On the determination of nontrivial equilibrium configurations close to a bifurcation point

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Abstract

Bifurcation phenomena of equilibrium states occur in both standard and complex materials. In this paper we study the equilibrium configurations close to a bifurcation point. In particular the attention is focused on bifurcations of pitchfork type [S.H. Strogatz, Non Linear Dynamics and Chaos, Addison-Wesley Publishing Company, 1994]. This problem is usually solved by using the Signorini’s compatibility of the solution expansion in a neighborhood of the critical point. We show how the same results can be reached in another way which involves just the linear term of the solution expansion. As a test, we analyze two bifurcation phenomena: the buckling of an elastic beam under an axial load and the magnetic field-induced optical switch in nematic liquid crystals.

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1. Introduction

Equilibrium configurations of continua are often obtained by minimizing a suitable energy functional. Sometimes, the related Euler–Lagrange equations admit trivial solutions which bifurcate in nontrivial ones, through a change in a controllable parameter. The classical example in the theory of elasticity goes back to Euler, who investigated the buckling of an elastic beam under a top load (see for example [2]). In this case the applied load is the parameter which induces the bifurcation between undeformed, in reality simply compressed, configuration (trivial solution) and the bent beam (nontrivial solution). Such bifurcation occurs of course also in more complex materials [3] such as liquid crystals [4] and magnetic films [5]. In particular, in the case of nematic liquid crystals an external magnetic or electric field can induce a phase transition from a homogeneous alignment of the molecules to an inhomogeneous one [4,6,7]. In fact this phase transition, known as a Freedericksz transition, is the basis for electromagneto-optic device design.

In this paper we investigate the equilibrium configurations close to the critical point of a supercritical pitchfork bifurcation (see, for example, [1,8]). The first step requires the linearization of the equilibrium equation in a neighborhood of the trivial solution. Within this approximation, allowed by an eigenvalue requisite, an infinity of solutions exist. Those are the trivial solution plus a countable infinity of nontrivial ones. The physically relevant

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solution corresponds to that with lowest energy. However, within the linear approximation, the solution is determined only up to a multiplicative factor.

To remove that indeterminacy, one may adopt Signorini’s compatibility conditions for higher approximations of the equilibrium equations. The solution is expanded as a power series in a small parameter $\epsilon$ and the approximation is pursued beyond the first order. Within the elasticity and liquid crystals theory, for examples of the efficacy of the process, see [9,7], respectively. We show how another technique can be implemented to remove the indeterminacy. This technique employs a direct expansion as a series in $\epsilon$ of the functional. Once the first-order linear equation is solved one minimizes the first functional term $O(\epsilon^n)$, $n > 2$. This minimization reduces to search for the minimum of a fourth-order polynomial in the indeterminate factor.

This paper is organized as follows. In Section 2, we choose the functional energy to study, we deduce the related Euler–Lagrange equation, and we perform the linear analysis of this equation. In Section 3 we describe both methods used to remove the indeterminacy left in the first approximation; also, we quote as an immediate example the buckling of Euler’s beam. Then in Section 4 we examine weak Fredericksz transition in nematic liquid crystals. We compare the results obtained with the second technique with results already obtained [7] by using the higher order approximation. Some conclusions are pointed out in Section 5.

2. Energy functional

We focus our attention on a functional energy, depending on a real function $u(x)$, its first derivative $u'(x)$ with $x \in [a, b]$ and a positive real parameter $\lambda$,

$$\mathcal{F}[u, u' | \lambda] = \int_a^b \mathcal{L}(u, u' | \lambda)\,dx + W^+ f(u(a)) + W^- f(u(b)), \quad (1)$$

where $W^\pm$ are two positive constants, and $f$ a smooth even function of $u$ with a local minimum for $u$ identically zero and strictly convex around this point. Let us assume the integrand of the form

$$\mathcal{L}(u, u' | \lambda) = \frac{1}{2}g(u)u^2 - \lambda h(u), \quad (2)$$

where $g$ and $h$ are smooth even functions of $u$; $g$ is strictly positive and $h$ is strictly convex around $u \equiv 0$.

