \Omega_P + \Omega_Q \]
\[ \Gamma_{\text{p}\cup\text{q}} = \Gamma_P + \Gamma_Q - 2 + \text{PQ}_\text{anti-shared} \]

For the union problem in Fig. 1:

\[ \Gamma_{\text{p}\cup\text{q}} = \text{Quad}(ABCD) + \text{Circle}(1234) - \text{arc}(143) + \text{seg}(1-3) \] - \{ \phi \} \]

2.4. Governing equations

In this section, we present a generalized mathematical statement for the CSA methodology. The statement encompasses the non-linear and linear domain of problems. The intention is to emphasize the generality of the methodology prior to its simplification (purely for ease of implementation) to the linear static set of problems (Section 2.4.4).

Fig. 2. Common conventions adopted in the mathematical formulation of the CSG procedure.
2.4.1. Statement

The balance of mechanical energy for a static system (neglecting the kinetic energy term) in the rate form is:

\[ W = P_E \]  

(1)

where

\[ W = \int_{\Omega} \sigma \varepsilon \, d\Omega \]  

(2)

\[ P_E = \int_{\Gamma} b \cdot u \, d\Gamma + \int_{\Omega} \sigma \varepsilon \, d\Omega \]  

(3)

are the rate of internal energy and the power of external work respectively. In the above equations, \( \sigma \) and \( \varepsilon \) denote the stress and the rate of strain tensors, respectively; \( b, t \) and \( \dot{u} \) denote the body force, traction and velocity (\( \dot{u} \) being the displacement) vectors, respectively. Throughout the paper, juxtaposition of tensorial quantities without the explicit use of indices implies an inner product between the quantities.

We now formulate the generalized variational statement for the Boolean operations of union and subtraction. The form of the statement for the intersection operation is similar to that for the subtraction problem and is not explicitly outlined.

2.4.2. Union

For the union problem, the balance of mechanical energy for the static problem can be stated as

\[ W_{P \cap Q} = P_{E_{P \cap Q}} \]  

(4)

\[ W_{Q \cap P} = P_{E_{Q \cap P}} \]  

(5)

\[ W_{P \cap Q} = P_{E_{P \cap Q}} \]  

(6)

In order to carry out the operations on the primitives, as is the goal of the methodology, the field within the intersection region needs to be defined in terms of the fields associated with the primitives. Thus, a key to the proposed methodology is the definition of the field within the intersection region for the union problem (\( \Omega_{P \cap Q} \)) as

\[ u_{P \cap Q} = c_p u_P + c_Q u_Q \]  

(7)

Here, \( c_P \), \( c_Q \), are constants to be selected and \( u_P \) and \( u_Q \) are the field definitions for the domains \( P \) and \( Q \), respectively. An assumption of this form also eliminates the necessity to minimize the overlap between the intersecting primitive domains as is done in the ‘Domain Composition’ method [9–11].

Since the balance of mechanical energy is satisfied at every instant, and since it is required for the power of external work at the boundaries of \( \Omega_{P \cap Q} \) to be equal and opposite to that at the corresponding boundaries of \( \Omega_{P \cap Q} \) and \( \Omega_{Q \cap P} \), an approximate, but equivalent, problem can be constructed as:

\[
\min \int_0^T \left( W_{P \cap Q} + W_{P \cap Q} + W_{Q \cap P} - \int_{\Gamma_{P \cap Q}} t_P \dot{u}_P \, d\Gamma \right) \\
- \int_{\Gamma_{P \cap Q}} t_Q \dot{u}_Q \, d\Gamma \right) \, d\tau 
\]

(8)

s.t. : \( c_p u_P + c_Q u_Q = u_P = u_Q \)

\( \Rightarrow c_p + c_Q = 1 \)  

(9)

(10)

Thus, any combination of values for the constants \( c_P \) and \( c_Q \) are possible as long as they satisfy Eq. (10).

