Embeddings of Simple Maximum Packings of triples with $\lambda$ Even

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Received 8 July 1992; revised 30 November 1993

Abstract

Let $\text{MPT}(v, \lambda)$ denote a maximum packing of triples of order $v$ and index $\lambda$. An $\text{MPT}(v, \lambda)$ is called simple if it contains no repeated triples. It is proved in this paper that for $v > 2$ and any even $\lambda$, the necessary and sufficient condition for the embedding of a simple $\text{MPT}(v, \lambda)$ in a simple $\text{MPT}(u, \lambda)$ is $u \geq 2v + 1$.

1. Introduction

A maximum packing of triples of order $v$ and index $\lambda$, denoted $\text{MPT}(v, \lambda)$, is a pair $(V, B)$ where $V$ is a $v$-set and $B$ is a collection of $3$-subsets (called blocks or triples) of $V$ such that: (i) each $2$-subset of $V$ is contained in at most $\lambda$ triples of $B$, (ii) if $C$ is any collection of $3$-subsets of $V$ satisfying (i), then $|B| > |C|$. An $\text{MPT}(v, \lambda)$ is called simple if it contains no repeated triples.

Let $(V, B)$ be an $\text{MPT}(v, \lambda)$, the leave of $(V, B)$ is a multigraph $(V, E)$ where an edge $\{x, y\} \in E$ with multiplicity $m$ if and only if the corresponding $2$-subset $\{x, y\}$ is contained in exactly $\lambda - m$ triples of $B$.

It is well known [1] that the leave of an $\text{MPT}(v, \lambda)$ is empty if and only if $\lambda(v - 1) \equiv 0$ (mod 2) and $\lambda v(v - 1) \equiv 0$ (mod 6); in this case, the $\text{MPT}(v, \lambda)$ is called a triple system and denoted $\text{TS}(v, \lambda)$.

The following lemma can be found in [4].

**Lemma 1.1.** If $\lambda \equiv 0$ (mod 6) and $v \neq 2$, or $\lambda \equiv 2$ or $4$ (mod 6) and $v \equiv 0$ or $1$ (mod 3), then the leave of an $\text{MPT}(v, \lambda)$ is empty. If $\lambda \equiv 2$ (mod 6) and $v \equiv 2$ (mod 3), then the

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leave of an MPT \((v, \lambda)\) is a double edge \(\{a, b\}, \{a, b\}\). If \(\lambda \equiv 4 \pmod{6}\) and \(v \equiv 2 \pmod{3}\), then the leave of an MPT \((v, \lambda)\) is one of the following graphs:

Type 1: A quadruple edge \(\{a, b\}, \{a, b\}, \{a, b\}, \{a, b\}\).
Type 2: Two disjoint double edges \(\{a, b\}, \{a, b\}, \{c, d\}, \{c, d\}\).
Type 3: Two joint double edges \(\{a, b\}, \{a, b\}, \{b, c\}, \{b, c\}\).
Type 4: A quadrilateral \(\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\).

The following lemma can be easily proved by Lemma 1.1 and elementary counting.

**Lemma 1.2.** If \(v > 2\), \(\lambda \equiv 0 \pmod{2}\) and a simple MPT \((v, \lambda)\) exists, then \(\lambda \leq v - 2\).

Now let \((X, A)\) be an MPT \((v, \lambda)\) and \((Y, B)\) be an MPT \((u, \lambda)\). If \(X\) is a proper subset of \(Y\) and \(A\) is a subcollection of \(B\), then \((X, A)\) is said to be embedded in \((Y, B)\). The following lemma can also be proved by Lemma 1.1 and simple counting.

**Lemma 1.3.** If \(v > 2\) and \(\lambda \equiv 0 \pmod{2}\), then a necessary condition for the embedding of an MPT \((v, \lambda)\) in an MPT \((u, \lambda)\) is \(u \geq 2v + 1\).

The concept of embeddings of maximum packings of triples is a generalization of the concept of embeddings of triple systems [2, 6]. The embedding problem for simple triple systems was completely settled in a recent paper [5] by the following theorem.

**Theorem 1.1.** For any given \(\lambda\), the necessary and sufficient conditions for the embedding of a simple TS \((v, \lambda)\) in a simple TS \((u, \lambda)\) are \(\lambda(u - 1) \equiv 0 \pmod{2}\), \(\lambda u(u - 1) \equiv 0 \pmod{6}\), \(u \geq 2v + 1\).

Embeddings of MPT \((v, 1)\) was discussed and partial results were obtained in [3]. In this paper we study the embedding problem for simple maximum packings of triples and give a complete solution for all even \(\lambda\).