In order to obtain the Euler–Lagrange equation related to the functional (1), we calculate its first variation defined as

$$d\mathcal{F}[u, u' | \lambda] = \frac{\partial}{\partial \epsilon} \mathcal{F}[u + \epsilon \varphi, u' + \epsilon \varphi' | \lambda] \bigg|_{\epsilon=0}, \quad (3)$$

whence

$$d\mathcal{F}[u, u' | \lambda] = \int_a^b \left( \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} \right) \varphi \,dx + \left[ \left( \frac{\partial \mathcal{L}}{\partial u'} + W^+ \frac{\partial f}{\partial u} \right) \varphi \right]_{x=a} - \left[ \left( \frac{\partial \mathcal{L}}{\partial u'} - W^- \frac{\partial f}{\partial u} \right) \varphi \right]_{x=b}. \quad (4)$$

Without loss of generality let us suppose $x \in [-1/2, 1/2]$. Also, we introduce the following notation: plus and/or minus superscripts denote function values in $x = 1/2$ and/or $x = -1/2$. From Eq. (4) and by the arbitrariness of $\varphi$ the equilibrium equation

$$gu'' + \frac{1}{2}g_u u'^2 + \lambda h_u = 0 \quad x \in (-1/2, 1/2) \quad (5)$$

follows together with the related boundary conditions

$$\pm g^\pm u'^\pm + W^\pm f_u^\pm = 0 \quad x = \pm 1/2. \quad (6)$$

Notice that boundary conditions of Dirichlet ($u^\pm = 0$) or Neumann ($u'^\pm = 0$) type can be reached in the limiting cases $g^\pm / W^\pm \to 0$ or $W^\pm / g^\pm \to 0$, respectively. In the general case, in $x = \pm 1/2$ the function $u$ will adjust itself in order to satisfy (6).
By construction, the equilibrium equation (5) together with the boundary conditions (6) admits the trivial solution \( u = 0 \). We consider the solution slightly perturbed with respect to \( u = 0 \). Let us assume the approximation
\[
\begin{align*}
    u(x) = \varepsilon u_1(x) + o(\varepsilon).
\end{align*}
\] (7)
The parameter \( \varepsilon \) is defined by \( \varepsilon = \sqrt{\lambda / \lambda_0} - 1 \ll 1 \), whence it follows easily that
\[
\lambda = \lambda_0 (1 + \varepsilon^2),
\] (8)
where the parameter \( \lambda_0 \) will be determined within the linear analysis. By replacing (7) and (8) into (5) and (6) we obtain, up to first order,
\[
\begin{align*}
    g^0 u''_1 + \lambda_0 \delta^0_{uu} u_1 &= 0, \quad x \in (-1/2, 1/2), \\
    \pm g^0 u'_1 \pm W^\pm f^0_{uu} u'_1 &= 0, \quad x = \pm 1/2,
\end{align*}
\] (9, 10)
where the superscript zero denotes functions evaluated in \( u = 0 \). The general solution of (9) is
\[
    u_1(x) = A_1 \cos(\Omega x) + B_1 \sin(\Omega x),
\] (11)
where \( \Omega^2 = \lambda_0 \delta_{uu} / g^0 \) and \( A_1 \) and \( B_1 \) are constants to be determined. Imposing the boundary conditions (10) yields a linear homogeneous system in the unknowns \( A_1 \) and \( B_1 \):
\[
\begin{bmatrix}
    -\Omega g^0 \sin\left(\frac{\Omega}{2}\right) + W^- f^0_{uu} \cos\left(\frac{\Omega}{2}\right) - \Omega g^0 \cos\left(\frac{\Omega}{2}\right) - W^- f^0_{uu} \sin\left(\frac{\Omega}{2}\right) \\
    -\Omega g^0 \sin\left(\frac{\Omega}{2}\right) + W^+ f^0_{uu} \cos\left(\frac{\Omega}{2}\right) + \Omega g^0 \cos\left(\frac{\Omega}{2}\right) + W^+ f^0_{uu} \sin\left(\frac{\Omega}{2}\right)
\end{bmatrix}
\begin{bmatrix}
    A_1 \\
    B_1
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0
\end{bmatrix}
\] (12)
whose solution is trivial unless the two equations are linearly dependent. Let us denote as \( \mathbb{M} \) the \( 2 \times 2 \) matrix on the left in (12). The existence of nontrivial solutions imposes \( \text{det}(\mathbb{M}) = 0 \) which gives the implicit equation for \( \Omega \) (and therefore for \( \lambda_0 \))
\[
(W^- + W^+) \Omega g^0 f^0_{uu} \cos(\Omega) + \left( W^- W^+ \left( f^0_{uu} \right)^2 - \Omega^2 \left( g^0 \right)^2 \right) \sin(\Omega) = 0
\] (13)
and allows us to calculate the ratio between the constants \( B_1 \) and \( A_1 \):
\[
K = \frac{B_1}{A_1} = \frac{-\Omega g^0 \sin\left(\frac{\Omega}{2}\right) + W^- f^0_{uu} \cos\left(\frac{\Omega}{2}\right)}{\Omega g^0 \cos\left(\frac{\Omega}{2}\right) + W^- f^0_{uu} \sin\left(\frac{\Omega}{2}\right)}.
\] (14)
Notice that (13) admits a countable infinity of solutions. The solution in the interval \([-\pi, \pi]\) determines the critical parameter \( \Omega_{cr} \) and correspondingly \( \lambda_{cr} \) and \( K_{cr} \). Finally we can put the linear solution in the form
\[
    u_1(x) = A_1 \left( \cos(\Omega_{cr} x) + K_{cr} \sin(\Omega_{cr} x) \right).
\] (15)
It is easy to check that \( K \neq 0 \) only in the case of asymmetric boundary conditions, \( W^+ \neq W^- \).