2.4.3. Subtraction

For the subtraction problem since the intersection region \( (\Omega_{P \cap Q}) \) is eliminated:

\[ W_{P \cap Q} = 0 = P_{E_{P \cap Q}} \]  

(11)

As in the union problem, we define the field within the intersection region \( \Omega_{P \cap Q} \) as

\[ u_{P \cap Q} = c_p u_P - c_Q u_Q \]  

(12)

to ensure the description of the compound problem in terms of the primitive fields. If for simplicity, we assume the body force in the intersection region to be zero, then from Eqs. (11) and (12) we get:

\[ \int_{\Gamma_{P \cap Q}} t(c_p u_P + c_Q u_Q) \, d\Gamma = 0 \]  

(13)

If the tractions on the boundary \( \Gamma_{Q \cap P} \) are forced to be zero, the condition is always satisfied. If not, we can always find a field \( u_Q \) such that the condition of Eq. (13) is satisfied, provided \( c_Q \neq 0 \).

2.4.4. Variational statement for the linear elastic static case

We now recast the variational statements for the Boolean operations formulated in Sections 2.4.2 and 2.4.3 for the linear elastic static problem. The implementation of the intersection operation is procedurally similar to the subtraction problem and hence has not been discussed separately. For a conservative system, assuming static conditions, one can define a potential of the form:

\[ \Pi = \int_0^T (W - P_E) \, d\tau \]  

(14)
where \( \tau \) is the time used only to identify the end state. In other words, the value of the integral depends only on the end states and not on the path since the system is conservative. Carrying out the time integration, we get:

\[
W = \int_0^T W \, dt = \int_0^T \int_\Omega \sigma \varepsilon \, d\Omega \, d\tau = \int_\Omega \frac{1}{2} \sigma \varepsilon \, d\Omega \tag{15}
\]

\[
V = \int_0^T P_\varepsilon \, d\tau = \int_0^T \int_\Gamma \tau u \, d\Gamma \, d\tau + \int_0^T \int_\Omega bu \, d\Omega \, d\tau
= \int_\Gamma tu \, d\Gamma + \int_\Omega bu \, d\Omega \tag{16}
\]

For a generalized union problem of the type shown in Fig. 4 involving two overlapping primitives \( P \) and \( Q \), the variational problem can now be stated as:

\[
\min W_{P\cap Q} + W_{P\cap Q} + W_{Q\cap P} - \int_{\Gamma_{\text{rew}}} t_{\text{up}} \, d\Gamma
\]

subject to:

\[
\begin{align*}
&u_{P\cap Q} = u_P \text{ on } \Gamma_{\text{gap}} \\
&u_{Q\cap Q} = u_Q \text{ on } \Gamma_{\text{gap}}
\end{align*}
\]

with \( u_{P\cap Q} = c_P u_P + c_Q u_Q \) as before, stationarity of the Lagrangian with respect to the Lagrange multipliers then implies:

\[
\delta L_P = c_P \delta u_P + c_Q \delta u_Q - u_P = 0 \text{ on } \Gamma_{\text{gap}}
\]

\[
\delta L_Q = c_P \delta u_P + c_Q \delta u_Q - u_Q = 0 \text{ on } \Gamma_{\text{gap}}
\]

with \( c_P + c_Q = 1 \) as derived in Section 2.4.2.

The variational statement for the subtraction problem can similarly be written as:

\[
\min W_p - \int_{\Gamma_{\text{rew}}} t_{\text{up}} \, d\Gamma - \int_{\Gamma_{\text{rew}}} t_{\text{up}} \, d\Gamma
\]

subject to:

\[
W_{P\cap Q} = 0
\]

Now, if the field in the intersection region were to be described as: \( u_{P\cap Q} = c_P u_P - c_Q u_Q \), stationarity of the Lagrangian with respect to \( u_Q \) implies:

\[
\delta L_{u_Q} = c_Q \int_{\Omega_{\text{rew}}} \varepsilon_0 D \delta \varepsilon_0 \, d\Omega - c_P c_Q \int_{\Omega_{\text{rew}}} \varepsilon_0 D \delta \varepsilon_0 \, d\Omega
\]

\[
= 0
\]

\[
\Rightarrow (c_Q \varepsilon_0 - c_P \varepsilon_0) = 0 \text{ if } c_Q \neq 0 \tag{20}
\]

Thus, we can always find a field \( u_Q \), given a field \( u_P \), such that \( W_{P\cap Q} = 0 \). Instead of explicitly imposing the condition \( W_{P\cap Q} = 0 \), we eliminate integration in this region since the subtraction operation is tantamount to elimination of ‘material’ from the region \( \Omega_{P\cap Q} \). This simplifies the procedure by eliminating the explicit constraint on the minimization problem in Eq. (19) [17].