2. Basic construction techniques

Let \(X\) be a finite set containing \(n\) elements, \(n \geq 2\). A collection \(P\) of 2-subsets (called pairs) of \(X\) is called a partial \(\lambda\)-factor if each element of \(X\) is contained in at most \(\lambda\) pairs of \(P\), and \(P\) is called a \(\lambda\)-factor if each element of \(X\) is contained in exactly \(\lambda\) pairs of \(P\). A \(\lambda\)-factor or partial \(\lambda\)-factor is called simple if it contains no repeated pairs.

Let \(X = \mathbb{Z}_n\), the multiset \(D(n, \lambda)\) with elements from \(\mathbb{Z}_n\) is defined as follows:

\[
D(n, \lambda) = \begin{cases} 
\{\lambda \ast d: 1 \leq d < n/2\} & \text{if } n \equiv 1 \pmod{2}, \\
\{\lambda \ast d: 1 \leq d < n/2\} \cup \{\frac{1}{2} \ast n/2\} & \text{if } n \equiv 0 \pmod{2}.
\end{cases}
\]
The elements of \( D(n, \lambda) \) are called differences. The notation \( \lambda \cdot d \) indicates that the difference \( d \) appears \( \lambda \) times in \( D(n, \lambda) \). When \( n \equiv 0 \pmod{2} \) and \( d = n/2 \), the notation \( \frac{1}{2} \cdot d \) means that from it we can form a 1-factor \( \{ \{ i, i + n/2 \} : 0 \leq i < n/2 \} \).

Let \( a, b, c \in D(n, \lambda) \), if \( a + b + c \equiv 0 \pmod{n} \) or one is the sum of the others, say \( a + b \equiv c \pmod{n} \), then \( (a, b, c) \) is called a difference triple. The following observations are useful in this paper.

Let \( (a, b, c) \) be a difference triple with \( a + b + c \equiv 0 \pmod{2} \) or \( a + b \equiv c \pmod{n} \). If \( (a, b, c) \) contains no repeated differences, then from \( (a, b, c) \) we can form a base block \( \{0, a, a + b\} \) or \( \{0, b, a + b\} \) and their orbits \( \{ \{ i, a + i, a + b + i \} : i \in \mathbb{Z}_n \} \) and \( \{ \{ i, b + i, a + b + i \} : i \in \mathbb{Z}_n \} \) are disjoint and each contains \( n \) distinct triples. If \( (a, b, c) \) contains a two times repeated difference \( a = b \not\equiv c \), then we can form a base block \( \{0, a, 2a\} \) and its orbit \( \{ \{ i, a + i, 2a + i \} : i \in \mathbb{Z}_n \} \) contains \( n \) distinct triples. If \( n \equiv 0 \pmod{3} \), \( a = b = c = n/3 \), then we can form a base block \( \{0, n/3, 2n/3\} \) and its orbit contains only \( n/3 \) distinct triples, so the difference \( n/3 \) is used once. For any \( d \in D(n, \lambda) \), if \( 2d \not\equiv 0 \pmod{2} \), then we can form a single 2-factor \( \{ \{ i, d + i \} : i \in \mathbb{Z}_n \} \), if \( n \equiv 0 \pmod{2} \) and \( d = n/2 \), the we can form two 1-factors \( F_1 = F_2 = \{ \{ i, i + n/2 \} : 0 \leq i < n/2 \} \), and in this case, from \( n/2 \) and any odd difference \( d \not\equiv 2 \), we can always form two simple 2-factors. It is also worth remarking that orbits obtained from distinct difference triples are disjoint and 2-factors obtained from distinct differences are disjoint.

The following lemma gives an example of the applications of the difference method to our embedding problem in the simplest case.

**Lemma 2.1.** Let \( \lambda \equiv 0 \pmod{2} \), \( u \equiv v \equiv 2 \pmod{3} \), \( v > 2 \), \( u \geq 2v + 1 \) and a simple MPT\((v, \lambda)\) exist. If we can select \( \lambda(u - 2v - 1)/2 \) differences from \( D(u - v, \lambda) \) to form \( \lambda(u - 2v - 1)/6 \) difference triples such that, (i) no difference triple \((u - v)/3, (u - v)/3, (u - v)/3\) appears, (ii) each difference triple without repeated differences appears at most twice, (iii) each difference triple with a twice repeated difference appears no more than once, and (iv) in the remaining \( \lambda v/2 \) differences, when \( u - v \equiv 0 \pmod{2} \), the number of odd differences is at least the same as the multiplicity of the difference \((u - v)/2\), then any MPT\((v, \lambda)\) can be embedded in a simple MPT\((u, \lambda)\).