3. Removing the indeterminacy

3.1. Compatibility at higher order approximations

Notice that \( A_1 \) still remains undetermined. This indeterminacy given by the first approximation can be rendered definite by using the conditions of compatibility for higher approximations.

Let us expand the solution as a power series in \( \varepsilon \):
\[
    u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots.
\] (16)
By substituting this expression into (5) and (6) and keeping the \( \varepsilon^2 \) terms we arrive at

\[
g_0u_2'' + \lambda_{cr}h^0_{uu}u_2 = 0
\]

and the boundary conditions

\[
\pm g^0(u_1')^{\pm} + W^\pm f^0_{uu}u_1^{\pm} = 0.
\]

The equation for \( u_2 \) and its boundary conditions are identical to those obtained for \( u_1 \). As before, they do not allow the evaluation of the amplitude \( A_1 \).

Pushing the perturbation algorithm to \( \mathcal{O}(\varepsilon^3) \) we obtain the equation for \( u_3 \):

\[
g_0u_3'' + \lambda_{cr}h^0_{uu}u_3 = -\frac{1}{2}g^0_{uu}\left(u_1'^2u_1'' + u_1u_1''ight) - \lambda_{cr}u_1\left(h^0_{uu} + \frac{1}{6}u_1^2h^0_{uuuu}\right),
\]

and the associated boundary conditions

\[
\pm g^0u_3^{\pm} + W^\pm f^0_{uu}u_3^{\pm} = \mp \frac{1}{2}g^0_{uu}\left(u_1^{\pm}\right)^2u_1^{\pm} - \frac{1}{6}W^\pm f^0_{uuuu}\left(u_1^{\pm}\right)^3.
\]

Eq. (17) can be solved taking into account (15). Its general solution is of the form

\[
u_3(x) = A_3 \cos(\Omega_{cr}x) + B_3 \sin(\Omega_{cr}x) + u^p_3(x),
\]

where \( u^p_3(x) \) is a particular solution. After some algebra, it can be shown that

\[
u^p_3(x) = \alpha u_1(x)^3 + \left(\frac{1}{2} + \beta A_1^2\right) x u_1'(x),
\]

where

\[
\alpha = \frac{h^0_{uuuu}}{48h^0_{uu}} - \frac{\beta_{uu}}{8g^0}, \quad \beta = \left(1 + K_2^2\right)\left(h^0_{uuuu} + \frac{\beta_{uu}}{16h^0_{uu}} - \frac{\beta_{uu}}{8g^0}\right).
\]