2.5. Field variable definition using non-uniform rational B-splines

NURBS are used as the mathematical basis to define both the geometry as well as for the analysis field variables which, in our case, are the displacements across the design domain. This achieves ‘representational’ integration of the modeling and analysis phases. The methodology, as is formulated, is not dependent on the NURBS basis. The reader is referred to well established literature for a detailed description of NURBS [18,19]. The equation describing a NURBS surface is [18]:

\[
S(\xi, \eta) = \frac{\sum_{i=0}^n \sum_{j=0}^m N_{ip}(\xi) N_{jq}(\eta) w_{ij} P_{ij}}{\sum_{i=0}^n \sum_{j=0}^m N_{ip}(\xi) N_{jq}(\eta) w_{ij}}
\]

\[
N_{ip}(\xi) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(\xi) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+p}}
\]

\[
\times N_{i+1,p-1}(\xi)
\]

\[
N_{ij}(\xi) = \begin{cases} 
1 & \text{if } \xi_i \leq \xi \leq \xi_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

where \( P_{ij} \) and \( w_{ij} \) are the \( i \)th and \( j \)th control point vector (made up of the coordinates \( x, y, z \)) and the weight associated with the \( i \)th control point, respectively. The above expression can be recast in the following form for any point in the system:

\[
S(\xi, \eta) = \sum_I N_I(\xi, \eta) P_I
\]

where \( P_I \) are the set of control points defining the system geometry, and

\[
N_I = \frac{N_{ip}(\xi) N_{jq}(\eta) w_{ij}}{\sum_{i=0}^n \sum_{j=0}^m N_{ip}(\xi) N_{jq}(\eta) w_{ij}}
\]

with the index \( I \) taking on all possible values of the combination of indices \( ij \). In this form, the similarity of the mathematical representation to the finite element method is clear, although the nature and continuity of the representation are quite different. Similarly, any field (e.g. displacements) can be approximated over the design domain as:

\[
\bar{u}(\xi, \eta) = \sum_I N_I(\xi, \eta) u_I
\]

where \( u_I \) are the field values at the appropriately chosen control points. The NURBS basis inherently possesses properties that are desirable in any analysis procedure:

1. Non-negativity: \( N_I(\xi, \eta) \geq 0 \)
2. Local support: the NURBS basis function is non-zero only over its ‘support’ region. For a NURBS basis of degree \( p \), the basis is non-zero over \( p + 1 \) knots.
3. Partition of unity: at any point within the domain,

\[
\sum_I N_I = 1
\]
4. \( p - k \) times differentiable where \( k \) is the knot multiplicity
5. At most \( (p+1)(q+1) \) non-zero basis functions in a rectangle \([\xi_i, \xi_{i+1}) \times [\eta, \eta_{i+1})\)

These properties are advantageous for a meshless analysis, and compare very favorably to element free Galerkin approaches [20,21]. Furthermore, NURBS are more flexible in representing local variations of the geometry as well as the field variables. Algorithms for knot refinement \((h\text{-adaptivity})\) and degree elevation \((p\text{-adaptivity})\) are well established [18] and are easier to implement algorithmically as compared to other meshless procedures or the finite element method.

2.6. Integration

Construction of the solution system requires integration of strain energy over the design domain in Problems (17) and (19). The constructive analysis procedure enables the integration over the body to be naturally carried out on the individual primitives by using background integration nets assigned to each primitive. Each primitive may possess a different quadrature net as shown in Fig. 3. Standard Gaussian integration is used for all the problems [16]. However, adaptive integration schemes can also be used. Such an adaptive integration scheme was implemented in earlier research by Renken and Subbarayan in their NURBS based droplet shape prediction tool [7]. The accuracy of the integration will depend on the number of cells and their distribution in each quadrature net, particularly at the boundary.