**Proof.** Let \( X = \{ \infty, 1, 2, \ldots, \infty \} \) and \((X, A)\) be a simple MPT\((v, \lambda)\). Let \((a, b, c)\) be a difference triple with \( a + b + c \equiv 0 \pmod{u - v} \) or \( a + b \equiv c \pmod{u - v} \). If \((a, b, c)\) appears only once, then form a base block \( \{0, a, a + b\} \). If \((a, b, c)\) appears twice, then it contains no repeated differences by (ii), we form two base blocks \( \{0, a, a + b\} \) and \( \{0, b, a + b\} \). By (i)–(iii), all the triples obtained from these base blocks are distinct. Let \( B_0 \) be the set of these triples. Now since \( v > 0 \) and a simple MPT\((v, \lambda)\) exists, then \( \lambda \leq v - 2 \) by Lemma 1.2, and so in the remaining \( \lambda v/2 \) differences, each difference appears at most \( v - 2 \) times, and when \( u - v \equiv 0 \pmod{2} \), the difference \((u - v)/2\) appears at most \((v - 2)/2\) times. Thus (and by (iv) if \( u - v \equiv 0 \pmod{2} \)) we can always form \( v \) simple \( \lambda \)-factors \( F_1, F_2, \ldots, F_v \) from these \( \lambda v/2 \) differences. Let \( Y = X \cup \mathbb{Z}_n \),
Lemma 2.2. If \( n \equiv 1 \pmod{3} \), then from the difference triple \( (1, 1, 2) \) we can form (i) \( 2(n-1)/3 \) distinct triples, a simple 2-factor and a double edge, or (ii) \( (n-4)/3 \) distinct triples, two simple 2-factors and two double edges.

Proof. (i) The double edge is \( \{0, n-1\} \), the triples are \( T_1 = \{3i+1, 3i+2, 3i+3\}; \ 0 \leq i \leq (2n-5)/3 \), and the simple 2-factor is \( F_1 = \{3i+1, 3i+2\}, \ \{3i+1, 3i+3\}, \ \{3i+2, 3i+3\}; \ 2(n-1)/3 \leq i \leq n-1 \}\ \{\{0, n-1\}, \{0, n-1\}\}. \\
(ii) The double edges are \( \{0, n-1\} \) and \( \{1, 2\} \), the triples are \( T_2 = \{3i+1, 3i+2, 3i+3\}; \ (n+2)/3 \leq i \leq (2n-5)/3 \), and the simple 2-factors are \( F_1 \) and \( F_2 = \{3i+1, 3i+2\}, \ \{3i+1, 3i+3\}, \ \{3i+2, 3i+3\}; \ 0 \leq i \leq (n-1)/3 \}\ \{\{1, 2\}, \{1, 2\}\}. \\

Lemma 2.3. If \( n \equiv 2 \pmod{3} \) and \( d = 1 \) or \( 3 \), then from the difference triple \( (d, d, 2d) \) we can form \( (n-2)/3 \) distinct triples, two simple 2-factors and a double edge.

Proof. The double edge is \( \{0, d\} \), the triples are \( T = \{3id, (3i+1)d, (3i+2)d\}; \ 2(n+1)/3 \leq i \leq n-1 \), and the simple 2-factors are \( F_1 = \{id, (1+i)d\}; \ 1 \leq i \leq n-1, i \neq 2 \}\ \{\{0, 2d\}, \{d, 3d\}\}, \ F_2 = \{3id, (3i+2)d\}, \ \{3i+1)d, (3i+3)d\}, \ \{3i+1)d, (3i+2)d\}; \ 1 \leq i \leq (n-2)/3 \}\ \{\{d, 2d\}, \{2d, 3d\}\}. \\

Lemma 2.4. If \( n \equiv 1 \pmod{3} \), \( d = 1 \) or \( 3 \), then from the difference triple \( (d, d, 2d) \) we can form \( (n+2)/3 \) distinct triples and two simple partial 2-factors each missing an edge \( \{ad, (a+1)d\} \) where \( a \in \mathbb{Z}_n \).