Substitution in the boundary conditions for \( u_3(x) \) gives a linear system in the unknown \( A_3 \) and \( B_3 \) of the form \( \mathbb{M}x = b \) where \( x = (A_3, B_3)^T \), \( b = (b^-, b^+)^T \) with

\[
b^\pm = \mp g^0u_3^{\pm} - W^\pm f^0_{uu}u_3^{\pm} \mp \frac{1}{2}g^0_{uu}\left(u_1^{\pm}\right)^2u_1^{\pm} - \frac{1}{6}W^\pm f^0_{uuuu}\left(u_1^{\pm}\right)^3.
\]

Let us denote as \( \mathbf{m}_1 \) and \( \mathbf{m}_2 \) the column vectors composing \( \mathbb{M} \). In order to solve this system we must have \( b \in \text{span}\{\mathbf{m}_1, \mathbf{m}_2\} = \text{span}\{\mathbf{m}_1\} = \text{span}\{\mathbf{m}_2\} \) since \( \mathbf{m}_1 \) and \( \mathbf{m}_2 \) are linearly dependent (\( \det(\mathbb{M}) = 0 \)). This means that we gather a third-order algebraic equation in \( A_1 \) from one of the two determinants \( \det(\mathbf{m}_1|b) = 0 \) or \( \det(\mathbf{m}_2|b) = 0 \).

By introducing the notation \( \varphi^\pm = u^\pm/\alpha, \gamma^\pm = \sqrt{\Omega_{cr}^2(g^0)^2 + (W^\pm)^2(f^0_{uu})^2}, \eta^\pm = \varphi^\pm \gamma^\pm[(\gamma^\pm)^2 + 2W^-g^0f^0_{uu}] + \gamma^-\varphi^\pm[(\gamma^-)^2 + 2W^+g^0f^0_{uu}], \eta^\pm = 12\alpha^2g^0f^0_{uu} + 3f^0_{uu}g^0_{uu} - g^0f^0_{uuuu} \),

a tedious but easy computation yields the equation for the amplitude:

\[
A_1^3\left[6\beta\varphi + 2\left(W^-\gamma^+(\gamma^-)^3 + W^+\gamma^-(\gamma^+)^3\right)\eta\right] + 3A_1\varphi = 0.
\]

Eq. (22) admits the trivial solution \( A = 0 \) and two other real opposite solutions. The former corresponds to the trivial solution which can be shown to be unstable. Due to the symmetry of the problem the other two represent the amplitudes of nontrivial solutions with the same energy.
3.2. Minimizing the $\mathcal{O}(\epsilon^4)$ functional term

Consider now a direct expansion of the functional (1) in powers of $\epsilon$, under the assumptions $u(x) = \epsilon u_1(x)$, $x \in [-1/2, 1/2]$ and $\lambda = \lambda_0(1 + \epsilon^2)$. So, up to the second order we obtain

$$
\mathcal{F}[\epsilon u_1, \epsilon u_1'|\lambda] = f^0 \left(W^- + W^+\right) - \lambda_0 h^0 + \epsilon^2 \int_a^b \left[\frac{1}{2} g_0 u_1'^2 - \frac{1}{2} \lambda_0 h_{uu}^0 u_1^2 - \lambda_0 h^0\right] dx
+ \epsilon^2 \frac{1}{2} f_{uu}^0 \left(W^- (u_1^-)^2 + W^+ (u_1^+)^2\right].
$$

(23)

It is easy to check that the $\mathcal{O}(1)$ term of the functional is constant, while the $\mathcal{O}(\epsilon^2)$ is minimized by the linear problem (9) and (10). Therefore, $u_1$ is still of the form (15).

