AN AUTO-REGRESSIVE MODEL FOR SIMULATION OF TIME-VARYING RAIN RATE

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ABSTRACT
An autoregressive (AR) model for computer generation of time-varying lognormally distributed rainfall rate is presented. The AR coefficients are derived from rain rate statistics and autocorrelation function obtained from measurement. Results from computer simulations are found to fit well with rain rate data from Barcelona.

INTRODUCTION
Recent advances in broadband multimedia services call for a shift of carrier frequencies towards those in the lower tens of gigahertz band, either for terrestrial [1] or earth-satellite applications [2]. Radio link performance in this spectral range is highly influenced by rain attenuation. Some rain fade mitigation techniques that work adaptively with respect to the time-varying rain fade, such as power control or data-link layer protocol, have been proposed and tested on real radio links with various results.

Performance assessment of newly-proposed adaptive fade compensation techniques or other schemes that involve adaptive elements for communications through rain, such as adaptive modulation, adaptive coding, or cell-site diversity on multiple links using scanned combining, require fade measurements from radio links, sometimes a multiplicity thereof, which are not always available to the researcher. In such situations, therefore, an algorithm to generate a time series of rain rate and/or attenuation for simulation on computers is necessary.

This paper presents a work on modelling the time variation of rainfall rate as an AR process. This is achieved by assuming that rain rate in logarithmic scale is normally distributed with known statistics and autocorrelation function. Since the same type of distribution is often used to describe rain attenuation, the same procedure can also be applied to simulation of rain attenuation.

AUTO-REGRESSIVE MODEL OF RAIN RATE
The underlying assumptions are described below. Firstly, point rain rate $r$ (mm/hr) is stationary and lognormally distributed (as reported in [3], [4] and many others) and hence $\eta = \ln r$ (the natural logarithm of rain rate) is normally distributed with known parameters, i.e. the median rain rate $r_m$ (or equivalently the mean $\mu_\eta$ of $\eta$) and the standard deviation $\sigma_\eta$ of $\eta$. When complete information of measured exceedance probabilities of rain rate is available, these two parameters can be derived from fitting a theoretical normal distribution to the measured statistics.

Secondly, time autocorrelation function of point rain rate is known. Given the normalized autocorrelation function $\phi_\eta(\tau)$ of the lognormally distributed rain rate $r$, where $\tau$ is time lag, the autocovariance function $\phi_a(\tau)$ for $\eta$ can be obtained.

The proposed solution is by synthesizing an AR process that behaves like a time series of $\eta$, a procedure commonly used to generate time-varying Rayleigh fading channels in simulations of wireless communication systems (see e.g. [5]). A sequence of zero-mean normally distributed $\eta(k) = \eta(\tau)$, where $k$ are integers and $\tau$ sampling time, is generated recursively following

$$\eta_0(k) = -\sum_{n=1}^{M} a_n \eta_0(k-n) + c_g(k)$$

where $a_n$’s with $n = 1, ..., M$ are the AR coefficients, $M$ the order of the process that depends on the maximum time lag of the autocorrelation function, $g(k)$ a computer-generated sequence of independent zero-mean unit-variance Gaussian random numbers computers, and $c$ a factor denoting the standard deviation of white noise sequence $c_g(k)$. Once sequence $\eta_0(k)$ having the desired number of samples is obtained, the rain rate sequence $r(k)$ can be obtained

$$r(k) = \exp(\eta_0(k) + \mu_\eta)$$

DERIVATION OF AR COEFFICIENTS
The AR coefficients are derived by solving the Yule-Walker equations [6]. For a real-valued process, the solution in matrix form is

$$a = -\Phi^{-1} \phi$$

where $a = [a_1, a_2, ..., a_M]^T$, $\phi = [\phi_\eta(1), \phi_\eta(2), ..., \phi_\eta(M)]^T$,

$$\Phi = \begin{bmatrix}
\phi_\eta(0) & \phi_\eta(1) & \cdots & \phi_\eta(M-1) \\
\phi_\eta(1) & \phi_\eta(0) & \cdots & \phi_\eta(M-2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_\eta(M-1) & \phi_\eta(M-2) & \cdots & \phi_\eta(0)
\end{bmatrix}$$

$$\Phi = \begin{bmatrix}
\phi_\eta(0) & \phi_\eta(1) & \cdots & \phi_\eta(M-1) \\
\phi_\eta(1) & \phi_\eta(0) & \cdots & \phi_\eta(M-2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_\eta(M-1) & \phi_\eta(M-2) & \cdots & \phi_\eta(0)
\end{bmatrix}$$
\( \phi_n(n) = \phi_n(n \tau) \) is the autocovariance function of \( \eta(k) \) (equivalently, the autocorrelation function of \( \eta(k) \)) at time lags integer multiples of sampling time \( \tau \). Factor \( c \) is obtained through

\[
c = \sqrt{\sum_{n=1}^{M} a_n \phi_n(n)}
\]

where \( a_0 = 1 \). So as for the AR process to be stable, the roots of its characteristic equation, i.e., \( 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_M z^{-M} = 0 \) must lie inside the unit circle in the \( z \)-plane [6].