Proof. The triples are \( T = \{(a+3i)d, (a+3i+1)d, (a+3i+2)d\}; \ 0 \leq i \leq (n-1)/3 \}, \) and the simple partial 2-factors are \( F_1 = \{(a+3i)d, (a+3i+1)d\}; \ (n+2)/3 \leq i \leq n-1 \}\ \{(a+3i+1)d, (a+3i+2)d\}; \ (n+2)/3 \leq i \leq 2(n-1)/3 \}, \ F_2 = \{(a+3i)d, (a+3i+2)d\}; \ (n+2)/3 \leq i \leq n-1 \}\ \{(a+3i+1)d, (a+3i+2)d\}; \ (2n+1)/3 \leq i \leq n-1 \}. \\

Lemma 2.5. If \( n \equiv 2 \pmod{3} \), \( a \in \mathbb{Z}_n \) and \( d = 1 \) or \( 3 \), then from the difference triple \( (d, d, 2d) \) and another difference \( 2d \), we can form \( 2(n+1)/3 \) distinct triples and two simple partial 2-factors each missing an edge \( \{ad, (a+1)d\} \).
Proof. The triples are $T = \{ (a + 3i)d, (a + 3i + 1)d, (a + 3i + 2)d \}: 0 \leq i \leq (2n - 1)/3$, and the simple partial 2-factors are

\[
F_1 = \{ (a + 3i + 2)d, (a + 3i + 3)d \} \cup \{ (a + 3i + 4)d \} \cap \{ (a + 3i + 3)d, (a + 3i + 4)d \}: 0 \leq i \leq (n - 5)/3
\]

Lemma 2.6. If \( n \equiv 1 \pmod{3} \), then from the difference triple \((1, 1, 2)\) and another difference 2, we can form \((2n + 4)/3\) distinct triples and two simple partial 2-factors each missing four points 1–4.

Proof. The triples are $T = \{ 3i + 1, 3i + 2, 3i + 3 \}: 0 \leq i \leq (2n + 1)/3$, and the simple partial 2-factors are

\[
F_1 = \{ 3i + 1, 3i + 2 \}, \{ 3i + 1, 3i + 3 \}, \{ 3i + 2, 3i + 3 \} : (2n + 1)/3 \leq i \leq n - 1
\]

Lemma 2.7. If \( n \equiv 2 \pmod{3} \) then from the difference triple \((1, 1, 2)\) and another difference 2, we can form \((n + 4)/3\) distinct triples, a simple 2-factor and two simple partial 2-factors each missing four points 1–4.

Proof. The triples are $T = \{ 3i + 1, 3i + 2, 3i + 3 \}: 0 \leq i \leq (n + 1)/3$. The simple 2-factor is

\[
F_0 = \{ 3i + 2, 3i + 4 \}: 1 \leq i \leq (n - 2)/3 \cup \{ 3i + 2, 3i + 3 \}
\]

3. Main result

With the preparations we have made in Section 2, we are now in a position to prove our main theorem.
**Theorem 3.1.** If \( v > 2 \) and \( \lambda \equiv 0 \pmod{2} \), then the necessary condition \( u \geq 2v + 1 \) for the embedding of a simple MPT\((v, \lambda)\) in a simple MPT\((u, \lambda)\) is also sufficient.

We remark that to prove the theorem, it is sufficient to consider the orders \( u \) and \( v \) satisfying \( 2v + 1 \leq u \leq 4v + 2 \). We also note that, since a simple MPT\((v, \lambda)\) exists, then, by Lemma 1.2, we have \( \lambda \leq v - 2 \).

For \( \lambda \equiv 0 \pmod{6} \) and all \( v \neq 2 \), or for \( \lambda \equiv 2 \) or \( 4 \pmod{6} \) and \( v \equiv 0 \) or \( 1 \pmod{3} \), any MPT\((v, \lambda)\) is in fact a TS\((v, \lambda)\), as a consequence of Theorem 1.1, we then have the following result.

**Lemma 3.1 (Shen \[5\]).** If \( \lambda \equiv 0 \pmod{6} \) and \( v \neq 2 \), or \( \lambda \equiv 2 \) or \( 4 \pmod{6} \) and \( u, v \equiv 0 \) or \( 1 \pmod{3} \), \( u \geq 2v + 1 \), then any simple MPT\((v, \lambda)\) can be embedded in a simple MPT\((u, \lambda)\).

Thus, to prove our main theorem, we need only to consider the cases when one of \( u \) and \( v \) is \( 2 \pmod{3} \) and \( \lambda \equiv 2 \) or \( 4 \pmod{6} \). The simplest case is \( u \equiv v \equiv 2 \pmod{3} \).

**Lemma 3.2.** If \( \lambda \equiv 2 \) or \( 4 \pmod{6} \), \( u \equiv v \equiv 2 \pmod{3} \), \( u \geq 2v + 1 \), then any simple MPT\((v, \lambda)\) can be embedded in a simple MPT\((u, \lambda)\).