To remove the indeterminacy we push the expansion of $\mathcal{F}$ to higher orders. The first non-null term of the functional expansion $\mathcal{O}(\epsilon^4)$ is the $\mathcal{O}(\epsilon^4)$ term

$$
\epsilon^4 \int_a^b \left[\frac{1}{4} g_{uu}^0 u_1'^2 - \frac{1}{2} \lambda_0 h_{uu}^0 u_1^2 - \frac{1}{24} \lambda_0 h_{uuuu}^0 u_1^4\right] dx + \epsilon^4 \frac{1}{24} f_{uuuu}^0 \left(W^- (u_1^-)^4 + W^+ (u_1^+)^4\right].
$$

(24)

We stress the fact that in (24) only the first term of the solution expansion has been taken into account. Notice that the form of $u_1$ is now known and a direct integration of (24) can be performed. The so-obtained expression is a polynomial expression for the only free parameter $A_1$. Therefore, minimization of the functional (24) reduces to calculating the minima of a polynomial function.

Using the identity $u_1'^2 + \Omega_{c1}^2 u_1^2 = A_1^2 (1 + K_{c1}^2) \Omega_{c1}^2$ and integrating by parts, the following representations of the integrals involved in (24) are obtained:

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} u_1' dx = \frac{1}{2} \left[A_1^2 (1 + K_{c1}^2) x - \frac{1}{\Omega_{c1}^2} u_1 u_1'\right]_{-\frac{1}{2}}^{\frac{1}{2}},
$$

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} u_1^4 dx = \frac{3}{4} A_1^2 (1 + K_{c1}^2) \int_{-\frac{1}{2}}^{\frac{1}{2}} u_1'^2 dx - \left[\frac{1}{4 \Omega_{c1}^2} u_1^3 u_1'\right]_{-\frac{1}{2}}^{\frac{1}{2}}.
$$

By the use of boundary conditions of $u_1$ and differentiating with respect to the amplitude we finally gather the equation for $A_1$:

$$
A_1^2 \left[12 g_{uu}^0 h_{uu}^0 (1 + K_{c1}^2) \Theta + g^0 \left(4 h_{uu}^0 f_{uuuu}^0 - h_{uuuu}^0 f_{uu}^0\right) \Gamma_4\right] - 6 A_1 h_{uu}^0 \left(2 g^0 + g_{uu}^0\right) \Theta = 0,
$$

(25)

where we have set

$$
\Gamma_2 = W^- (u_1^-) + W^+ (u_1^+), \quad \Gamma_4 = W^- (u_1^-) + W^+ (u_1^+),
$$

\[ \Theta = (1 + K_{c1}^2) \Omega_{c1}^2 g^0 + f_{uu}^0 \Gamma_2. \]

If $A_1$ has to realize the minimum, the derivative of this expression must be zero. In this way, we arrive at an algebraic equation in $A_1$ and $A_1^3$ which gives the same result as Eq. (22).

A quick check of the usefulness of these techniques is provided by the buckling of Euler’s beam clamped at one end under an external axial load. In this case it can be shown that the functional takes the form

$$
\mathcal{F}[\vartheta', \vartheta|\lambda] = \int_{-1/2}^{1/2} \left[\frac{1}{2} (\vartheta')^2 + \lambda \cos \vartheta\right] dx,
$$

(26)

which must be minimized with the boundary conditions $\vartheta(-1/2) = 0$ and $\vartheta'(1/2) = 0$. The first-order approximation is the well known

$$
\vartheta_1(x) = \pm 2 \left[\cos \left(\frac{\pi}{2} x\right) + \sin \left(\frac{\pi}{2} x\right)\right].
$$

(27)
4. Freedericksz transition in nematic liquid crystals

A nematic liquid crystal [4] is a system of rod-like molecules whose centers of mass do not exhibit any positional order. The interaction between neighbouring molecules tries to make them parallel with one another, and induces a partial ordering at mesoscopic scales. This effect competes against the distortions induced by external mechanical actions, electric or magnetic fields, and the disordering thermal effects. The average alignment of the molecules is represented by a unit vector \( \mathbf{n} \), called the director, where \( \mathbf{n} \) is physically equivalent to \(-\mathbf{n}\). In the classic theory, a local stored energy function depending on \( \mathbf{n} \) and its gradient is assumed. The director \( \mathbf{n} \) adjusts throughout the sample in order to minimize that energy according to the boundary conditions.

