Generalized Consistency and Intensity Vectors for Comparison Matrices

L. D’Apuzzo,1,* G. Marcarelli,2,† M. Squillante2,‡
1Dipartimento di Costruzioni e Metodi Matematici in Architettura, Università di Napoli, via Monteoliveto 3, 80134 Napoli, Italy
2Dipartimento di Analisi dei Sistemi Economici e Sociali, Facoltà di Scienze Economiche e Aziendali, Università del Sannio, via Nazionale delle Puglie 82, 82100 Benevento, Italy

A crucial problem in a decision-making process is the determination of a scale of relative importance for a set \( X = \{x_1, x_2, \ldots, x_n\} \) of alternatives either with respect to a criterion \( C \) or an expert \( E \). A widely used tool in Multicriteria Decision Making is the pairwise comparison matrix \( A = (a_{ij}) \), where \( a_{ij} \) is a positive number expressing how much the alternative \( x_i \) is preferred to the alternative \( x_j \). Under a suitable hypothesis of no indifference and transitivity over the matrix \( A = (a_{ij}) \), the actual qualitative ranking on the set \( X \) is achievable. Then a vector \( w \) may represent the actual ranking at two different levels: as an ordinal evaluation vector, or as an intensity vector encoding information about the intensities of the preferences. In this article we focus on the properties of a pairwise comparison matrix \( A = (a_{ij}) \) linked to the existence of intensity vectors. © 2007 Wiley Periodicals, Inc.

1. INTRODUCTION

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of alternatives. A scale of relative importance for the alternatives either with respect to a criterion \( C \) or an expert \( E \) can be obtained by starting from explicit comparisons of each alternative with every other: a real positive number \( a_{ij} \) is assigned to each pair \( (x_i, x_j) \) expressing how much the alternative \( x_i \) is preferred to the alternative \( x_j \).¹ We assume that

- \( a_{ij} > 1 \) if and only if \( x_i \) is preferred to \( x_j \), whereas \( a_{ij} < 1 \) indicates the reverse preference
- \( a_{ij} = 1 \) indicates indifference between \( x_i \) and \( x_j \)
- the pairwise comparison matrix \( A = (a_{ij}) \) is reciprocal, that is

\[
 r) \quad a_{ji} = \frac{1}{a_{ij}} \quad \forall i, j \in \{1, 2, \ldots, n\} \quad (reciprocity)
\]

*Author to whom all correspondence should be addressed: e-mail: liviadap@unina.it.
†e-mail: marcarel@unisannio.it.
‡e-mail: squillan@unisannio.it.

As a consequence, $a_{ii} = 1 \forall i \in \{1, 2, \ldots, n\}$.

Several tools have been indicated in literature in order to derive from $A$ a positive vector $w = (w_1, w_2, \ldots, w_n)$ representing the priorities on $X$.\(^1\)\(^-\)\(^4\) Let us mention the eigenvector method, for which, if $\lambda_{\text{max}}$ is the greatest eigenvalue of $A$, then each positive solution of the equation $A w = \lambda_{\text{max}} w$ is a priority vector,\(^3\) the arithmetic and the geometric mean operators that translate the matrix $A$ in a vector, when they are applied to each row of the matrix, and the logarithmic least squares method.\(^5\) In this context a dominance vector, providing the weights of the alternatives, is a priority vector whose components sum to 1. Nevertheless, it has been shown that, if $A$ does not verify the condition

\[
\forall i, j, h = 1, \ldots, n: a_{ih} a_{hj} = a_{ij} \quad (\text{consistency})
\]

then it may happen that a vector $w$, usually proposed as priority vector, does not agree with the dominance ratios expressed by the entries of $A$. In other words, it may happen that $w_i > w_j$ whereas $a_{ij} < 1$ indicates that the alternative $x_j$ is preferred to the alternative $x_i$.\(^6\)

To get round this difficulty, Basile and D’Apuzzo\(^6\) have indicated a different procedure for stating a weighted ranking:

1. to determine the actual ranking, that is a qualitative ranking agreeing with the preference ratios $a_{ij}$
2. to look for vectors $w = (w_1, w_2, \ldots, w_n)$ representing the actual ranking.

Basile and D’Apuzzo\(^6\)\(^-\)\(^9\) apply this procedure to the $\text{Rts}$ matrices, which are reciprocal matrices verifying the conditions

\[
\text{r)} \quad a_{ij} > 1 \quad \text{for } i \neq j \quad \text{(no indifference)}
\]

\[
\text{t)} \quad a_{ij} > 1 \text{ and } a_{jk} > 1 \Rightarrow a_{ik} > 1 \quad \text{(transitivity)}
\]

The conditions $\text{r}$, $\text{t}$, and $\text{s}$ ensure that the asymmetric relation $>$ on $X$ defined by

\[
x_i > x_j \iff a_{ij} > 1 \tag{1}
\]

is a strict preference order (i.e., a transitive, asymmetric, and complete relation). Hence, if $A = (a_{ij})$ is an $\text{Rts}$ matrix, then a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ is available so that the elements of $X$ are ordered in the decreasing chain

\[
x_{i_1} > x_{i_2} > \cdots > x_{i_n} \tag{2}
\]

representing the actual ranking on $X$. In Section 2 of this article we report a characterization given in Ref. 9 of the $\text{Rts}$ matrices that allows us to immediately read the actual ranking.