2.7. Boundary conditions and constraints

The application of the boundary conditions in finite elements or boundary elements employing the standard Lagrangian interpolation is simple since nodes can be placed at the point of application of the boundary condition. However, in the case of a NURBS representation, the control points need not coincide with the boundary. Also, unlike in finite elements, the value of the shape function corresponding to a control point is not unity at that node. Thus, even if the control point were to be coincident with the location of the boundary condition, direct application of the boundary condition is not possible since the specified field value will be distributed to control points influencing the point under consideration. The same is true for the constraints that arise due to Boolean operations on the primitive fields.

2.7.1. Detection of control points

Once the boundary of the domain resulting from the Boolean operations is determined, it is necessary to find the control points influencing the boundary for application of the boundary constraints in terms of the field variable unknowns associated with the control points of the primitives. Furthermore, control points influencing neither the domain nor the boundary of design need to be eliminated to prevent degeneration of the solution system.

The control points influencing the domain are easily determined by noting the span of the design domain within the knot space of the NURBS field. The determination of the control points influencing the boundary requires the use of a point inversion algorithm, which determines the position of a point within the parameterized space, \((\xi, \eta)\), given the Cartesian co-ordinates of the point, \(P(x, y, z)\) [18].

Fig. 3. A schematic of the procedure for numerical quadrature over fixed, background nets assigned to each primitive.
For the subtraction problem, apart from control point detection, integration points which do not belong to the material domain need to be eliminated. Classification of integration points as inside or outside the domain is done in the present study using the algorithm described by Mortenson [15].

2.8. Formulation of the discretized solution system

For the sake of simplicity, we have demonstrated the developed methodology only on linear elastic problems. As discussed in Section 2.4 the methodology is more general and is valid for non-linear problems as well. In the present study, a Lagrange multiplier scheme is adopted for the application of the constraints on the primitives as well as the boundary conditions. This is briefly outlined below. Considering the essential boundary conditions defined on $\Gamma_u$, the principle of minimum potential energy may be stated (using Eqs. (14)–(16)) as:

$$\text{min} \quad H = W - V$$

s.t. : $$u = \tilde{u} \quad \text{on} \quad \Gamma_u$$

which is equivalent to finding the extremum of:

$$H' = W - V - \int_{\Gamma_u} \lambda (u - \tilde{u}) d\Gamma$$

(27)

where, $\lambda$ is the Lagrange multiplier field.

Discretizing the displacement and Lagrange multipliers as:

$$\tilde{u} = \sum_{I} N_I u_I \quad \text{over} \quad \Omega$$

$$\tilde{\lambda} = \sum_{K} \Phi_K \lambda_K \quad \text{over} \quad \Gamma_u$$

(28)

Noting that $\epsilon(\xi, \eta) = \sum B_I u_I$, the discretized form yields:

$$H' = \frac{1}{2} \tilde{u}^T \left( \int_{\Omega} B_I^T D B_J \ d\Omega \right) \tilde{u}$$

$$- \tilde{u}^T \left( \int_{\Omega} N_I^T b \ d\Omega + \int_{\Gamma_u} N_I^T \tilde{t} d\Gamma \right)$$

$$- \lambda^T \left( \int_{\Gamma_u} \Phi_K^T N_I d\Gamma \right) u_I + \lambda^T \left( \int_{\Gamma_u} \Phi_K^T \tilde{u} d\Gamma \right)$$

(29)

where, $N_I$ are the NURBS basis functions for the displacement field, and $\Phi_K$ are the NURBS basis for discretization of the Lagrange multipliers. $B = dN/d(\xi, \eta)$ is the strain displacement matrix. It is important to note that, $N$ and $\Phi$ are chosen to be independent of one another since displacements are discretized over the domain while the Lagrange multipliers are discretized only over the boundaries of interest. Stationarity of Eq. (29) with respect to $\lambda$ and $u$ gives:

$$Ku - G^T \lambda = f$$

(30)