In usual magneto-optic devices the liquid crystal is confined between two parallel plates and its molecular orientation can be driven through a magnetic field oriented orthogonally to the plates. In the presence of a magnetic field the liquid crystal tends to align its molecules along or normal to the direction of the field, depending on the orientation can be driven through a magnetic field oriented orthogonally to the plates. In the presence of a magnetic field the liquid crystal tends to align its molecules along or normal to the direction of the field, depending on the magnetic field the liquid crystal tends to align its molecules along or normal to the direction of the field, depending on the

In the absence of external actions, the preferred direction of the molecules on the boundary minimizes the anchoring energy. The preferred direction is called the easy axis. In the presence of external actions, the direction of the molecules on the boundary is unknown.

We consider a nematic liquid crystal confined between two parallel plates placed at \( Z = -d/2 \) and \( Z = d/2 \), subject to weak anchoring at the external surfaces. The easy axis is assumed to lie in the boundary planes, and will be labeled as the \( X \)-axis. So, in the absence of any external action, the nematic is in homogeneous planar alignment, \( n = (1, 0, 0) \). A homogeneous magnetic field \( \mathbf{H} \) is applied orthogonally to the delimiting plates. Above a critical value of the magnetic field, when the external field exceeds the elastic strength, the nematic switches from a homogeneous alignment to an inhomogeneous one. This effect is called the Freedericksz transition.

We assume plane deformations of the director field. In that case, the following representation of the director is possible: \( \mathbf{n} = (\cos \theta, 0, \sin \theta) \). The angle \( \theta \) is determined by the director \( \mathbf{n} \) and the \( X \)-axis and it will be a function of the \( Z \) coordinate only. With these hypotheses, the distortion energy per unit area \([4,6]\) takes the form

\[
\mathcal{G}_F = \frac{1}{2} \int_{-d/2}^{d/2} \kappa \left( \frac{d\theta}{dZ} \right)^2 dZ,
\]

where \( \kappa \) is a positive elastic constant. According to [4], the director–field interaction is described by the energy

\[
\mathcal{G}_I = -\frac{1}{2} \int_{-d/2}^{d/2} \chi_a H^2 \sin^2 \theta dZ,
\]

where the quantity \( \chi_a \) measures the diamagnetic anisotropy. It is easy to check that, whenever \( \chi_a \) is positive, the molecules prefer to align their axes along the magnetic field direction. We assume \( \chi_a > 0 \) in order to create competition between the magnetic field and the surface anchoring.

The anchoring at the boundaries is described by the Rapini–Papoular energy [10]:

\[
\mathcal{G}_A = \frac{w^\pm}{2} \sin^2 \theta \quad \text{at} \quad Z = \pm \frac{d}{2},
\]

where both anchoring strengths \( w^+ \) and \( w^- \) are positive. With this choice of \( w^\pm \), the anchoring energy is minimized when \( \mathbf{n} \) is parallel to the plates. It favours the planar easy axes.

The total energy per area unit is then \( \mathcal{G} = \mathcal{G}_F + \mathcal{G}_I + \mathcal{G}_A \). By introducing the scaled variable \( z = Z/d \) and defining

\[
\lambda = \frac{\chi_a}{\kappa} (Hd)^2, \quad \beta^\pm = \frac{d\omega^\pm}{\kappa},
\]

the rescaled total free energy can be written as

\[
\mathcal{F}[\theta', \theta|\lambda] = \frac{d}{\kappa} \mathcal{G} = \int_{-1/2}^{1/2} \left[ \frac{1}{2} \theta'^2 - \lambda \sin^2 \theta \right] dz + \frac{\beta^+}{2} \sin^2 \theta^+ + \frac{\beta^-}{2} \sin^2 \theta^-.
\]

Let us consider now the case with symmetrical boundary conditions \( \beta = \beta^+ = \beta^- \). Proceeding in the same way as for (23), we will consider the expansion of the functional when \( \theta(x) = \epsilon \theta_1(x) \) and \( \lambda = \lambda_0 (1 + \epsilon^2) \). The \( \mathcal{O}(\epsilon^2) \) term
provides an equation for $\theta_1(x)$ and the boundary conditions
\[
\theta_1'' + \lambda_0 \theta_1 = 0 \quad \text{and} \quad (\theta_1')^\pm \pm \beta \theta_1^\pm = 0,
\]
whose solution is $\theta_1(x) = A_1 \cos(\sqrt{\lambda_0} x)$, where $\lambda_0$ obeys the equation
\[
\frac{\sqrt{\lambda_0}}{\beta} = \cot \frac{\sqrt{\lambda_0}}{2}.
\] (33)

The critical dimensionless magnetic field $\lambda_{cr}^{1/2}$ is a solution of Eq. (33) in the interval $[-\pi, \pi]$. However, in this range there are two opposite solutions which correspond to the fact that the transition can be induced by magnetic fields in both positive and negative directions of the $Z$-axis.