Once the actual ranking on $X$ is stated, the second problem has to be solved.

Let $R^n_+$ denote the set $[0, +\infty[^n$. Then a vector $w = (w_1, w_2, \ldots, w_n) \in R^n_+$ may represent the actual ranking (2) at different levels.
1. as a ordinal evaluation vector or as a coherent priority vector, that is, as a vector verifying the condition \( w_i > w_j \iff a_i > a_j \), that, by (1), can be written as

\[
w_i > w_j \iff a_{ij} > 1
\]  

(3)

2. as a value ratio vector or as an intensity vector, representing the intensity of preferences by means of the equivalence

\[
\frac{w_j}{w_i} > \frac{w_s}{w_r} \iff a_{ij} > a_{rs}
\]  

(4)

then

\[
\frac{w_j}{w_i} = \frac{w_s}{w_r} \iff a_{ij} = a_{rs}
\]  

(5)

3. as a consistent vector, that is, as a vector verifying the desirable condition

\[
\frac{w_j}{w_i} = a_{ij} \quad \forall i, j \in \{1, 2, \ldots, n\}
\]  

(6)

It is well known that a consistent vector exists if and only if \( A = (a_{ij}) \) verifies the condition of consistency \( c \) and it is trivial that a consistent vector is an intensity vector.

We focus on the intensity vectors. By choosing \( r = s \) in (4), we can see that an intensity vector is a coherent priority vector.

The problem of finding ordinal evaluation vectors for inconsistent matrices has been widely investigated by Basile and D’Apuzzo.6,7,9 They introduce the condition

\[
\text{we) } a_{ij} > 1 \text{ and } a_{jk} > 1 \implies a_{ik} > a_{ij} \vee a_{jk} \quad \text{(weak consistency)}
\]

and characterize the class \( F \) of the aggregation operators

\[
F: u = (u_1, u_2, \ldots, u_n) \in R^n_+ \rightarrow F(u) \in ]0, +\infty[
\]  

(7)

that, when applied to the rows \( a_1, \ldots, a_n \) of a weakly consistent matrix, provide

\[
w_F = (F(a_1), F(a_2), \ldots, F(a_n))
\]  

(8)

as a coherent priority vector (see Refs. 7 and 9). In Section 2.1 of this article we report characterizations of a weakly consistent matrix together with a characterization of the class \( F \). An \( \text{Rts} \) matrix is weakly consistent if and only if each column is a coherent priority vector9; so a weakly consistent matrix \( A \) is also called a priority matrix. The matrix \( A \) is called an intensity matrix if and only if its columns are intensity vectors. It is straightforward that a consistent matrix is an intensity matrix and an intensity matrix is a priority matrix. The reverse implications do not hold (see Examples 2 and 4 in the Appendix).
A first approach to the problem of finding intensity vectors has been proposed in Ref. 10, where a characterization of intensity matrices has been given. In this article we carry on the investigation started in Ref. 10 and focus our attention on properties of an \textbf{Rts} matrix linked to the existence of an intensity vector. We also show that the existence of an intensity vector requires that the matrix $A = (a_{ij})$ be a priority matrix. We carry out further investigation on intensity matrices and give a characterization of this type of matrix by means of a relation of partial order $\triangleright$ introduced in $R^*_n = ]0, +\infty[^n$ that improves a result of Ref. 10.

2. PRELIMINARIES

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives and $A = (a_{ij})$ the related pairwise comparison matrix. From now on we assume the following.

**Assumption 1.** $A = (a_{ij})$ is an \textbf{Rts} matrix and (2) represents the actual ranking. We denote by

- $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\}$ the set of the rows of $A$
- $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\}$ the set of the columns
- $n(\tilde{a}_i)$ the number of the components of $\tilde{a}_i$ greater than 1
- $n_0(\tilde{a}_i)$ the number of the components of $\tilde{a}_i$ greater than or equal to 1
- $\triangleright$ the strict partial order on $R^*_n$ defined by
  \begin{equation}
  u = (u_1, \ldots, u_n) \triangleright v = (v_1, \ldots, v_n) \iff u_j > v_j \quad \forall j = 1, 2, \ldots, n
  \end{equation}
- $\triangleright$ the partial order
  \begin{equation}
  u \triangleright v \iff (u \triangleright v \text{ or } u = v)
  \end{equation}
- $\ll$ and $\preceq$ the relations opposite to $\triangleright$ and $\triangleright$, respectively.