$Gu = d$

where

$$K_{ij} = \int_{\Omega} B_I^T D B_J \ d\Omega$$

$$G_{kl} = \int_{\Gamma_u} \Phi_K^T N_l d\Gamma$$

The force and the specified boundary conditions which constitute the RHS of the equation are calculated as:

$$f_i = \int_{\Omega} N_I^T b \ d\Omega + \int_{\Gamma_u} N_I^T \tilde{t}$$

(32)

$$d_K = \int_{\Gamma_u} \Phi_K^T \tilde{u} d\Gamma$$

(33)

This can be expressed as:

$$\begin{bmatrix} K & -G^T \\ -G & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ -d \end{bmatrix}$$

(34)

The solution to the union problem involves the application of displacement constraints on $\Gamma_{P\cap Q}$. Insight into the formulation of the solution system for this problem can be obtained from the discretized form of the Lagrangian for Problem (17). Ignoring the applied boundary conditions for clarity:

$$L' = \frac{1}{2} u_p^T K_{P\cap Q} u_p + \frac{1}{2} u_q^T K_{Q\cap P} u_q + \frac{1}{2} (c_p u_p^T K_{P\cap Q} u_p$$

$$+ c_q u_q^T K_{Q\cap P} u_q + c_p c_q u_p^T K_{P\cap Q} u_q)$$

$$- \lambda_p^T \int_{\Gamma_{Q\cap P}} \Phi_p (1 - c_p) N_p u_p - c_q N_q u_q) d\Gamma$$

$$- \lambda_q^T \int_{\Gamma_{Q\cap P}} \Phi_q (1 - c_q) N_p u_p - c_p N_p u_p) d\Gamma$$

(35)

Using $c_P + c_Q = 1$ and absorbing $(1 - c_P)$ and $c_P$ into the Lagrange multipliers $\lambda_P$ and $\lambda_Q$, we get:

$$L' = \frac{1}{2} u_p^T K_{P\cap Q} u_p + \frac{1}{2} u_q^T K_{Q\cap P} u_q + \frac{1}{2} (c_q u_q^T K_{P\cap Q} u_p$$

$$+ c_q u_q^T K_{Q\cap P} u_q + c_p c_q u_p^T K_{P\cap Q} u_q)$$

$$- \lambda_p^T \int_{\Gamma_{Q\cap P}} \Phi_p (N_p u_p - N_q u_q) d\Gamma$$

$$- \lambda_q^T \int_{\Gamma_{Q\cap P}} \Phi_q (N_p u_p - N_q u_q) d\Gamma$$

(36)

where the subscripts $P$ and $Q$ in general refer to the primitives to which the discretizations belong. The subscripts on the stiffness terms $K_{P\cap Q}$ refer to the domain of integration and the subscripts on $\lambda$s refer to the domain on which the constraint is imposed. Thus, the first order necessary conditions involving the derivative of the above Lagrangian with respect to the independent quantities of
displacement field and the boundary condition Lagrange in Eq. (40), the range-space approach \[22\] gives the decomposition of the stiffness matrix 

\[
\begin{bmatrix}
K_{PQ} + c_P^2 K_{PQ} & c_P c_Q K_{PQ} & -G_{PP} & G_{QP} \\
-c_P c_Q K_{PQ} & K_{Q-P} + c_Q^2 K_{PQ} & G_{PP} & -G_{QP} \\
-G_{PP} & G_{QP} & 0 & 0 \\
G_{QP} & -G_{QP} & 0 & 0
\end{bmatrix}
\]

In general, the solution systems (Eqs. (34) and (37)) have the same block form.