The value of $A_1$ can be determined by using $O(\varepsilon^4)$. Taking into account the expression found for $\theta_1(x)$ and performing the integration, the $O(\varepsilon^4)$ term of the functional is
\[
e^4 \left( A_1^4 \lambda_{cr} \frac{\beta^4 + 2 \beta^3 + 2 \beta^2 \lambda_{cr} - 2 \beta \lambda_{cr} + \lambda_{cr}^2}{16(\beta^2 + \lambda_{cr})^2} - A_1^2 \lambda_{cr} \frac{\beta^2 + 2 \beta + \lambda_{cr}}{4(\beta^2 + \lambda_{cr})} \right).\] (34)

Minimization of the energy functional is then possible up to this term by means of a minimization of (34) with respect to $A_1$. Imposing that the derivative with respect to $A_1$ of (34) is equal to zero, it is easy to obtain a third-order polynomial equation in $A_1$ whose solutions are
\[
A_1 = \pm \sqrt{-\frac{2(\beta^2 + \lambda_{cr})(\beta^2 + 2 \beta + \lambda_{cr})}{\beta^4 + 2 \beta^3 + 2 \beta^2 \lambda_{cr} - 2 \beta \lambda_{cr} + \lambda_{cr}^2}}.
\] (35)

The double sign is due to the fact that the liquid crystal molecules can arrange themselves into two symmetric configurations having the same total energy, one tilted in a clockwise direction and the other tilted by the same angle in a counterclockwise direction. It has to be noted that in addition to these solutions there is the trivial solution $A_1 = 0$ which corresponds to the case $\theta(x) \equiv 0$.

The comparison with the results obtained in [7] requires some attention. The case studied in [7] involves the nematic anisotropy (here it is assumed to be zero) and assumes complete coupling with the external field (here the magnetic field is considered homogeneous and, therefore, unaffected by the director distortion). With reference to the notation of Section 3.2 of [7], the correct comparison must be performed in the limit of one constant approximation ($\alpha = 0$) and partial coupling ($\eta = 0$). With these limits the results are identical provided we replace the magnetic field with the electric field and the diamagnetic anisotropy with the dielectric anisotropy.

5. Concluding remarks

We have presented two techniques for determining the nontrivial solutions near a bifurcation point for a typical problem in continuum mechanics. The techniques turn out to be equivalent, but, in our opinion, the method reported in Section 3.2 is less expensive (in terms of computation) than the one exploited in Section 3.1. The simplicity of the problem treated allowed us to make a direct comparison between the results deriving from the two techniques. On the other hand, we do not investigate analytic reasons for this equivalence, that we conjecture to hold for more complex systems.

In fact, there are a lot of phenomena which require a generalization of the study performed in this work, for example the study of the Freedericksz transition induced by an electric field, where the nematic director equation is fully coupled with Maxwell equations for the electric potential [11,7]. In this case the equilibrium is given by solving a coupled system of two ordinary differential equations. In this particular phenomenon the critical field is not modified by the coupling, while the amplitude turns out to be affected.

Another generalization could concern the structure of the differential operator involved in the nonlinear equation under study. For example, study of the buckling of elastic plates [2] involves the biharmonic operator; therefore the equilibrium involves a partial differential equation of fourth order. Other more complicated systems involve the coupling of partial differential equations. This is the case for the Helfrich–Hurault effect induced by an electric field in lamellar phases [12]. In this case both the critical field and the amplitude are affected by the complete coupling.
The listed phenomena (and many others) are sometimes investigated by using other techniques, but to our knowledge the equivalence is not a priori obvious.

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References