**Proof.** We prove the lemma by partitioning \( D(u - v, \lambda) \) into difference triples and differences satisfying the conditions of Lemma 2.1.

If \( u - v \equiv 0 \pmod{6} \), let \( u - v = 6s \), since \( 2v + 1 \leq u \leq 4v + 2 \) and \( \lambda = 2\lambda_0 \leq v - 2 \) then \( 2s \leq v \leq 6s \) and \( \lambda_0 \leq (v - 2)/2 \). For \( 1 \leq k \leq 2s - 1 \), select \( 3(2ks - 1 - k(k - 1)/2) \) differences of \( D(u - v, 2k) \) to form the following collection \( T(k) \) of difference triples:

\[
(2i, 3s - i, 3s - i), \quad 1 \leq i \leq s - 1,
\]
\[
(2i - j, 3s - i, 3s + j - i), \quad 1 \leq j \leq k - 1, \quad \lfloor 1 + j/2 \rfloor \leq i \leq s,
\]
\[
(2i - j, s + i, s + j - i), \quad 1 \leq j \leq k, \quad \lfloor 1 + j/2 \rfloor \leq i \leq s.
\]

If \( u - v \equiv 3 \pmod{6} \), let \( u - v = 6s + 3 \), then \( 2s + 1 \leq v \leq 6s + 2 \), \( \lambda = \lambda_0 \leq v - 2 \). For \( 1 \leq k \leq 2s - 1 \), let \( T(k) \) be the collection of the following difference triples:

\[
(2i - j, 3s - i + 1, 3s + j - i + 2), \quad 0 \leq j \leq k - 1, \quad \lfloor 1 + j/2 \rfloor \leq i \leq s,
\]
\[
(2i - j, s + i, s + j - i), \quad 1 \leq j \leq k, \quad \lfloor 1 + j/2 \rfloor \leq i \leq s.
\]

We take all the difference triples of \( T(\lambda_0) \) if \( \lambda_0 < 2s - 1 \) and take all the difference triples of \( T(2s - 1) \) if \( \lambda_0 \geq 2s - 1 \). Let \( n \) be the number of the remaining differences of \( D(u - v, \lambda) \), then \( n \equiv \lambda v/2 \pmod{3} \) and \( n \leq \lambda v/2 \). Decompose \( (\lambda v/2 - n)/3 \) difference triples into differences, then we have exactly \( \lambda v/2 \) differences. It can be checked that all the conditions (i)–(iv) of Lemma 2.1 are satisfied. The conclusion then follows from Lemma 2.1. \( \square \)
Lemma 3.3. If $\lambda \equiv 2$ or $4 \pmod{6}$, $u \equiv 2 \pmod{3}$, $v \equiv 1 \pmod{3}$ and $u \geq 2v + 1$, then any simple $\text{MPT}(v, \lambda)$ can be embedded in a simple $\text{MPT}(u, \lambda)$.

Proof. If $u - v \equiv 1 \pmod{6}$, let $u - v = 6s + 1$, then $2s \leq v \leq 6s$, $\lambda = 2\lambda_0 \leq v - 2$. For $1 \leq k \leq 2s - 1$, choose differences of $D(6s + 1, 2k)$ to form the following collection $T(k)$ of difference triples:

\[
\begin{align*}
(1, 1, 2), (s, 2s - 1, 3s - 1), \\
(2i, 3s - i, 3s - i + 1), (2i - 1, s + i, s - i + 1) & \quad \text{if } 2 \leq i \leq s - 1, \\
(2i - j, 3s - i, 3s + j - i + 1), (2i - 1, j, s + j - i + 1) & \quad \text{if } 1 \leq j \leq k - 1, \quad \left\lfloor 1 + j/2 \right\rfloor \leq i \leq s, \\
(2i - j, s + i, s + j - i), & \quad 2 \leq j \leq k, \quad \left\lfloor 1 + j/2 \right\rfloor \leq i \leq s.
\end{align*}
\]

If $u - v \equiv 4 \pmod{6}$, let $u - v = 6s + 4$, then $2s + 1 \leq v \leq 6s + 1$ and $\lambda = 2\lambda_0 \leq v - 2$. For $1 \leq k \leq 2s - 1$, choose differences of $D(6s + 4, 2k)$ to form the following collection $T(k)$ of difference triples:

\[
\begin{align*}
(1, 1, 2), (s, 2s, 3s), \\
(2i, 3s - i + 3, 3s - i + 1) & \quad 2 \leq i \leq s, \\
(2i - 1, s + i, s - i + 1), & \quad 2 \leq i \leq s - 1, \\
(2i - j, 3s - i + 3, 3s + j - i + 1), (2i - j, s + i, s + j - i) & \quad 2 \leq j \leq k, \quad \left\lfloor 1 + j/2 \right\rfloor \leq i \leq s.
\end{align*}
\]