Using the system in Eq. (34) and the inverse formulated in Eq. (33), the range-space approach \[22\] gives the displacement field and the boundary condition Lagrange multipliers as the solution to the following:

\[
Ku = (f + G^T \lambda) \\
(GK^{-1}G^T) \lambda = -(GK^{-1}f - d)
\]

The calculation of \(u\) and \(\lambda\) involves the storage of the Cholesky factors of \(K\) and \((GK^{-1}G^T)\). The banded Cholesky decomposition of the stiffness matrix \(K\) requires \(1/2(np^2 + 7p + 1)\) flops where \(p\) is the bandwidth of \(K\) and \(n\) are the degrees of freedom involved. The factorization of \((GK^{-1}G^T)\) does not require the explicit assembly of the matrices; it is equivalent to solving \(KA = G^T\) and requires \(1/2(np^2 + 7p + 1) + 2m(p + 1)\) flops which include the cost of the fore mentioned Cholesky decomposition of \(K\). Here, \(m\) are the number of constraints and \(G \in \mathbb{R}^{m \times n}\). For \(m \ll n\), the size of the matrix \(GK^{-1}G^T\) is \(m \times m\) and small as compared to the size of the \(K\) matrix, thus facilitating efficient storage and computation. Since the inversion involves the stiffness matrix directly rather than the augmented matrix of Eq. (34), using the banded nature of the stiffness matrix reduces the cost of the inversion procedure. The total computational cost of solving the system is \(n(1/2p^2 + 2mp + m^2 + 5m + 2p) - p^3(m + 1) + \frac{7}{2}p + 5\) flops.

If \(m = n\), then one would prefer the null-space method. Assuming that \(G\) has full row rank, one can carry out the QR factorization of \(G^T \{23\}: G^T = QR\). Where \(Q\) is orthogonal and \(R\) is upper triangular. The displacement solution \(u\) can then be written using range-space and null-space components of \(G\):

\[
u = Yu_t + Zu_c
\]

Where, \(Y \in \mathbb{R}^{n \times m}\) and \(Z \in \mathbb{R}^{n \times a-m}\) are the orthonormal basis for range-space and null-space of \(G\) and \(u\), and \(u_c\) are the range space and null space components of \(u\), respectively. The constraint equations completely determine the range space component as shown below leaving the null space component of \(u\), namely \(u_c\), free:

\[
d = Gu = GYu_t + G Zu_c = GYu_c
\]

Since \(Z\) represents the basis for the null space of \(G, GZ = 0\) in the above equation.

It is easy to show that the first \(m\) columns of \(Q\) determined through \(QR\) decomposition of \(G\) represent the orthogonal basis for the range space \(Y\) and the next \(n - m\) columns represent the null-space basis \(Z\):

\[
GQ = G[Y\|Z] = [GY\|GZ] = R^T = [L\|0]
\]

\[
GY = L
\]

where \(L\) is \(m \times m\) lower triangular portion of \(R^T\). Therefore, Eq. (44) can be solved for \(u_c\) through forward substitution. The null-space component of the displacement \(u_c\) is determined by solving:

\[
(Z^T K Z) u_c = Z^T (f - K Y u_t)
\]

The projected matrix \(Z^T K Z\) is invertible even if \(K\) is not. The major cost of the null-space method lies in the \(QR\) factorization, which requires \(nm^2\) flops. The total cost is \(nm^2 + \frac{1}{6} (n-m)^3 + \frac{3}{2} (n-m)(m+1)\).

The tradeoff between the range-space method and the null-space method depends on the dimensions of \(Z^T K Z\) and the fore mentioned \((GK^{-1}G^T)\) matrices. Clearly, for \(m \ll n\) the range-space method is preferred. Thus, using a range space...
method, as is used in solution of quadratic programming
problems [22], advantage can be taken of the nature of the $K$
matrix (positive definite and banded) as well as the
small number of constraints when compared to the size of
$K$, thereby considerably reducing the floating point operations
[23]. This makes the methodology comparable, if not as
efficient, in computational expenditure to the finite element
method.

3. Validation problems

The power of the proposed methodology is demon-
strated through the basic Boolean operations. A powerful
general purpose JAVA language implementation of the
proposed methodology [24] was used to obtain the
solutions. The chosen validation problems were, in all
cases, problems of stress concentration with known
analytical solutions. These problems, due to the gradients
in stress, are ideal (and challenging) candidates for the
evaluation of accuracy and efficiency of the proposed
procedure. In the absence of overlap between the
primitive domains, the union of primitives is similar in
concept to an analysis with a decomposed domain. This
allows a direct comparison of the NURBS-based
discretization used in the present paper against the
Lagrangian interpolation of the classical finite element
analysis (FEA). Therefore, we solve a particle inclusion
problem with these characteristics using the ANSYS
finite element software. In the comparisons, the
computational efficiency was judged by the number of
uniformly distributed nodes or control points (without
any special arrangement that would enhance accuracy in
the stress concentration problems) that yielded a solution
with 5% or less error on the known analytical solution. It
should be noted that during the course of this research
we have found that NURBS can reproduce the fields
accurately with far fewer degrees of freedom when
augmented with procedures such as adaptive integration.
However, these issues are still under investigation.