**AUTOCOVARIANCE FUNCTION OF \( \eta \)**

By fitting exceedance probabilities of \( \eta \) to the theoretical normal distribution, \( \mu_\eta \) and \( \sigma_\eta \) can be obtained. The mean \( \mu_r \) and variance \( \sigma_r^2 \) of \( r \) are then

\[
\sigma_r^2 = \exp\left(2\mu_\eta + \sigma_\eta^2\right)\exp\left(\sigma_\eta^2\right) - 1
\]

\[
\mu_r = \exp\left(\mu_\eta + \frac{\sigma_\eta^2}{2}\right)
\]

When the normalized autocorrelation function \( \rho_r(n) \) of rain rate is known, the normalized autocovariance function of \( \phi_r(n) \) can be computed as

\[
\phi_r(n) = \left(\frac{\sigma_r^2 + \mu_r^2}{\sigma_r^2}\right)\rho_r(n) - \mu_r^2
\]

and the autocovariance function \( \phi_\eta(n) \) of \( \eta \) required in (3)–(5) is

\[
\phi_\eta(n) = \ln\left[1 + \phi_r(n)\left(\exp\left(\sigma_\eta^2\right) - 1\right)\right].
\]

**SIMULATION RESULTS**

The AR model is verified against statistics of real data presented in a series of publications authored by Vilar \textit{et al}, which reports detailed statistical analysis of rain rate and duration of rain rate exceedance in Barcelona, Spain over a period of 49 years ([4], [8], [9]). For this verification purpose, as many as 30 time sequences of rain rates, each comprising 100,000 samples with one-minute intervals, are generated on computer using the Barcelona rain rate statistics and normalized autocorrelation function. The order of the AR process is chosen to be 40, making maximum use of information available from the Barcelona data. Each of the 30 resulting sequences is subsequently used to compute the complementary cumulative distribution (CCDF) and the autocorrelation function of rain rate as well as the CCDF of rain rate exceedance duration. These are then averaged over the thirty sequences.

The CCDF of rain rate measured in Barcelona and the lognormal CCDF obtained from simulation are compared in Fig. 1. The two curves lie close to each other, with largest errors occurring in the range of time percentages lower than 0.05%, equivalent to rain rates higher than approximately 160 mm/hr.

![Fig. 1 CCDF of rain rate measured in Barcelona and the simulation result.](image1)

The average autocorrelation function resulting from the simulation is found very close to the one derived from measurement for all time lags. This comparison is depicted in Fig. 2.

![Fig. 2 Normalized autocorrelation function of rain rate measured in Barcelona and the simulation result.](image2)

Finally, comparison between the measurement and simulation results is made for the CCDF’s of rain rate exceedance duration for various thresholds, as depicted in Figs. 3 – 6. It can be seen that for higher rain rate thresholds, i.e., 80 mm/hr or higher, the difference between the measurement and simulation results are larger. Given a rain rate threshold, the simulation tends to produce a higher probability of exceedance for the same duration. For instance, when a threshold of 120 mm/hr is considered,
the probability of having the threshold exceeded for 10 minutes is 0.2% for the measurement result and approximately 1% according to the simulation. Equivalently, for the same probability level, the event of rain rate exceeding the threshold tends to last longer in the simulation. However, this observation does not hold true for thresholds lower than 80 mm/hr, as shown by the proximity between the pairs of curves in Figs. 3 and 4 for thresholds of 20 and 40 mm/hr.

DISCUSSIONS AND CONCLUSIONS
Comparisons made between the results of measurement and simulation in the previous section demonstrate that the AR model of rain rate is potentially useful for assessing performance of adaptive techniques operating under rainy conditions. However, the unfortunate absence of statistics of rain rate slope when exceeding the threshold from Barcelona data prevents the comparative study from being complete. Otherwise, the result can further verify whether or not the AR model is equally useful for simulation of rain attenuation, for which statistics of both fade duration and slope are of importance. The previous statement is based on the fact that the same measurement-based statistical models used for rain rate have usually been applied to rain-induced attenuation for frequencies higher than 10 GHz (see, e.g. [4], [7]). Whenever complete statistics of rain fade duration and slope are available, the fitness of the AR model for rain attenuation can be tested.

Fig. 3 CCDF of rain rate exceedance duration for a threshold of 20 mm/hr.

Fig. 4 CCDF of rain rate exceedance duration for a threshold of 40 mm/hr.

Fig. 5 CCDF of rain rate exceedance duration for a threshold of 80 mm/hr.

Fig. 6 CCDF of rain rate exceedance duration for a threshold of 120 mm/hr.
It should be noted, however, that despite the above discussed potential of the AR model, it should never be used to simulate rain events for the purpose of examining yearly or monthly statistics since in its current form the AR model parameters are derived based only on long-term average statistics of rain rate. The model also does not portray changes between the clear-sky and rainy states since it is not devised to consider the presence of these different states.

REFERENCES