It can be checked that in both cases, $T(k)$ satisfies the conditions (i)–(iv) of Lemma 2.1. Take all the difference triples of $T(\lambda_0)$ if $\lambda_0 < 2s - 1$ and take all the difference triples of $T(2s - 1)$ if $\lambda_0 \geq 2s - 1$. Let $n$ be the number of differences of $D(u - v, \lambda)$ not contained in the above difference triples, then $n < \lambda_0 v$ and $\lambda_0 v - n \equiv \lambda_0 \pmod{3}$. By Lemma 2.2, from the difference triple $(1, 1, 2)$, we form a simple $2$-factor and a double edge if $v_0 \equiv 1 \pmod{3}$, or two simple $2$-factors and two double edges, and appropriate number of triples, and then decompose $\left\lfloor (\lambda_0 v - n)/3 \right\rfloor$ difference triples into differences to form exactly $v$ simple $\lambda$-factors. The conclusion then follows. \hfill $\square$

In the proof of the following lemmas, as a first step, we always choose differences of $D(u - v, 2k)$ to form $T(k)$ of difference triples satisfying the conditions (i)–(iv) of Lemma 2.1 and containing the difference triple $(1, 1, 2)$ or $(1, 1, 2)$ and $(3, 3, 6)$. Since the constructions are similar to those of [5], and examples are shown in Lemmas 3.2 and 3.3, we omit the details in the proof of the following lemmas. The interested reader may refer to [5].

Lemma 3.4. If $\lambda \equiv 2$ or $4 \pmod{6}$, $u \equiv 2 \pmod{3}$, $v \equiv 0 \pmod{3}$, $u \geq 2v + 1$, then any simple $\text{MPT}(v, \lambda)$ can be embedded in a simple $\text{MPT}(u, \lambda)$. 
Proof. If \( \lambda \equiv 2 \pmod{6} \), assume \((1, 1, 2) \in T(k)\). By Lemma 2.3, we can form \((2n - 2)/3\) distinct triples, two simple 2-factors and a double edge from \((1,1,2)\). Since \( u - v \equiv 2 \pmod{3} \) and \( v \equiv 0 \pmod{3} \), then the number of remaining differences of \( D(u - v, \lambda) \) is \( n = \lambda_0(u - v - 1) \equiv 1 \pmod{3} \), and so we can decompose \( \{\lambda_0v - (n + 2)\}/3 \) difference triples into differences such that the total number of differences is \( \lambda_0v - 2 \). From these differences and the two simple 1-factors formed from \((1,1,2)\), we can form exactly \( v \) simple \( \lambda \)-factors. The conclusion then follows.

If \( \lambda \equiv 4 \pmod{6} \), assume \((1,1,2), (3,3,6) \in T(k)\). By Lemma 2.3 we can form \(2(n - 2)/3\) distinct triples, four simple 2-factors and two double edges from \((1,1,2)\) and \((3,3,6)\). The conclusion then follows in a similar way.

Lemma 3.5. If \( \lambda \equiv 2 \pmod{6} \), \( u \equiv 0 \) or \( 1 \pmod{3} \), \( v \equiv 2 \pmod{3} \), \( u \geq 2v + 1 \), then any simple \( MPT(v, \lambda) \) can be embedded in a simple \( MPT(u, \lambda) \).

Proof. If \( u \equiv 0 \pmod{3} \), assume \((1,1,2) \in T(k)\). By Lemma 2.4, we can form \((n + 2)/3\) distinct triples and two simple partial 2-factors \( P_1 \) and \( P_2 \) each missing an edge \{0,1\} from the difference triple \((1,1,2)\). If \( u \equiv 1 \pmod{3} \), assume \((1,1,2) \in T(k)\) and at least one difference 2 is not contained in \( T(k) \). By Lemma 2.5 from the difference triple \((1,1,2)\) and the difference 2, we can form \(2(u - v + 1)/3\) distinct triples and two partial 2-factors \( P_1 \) and \( P_2 \) each missing an edge \{0,1\}. Now let \( n \) be the number of the remaining differences of \( D(u - v, \lambda) \), decompose \((\lambda_0v - n - 2)/3\) difference triples into differences, then we have exactly \( \lambda_0v - 2 \) differences and from these differences we can form \( v - 2 \) simple \( \lambda \)-factors \( F_3, F_4, \ldots, F_v \), and two simple \((\lambda - 2)\)-factors \( F_1 \) and \( F_2 \) such that \( P_1 \cap F_1 = \emptyset, P_2 \cap F_2 = \emptyset \). Let \( \{\infty_1, \infty_2, \ldots, \infty_v\} \) be the point set of the simple \( MPT(v, \lambda) \) with the double edge \{\( \infty_1, \infty_2 \)\} as a leave. Form the following triples:

\[
\{\infty_1, \infty_2, 0\}, \{\infty_1, \infty_2, 1\}, \\
\{\infty_1, a, b\} \text{ for each } \{a,b\} \in P_1 \cup F_1, \\
\{\infty_2, a, b\} \text{ for each } \{a,b\} \in P_2 \cup F_2, \\
\{\infty_i, a, b\} \text{ for each } \{a,b\} \in F_i, \quad 3 \leq i \leq v.
\]

This completes the proof.

Lemma 3.6. If \( \lambda \equiv 4 \pmod{6} \), \( u \equiv 0 \pmod{3} \), \( v \equiv 2 \pmod{3} \), \( u \geq 2v + 1 \), the any simple \( MPT(v, \lambda) \) can be embedded in a simple \( MPT(u, \lambda) \).

Proof. Let \( X = \{\infty_1, \infty_2, \ldots, \infty_v\} \) and \((X, A)\) be the simple \( MPT(v, \lambda) \).

Case 1: If the leave of \((X, A)\) is of type 1 where the quadruple edge is \{\( \infty_1, \infty_2 \}\), then we assume \((1,1,2) \in T(k)\) and the difference 2 is contained in at most \( \lambda - 1 \) difference triples of \( T(k) \). By Lemma 2.6, from the difference triple \((1,1,2)\) and the difference 2 we can form \(2(u - v + 2)/3\) distinct triples and two simple partial 2-factors \( P_1 \) and \( P_2 \) each missing four points 1–4. Let \( n \) be the number of the remaining
differences of $D(u - v, \lambda)$. Decompose $(\lambda_0 v - n - 2)/3$ difference triples into differences, then we have exactly $\lambda_0 v - 2$ differences. From these differences we can form $v - 2$ simple $\lambda$-factors $F_3, F_4, \ldots, F_v$ and two simple $(\lambda - 2)$-factors $F_1$ and $F_2$ such that $P_1 \cap F_1 = \emptyset$, $P_2 \cap F_2 = \emptyset$. Form the following triples:

\[
\begin{align*}
\{1, 1, 1, 1\}, & \{1, 1, 1, 2\}, \{1, 1, 2, 2\}, \{1, 1, 2, 3\}, \{1, 1, 2, 4\}, \\
\{1, 1, a, b\} & \text{ for each } \{a, b\} \in P_1 \cup F_1, \\
\{1, 2, a, b\} & \text{ for each } \{a, b\} \in P_2 \cup F_2, \\
\{1, 4, a, b\} & \text{ for each } \{a, b\} \in P_1 \cup F_1, 3 \leq i \leq v.
\end{align*}
\]

The conclusion then follows.

**Case 2:** If the leave of $(X, A)$ is of type 2 and the two disjoint double edges are $\{1, 1, 1, 2\}$ and $\{3, 3, 6\} \in T(k)$. Let $a = 1$ in Lemma 2.4, then from the difference triples $(1, 1, 2)$ and $(3, 3, 6)$ we obtain $2(u - v + 2)/3$ distinct triples, two simple partial 2-factors $P_1$ and $P_2$ each missing an edge $\{1, 2\}$ and two simple 2-factors $P_3$ and $P_4$ each missing an edge $\{3, 6\}$. Let $n$ be the number of the remaining differences of $D(u - v, \lambda)$. Decompose $(\lambda_0 v - n - 4)/3$ difference triples into differences, we have exactly $\lambda_0 v - 4$ differences. From these differences we can form $v - 4$ simple $\lambda$-factors $F_4, F_5, \ldots, F_v$ and four simple $(\lambda - 2)$-factors $F_1, F_2, F_3$ and $F_4$ such that $P_i \cap F_i = \emptyset$ for $1 \leq i \leq 4$. Form the following triples:

\[
\begin{align*}
\{1, 1, 1, 1\}, & \{1, 1, 1, 2\}, \{1, 1, 2, 2\}, \{1, 1, 2, 3\}, \{1, 1, 2, 4\}, \\
\{1, 2, a, b\} & \text{ for each } \{a, b\} \in P_1 \cup F_1, 1 \leq i \leq 4, \\
\{1, 4, a, b\} & \text{ for each } \{a, b\} \in P_1 \cup F_1, 3 \leq i \leq v.
\end{align*}
\]

The conclusion then follows.