3.1. Union

Two problems were solved to demonstrate the union
procedure. The definition of the overlapping domain
problem is shown in Fig. 4. The resultant geometry
obtained through the Boolean operations on the quarter
elliptical primitive and the primitive with an elliptical
hole is a plane stress problem with uniform stress.
Although this appears to be a trivial problem, the
solution enables one to observe the variations in the
$s_{22}$ component of the stress which should be uniform
everywhere to the limit of integration precision. The
solution procedure was validated and a uniform stress
distribution was indeed obtained; the maximum error in
$s_{22}$ was less than 0.2%. It should be noted that in
general the CSA procedure allows different degrees of
approximation for the NURBS entities used to represent
the fields over the primitives. Thus, in the above union
problem, the surface used to model the quarter elliptical
primitive was of degree 2, while the surface used to model the plate was of degree 3.

A bimaterial problem involving a circular inclusion in a finite width plate was next solved. The problem definition is shown in Fig. 5. As discussed earlier, in order to facilitate a comparison with the finite element method, this union problem was implemented using a non-overlapping domain formulation. For a Young’s modulus ratio $E_{\text{incl}}/E_{\text{matrix}} = 0.5$, the analytical stress concentration factor is 1.5 [25]. To ensure a valid comparison with the finite element method, 8-noded quadrilateral elements were used in FEA, while in CSA, surfaces of degree 2 were used to model the primitives. Mapped meshing was used in FEA and in the CSA, the control points were placed at locations near the finite element nodal locations. The distribution of control points in CSA, the distribution of the elements in FEA, and the stress distribution are shown in Fig. 6. The +s are the control points in the inclusion and the stars are the control points in the matrix. The number of control points/nodes required to achieve a comparable error of 5% for the CSA and the FEA were 88 and 478, respectively. The stress concentration factors obtained using the two procedures were 1.45 and 1.43, respectively.

3.2. Subtraction

An illustration of the Boolean subtraction operation is shown in Fig. 7. The problem is to determine the stress concentration due to an elliptical hole (ratio 2:1) in a finite width plate of dimensions 20 × 10 units. A uniform traction of unit value was applied to the top edge of the plate. The material properties used were $E = 100$ and $\nu = 0.3$. The maximum stress determined through CSA was 5.85, which is in good agreement with the analytical stress concentration factor of 5.9 for an elliptical hole (ratio 2:1) in a finite width plate [26]. A total of 421 control points were required to achieve the above-mentioned accuracy. The distribution of the control points is shown in Fig. 8. The resulting $\sigma_{22}$ stress contours are shown in Fig. 9.

3.3. Shape optimization

As was mentioned in Section 1, one motivation for the development of the CSA procedure is to enable efficient re-analysis in problems such as shape optimal...
design where the geometry is iteratively changed. Here, we illustrate this advantage using the problem of the elliptical hole in a finite width plate considered in Section 3.2. The example problem is defined in Fig. 10; the goal is to find the orientation $\theta$ of the major axis of the ellipse with respect to the positive $x$-axis (keeping the size of the ellipse fixed) that minimizes the maximum Von–Mises stress in the plate. The chosen material

![Fig. 8. The distribution of control points used in the solutions of the subtraction problem and the shape optimization problem.](image)

![Fig. 10. The definition of the shape optimal design problem. The goal is to determine the optimal orientation of the elliptical hole that yields the least Von–Mises stress in the plate.](image)

![Fig. 9. $\sigma_{22}$ distribution obtained for the subtraction problem.](image)

![Fig. 11. Von–Mises stress distribution obtained for the shape optimal design problem in (a) initial state and (b) optimized state.](image)
properties and the sizes of the plate and the hole were identical to those used in the previous subsection. An identical distribution of control points was also used. The initial value of \( \theta \) was 0 radians corresponding to the subtraction problem discussed earlier. The optimal value was determined as \( \frac{\pi}{2} \) in one iteration. The Von–Mises stress distribution in the final configuration is shown in Fig. 11. The maximum Von–Mises stress value was 2.45 in the final state.