**Case 3:** If the leave of $(X, A)$ is of type 3 and the double edges are $\{1, 1, 1, 2\}$ and $\{3, 3, 3\}$, assume $(1, 1, 2), (3, 3, 6) \in T(k)$. Let $a = 0$ in Lemma 2.4, then from the difference triples $(1, 1, 2)$ and $(3, 3, 6)$ we obtain $2(u - v + 2)/3$ distinct triples, two simple partial 2-factors $P_1$ and $P_3$ each missing an edge $\{0, 1\}$ and two simple 2-factors $P_2$ and $P_4$ each missing an edge $\{0, 3\}$. Let $n$ be the number of the remaining differences of $D(u - v, \lambda)$. Decompose $(\lambda_0 v - n - 4)/3$ difference triples into differences, we have exactly $\lambda_0 v - 4$ differences. From these differences we can form $v - 3$ simple $\lambda$-factors $F_4, F_5, \ldots, F_v$, two simple $(\lambda - 2)$-factors $F_1$ and $F_2$, and a simple $(\lambda - 4)$-factor $F_3$ such that $P_i \cap F_i = \emptyset$ for $1 \leq i \leq 4$. Form the following triples:

\[
\begin{align*}
\{1, 1, 3, 0\}, & \{1, 1, 3, 1\}, \{1, 2, 3, 0\}, \{1, 2, 3, 3\}, \\
\{1, 4, a, b\} & \text{ for each } \{a, b\} \in P_1 \cup F_1, \\
\{1, 4, a, b\} & \text{ for each } \{a, b\} \in P_1 \cup F_1, 4 \leq i \leq v.
\end{align*}
\]

This completes the proof.
Case 4: If the leave of \((X, A)\) is of type 4 and the quadrilateral is \(\{\infty_1, \infty_2\}, \{\infty_2, \infty_3\}, \{\infty_3, \infty_4\}, \{\infty_4, \infty_1\}\), assume \((1, 1, 2), (3, 3, 6) \in T(k)\). Let \(a = 0\) in Lemma 2.4, then we obtain \(2(u - v + 2)/3\) distinct triples, two simple partial 2-factors \(P_1\) and \(P_2\) each missing an edge \([0, 1]\) and two simple partial 2-factors \(P_3\) and \(P_4\) each missing an edge \([0, 3]\). Let \(n\) be the number of the remaining differences of \(D(u - v, \lambda)\). Decompose \((\lambda_0v - n - 4)/3\) difference triples into differences, we then have exactly \(\lambda_0v - 4\) differences. From these differences, we can form \(v - 4\) simple \(\lambda\)-factors \(F_5, F_6, \ldots, F_v\) and four simple \((\lambda - 2)\)-factors \(F_1, F_2, F_3\) and \(F_4\) such that \(P_i \cap F_i = \emptyset\), \(1 \leq i \leq 4\). Form the following triples:

\[
\begin{align*}
\{\infty_1, \infty_2, 0\}, \{\infty_2, \infty_3, 0\}, \{\infty_1, \infty_2, 1\}, \{\infty_3, \infty_4, 3\}, \\
\{a, b\} & \text{ for each } \{a, b\} \in P_i \cup F_i, \ 1 \leq i \leq 4, \\
\{a, b\} & \text{ for each } \{a, b\} \in F_j, \ 5 \leq j \leq v.
\end{align*}
\]

This completes the proof. \(\square\)

Similarly, from Lemmas 2.5 and 2.7, we can prove the following lemma.

Lemma 3.7. If \(\lambda \equiv 4 (\text{mod } 6)\), \(u \equiv 1 (\text{mod } 3)\), \(v \equiv 2 (\text{mod } 3)\), \(u \geq 2v + 1\), then any simple \(\text{MPT}(v, \lambda)\) can be embedded in a simple \(\text{MPT}(u, \lambda)\).

Combining Lemmas 3.1–3.7, we have completely proved Theorem 3.1.

Acknowledgements

This work was done while the third author was visiting the University of Catania. The author would like to thank GNSAGA of CNR Italy for financial support and the University of Catania for the hospitality.

Research of the first and second author is supported by GNSAGA of CNR.

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