4. Concluding remarks

In the present paper, we proposed a hierarchical analysis procedure that is analogous to geometry creation using CSG. The approach is novel and useful for the following reasons:

1. The analysis is truly integrated with the design phase since the analysis is procedurally similar to the geometry construction, and therefore, arguably, the changes in primitive shapes are more ‘naturally’ handled during the analysis phase.
2. Since the analysis is carried out on the fields defined over the primitives, modifications to the final geometry do not require remeshing of the final geometry. This considerably reduces the computational cost of reanalysis. This is in general true of a large class of problems including fracture analysis and large deformation problems where significant geometry changes (caused by crack extension or large deformation) between sequential analyses steps necessitate remeshing. The procedure developed in the present paper is being applied to these class of problems as well [27].
3. The developed approach enables one to use the same mathematical representation for analysis fields as is used to represent the geometry. As mentioned in the paper, arguably, NURBS basis have advantages over other basis proposed for meshless analysis. Even though a NURBS basis was used to represent the fields, the CSA procedure by itself is not dependent on its use.

Our current research is aimed at applying the developed method to solve a variety of optimal design problems involving changing topologies and shapes [27]. The results from this research will be reported shortly.

Acknowledgements

The authors are thankful for partial support for this project from the National Science Foundation under grant ECS-9734349.

References

Devendra Natekar received his BE degree in Mechanical Engineering (1996) from the University of Pune, India and MS and PhD (2002) degrees in Mechanical Engineering from University of Colorado, Boulder. His research is primarily in the field of Engineering Mechanics. He is currently employed at Intel Corporation.

Xuefeng Zhang is a doctoral student in Mechanical Engineering at Purdue University. His received his BS degree in Mechanical and Electrical Engineering (1997), ME degree in Precision Instrumentation (2000) from the University of Science and Technology of China, and MS in Mechanical Engineering from University of Colorado at Boulder (2002). His research interests are in Computer Aided Design and Computational Mechanics in Optimal Design.

Ganesh Subbarayan is an Associate Professor of Mechanical Engineering at Purdue University. Prior to joining Purdue University, he was a member of the faculty of Mechanical Engineering at University of Colorado (1/94–9/02) and a development staff member at IBM Corporation (10/90–11/93). His education includes a BTech degree in Mechanical Engineering (1985) from the Indian Institute of Technology, and MS and PhD degrees in Mechanical Engineering from Cornell University, completing the last degree in 1990. Dr Subbarayan’s core research interests are in Computational Mechanics and in Optimal Design, with applications to Micro/nano systems Integration and Color Management. Dr Subbarayan is a recipient of the NSF CAREER award, the NSF Research Initiation Award, the 2003 Charles E. Ives Outstanding Paper Award from the Journal of Imaging Science and Technology, the 2002 Highly Commended Award from Soldering and Surface Mount Technology journal, the 2000 Best Paper Award, the 1996 Peter A. Engel Best Paper Award from ASME Journal of Electronic Packaging, and an IBM Invention Achievement Award. He has served on the program committees of several conferences including the ASME/Pacific Rim International Intersociety Conferences on Electronics Packaging (1997 program co-chair, 1999 reliability track chair, 2001 Modeling and Characterization track chair), ASME International Mechanical Engineering Conference and Exhibition (2002 Program Chair, EPP Division) and Intersociety Conference on Thermal and Thermomechanical Phenomenon in Electronic Systems (2002 program co-chair, Mechanics). Prior to joining the editorial board of IEEE Transactions on Advanced Packaging as the editor-in-chief, he served as an Associate Editor of IEEE Transactions on Components, Packaging and Manufacturing Technology. He is currently a member of ASME, IEEE and IS&T.