Moreover, for $\underline{1} = (1, 1, \ldots, 1)$, $u = (u_1, \ldots, u_n)$, and $v = (v_1, \ldots, v_n)$, we denote by

- $u/v$ the vector $(u_1/v_1, \ldots, u_n/v_n)$
- $1/v$ the vector $1/v = (1/v_1, 1/v_2, \ldots, 1/v_n)$.

We stress that the condition of reciprocity $r$ can be expressed as

$\begin{align*}
\rho' \quad a_k = \frac{1}{a_k} \quad \forall k = 1, 2, \ldots, n
\end{align*}$

and, because of the assumption $s$ for the matrix $A = (a_{ij})$,

\begin{equation}
 n_0(\tilde{a}_i) = n(\tilde{a}_i) + 1
\end{equation}

The following proposition indicates a characterization of the \textbf{Rts} matrices that has been given in Ref. 9.
Proposition 1. \( A = (a_{ij}) \) is an Rts matrix if and only if there exists a permutation \( \alpha = (i_1, i_2, \ldots, i_n) \) of \((1, 2, \ldots, n)\) such that
\[
n(a_{i_1}) = n - 1 > n(a_{i_2}) = n - 2 > \cdots > n(a_{i_n})
= n - h > \cdots > n(a_{i_n}) = 0
\]
or, equivalently,
\[
n_0(a_{i_1}) = n > n_0(a_{i_2}) = n - 1 > \cdots > n_0(a_{i_n})
= n - h + 1 > \cdots > n_0(a_{i_n}) = 1
\]
Moreover
\[
x_i > x_j \iff n(a_i) > n(a_j) \iff n_0(a_i) > n_0(a_j)
\]
For a proof, see Lemma 3.1 and Theorem 3.1 in Ref. 9. Hence, under Assumption 1, the same permutation \( \alpha = (i_1, i_2, \ldots, i_n) \) provides Inequalities (11) and (12) and the actual ranking (2). As a consequence, the vector
\[
g_0(A) = (n_0(a_1), n_0(a_2), \ldots, n_0(a_n))
\]
is a coherent priority vector, and a positive vector \( w = (w_1, w_2, \ldots, w_n) \) is a coherent priority vector if and only if \( w = (\phi(n_0(a_1)), \phi(n_0(a_2)), \ldots, \phi(n_0(a_n))) \), with \( \phi \) a strictly increasing positive function on \( \mathbb{R} \).

Anyway \( g_0(A) \) does not take the effective values of the \( a_{is} \)'s into account. So we shall consider coherent priority vectors and intensity vectors obtained by aggregating the elements of the matrix by means of the operator (7).

Definition 1. Let \( F \) be a functional (7) and \( w_F \) the vector (8). Then
\begin{itemize}
  \item \( F \) is an ordinal evaluation functional if and only if \( w_F \) is a coherent priority vector
  \item \( F \) is an intensity functional if and only if \( w_F \) is an intensity vector.
\end{itemize}

2.1. Priority Matrix and Ordinal Evaluation Functionals

By definition \( A = (a_{ij}) \) is a priority matrix if and only if each column \( a^h \) is a coherent priority vector; so, by the equivalence (3) defining a coherent priority vector, \( A = (a_{ij}) \) is a priority matrix if and only if
\[
a_{ij} > 1 \iff (a_{ih} > a_{jh} \ \forall h = 1, 2, \ldots, n)
\]

Proposition 2. \( A = (a_{ij}) \) is a priority matrix if and only if
\[
a_{ij} > 1 \iff a_i \triangleright a_j \iff a^i \triangleright a^j
\]

Proof. This follows by (15), the definition of \( \triangleright \), and Condition \( r' \).
Theorem 1. The following statements are equivalent:

(i) \( A = (a_{ij}) \) is a weakly consistent matrix
(ii) \( \triangleright \) is a strict simple order on \( \tilde{A} = \{a_1, a_2, \ldots, a_n\} \)
(iii) \( A = (a_{ij}) \) is a priority matrix.

Hence, by Proposition 2, Theorem 1, item ii, and equivalence (1) defining the relation \( \triangleright \), \( A \) is a priority matrix if and only if \( \tilde{A} = \{a_1, a_2, \ldots, a_n\} \) is ordered by \( \triangleright \) in accordance with the actual ranking (2), that is,
\[
a_{i_1} \triangleright a_{i_2} \triangleright \cdots \triangleright a_{i_n},
\]
whereas \( \tilde{A} \) is ordered as follows: \( a^{i_1} \triangleleft a^{i_2} \triangleleft \cdots \triangleleft a^{i_n} \).

Corollary 1. The following conditions are equivalent:

(1) \( A = (a_{ij}) \) is a weakly consistent matrix
(2) \( a_{ik} > a_{kh} \) for some \( h \)
(3) \( a_{ik} > a_{ai} \) for some \( i \)
(4) \( a_{ik} = a_{ah} \) for some \( h \)

Moreover Condition 1 implies

(4) \( a_{ik} = a_{ah} \) for some \( h \) \( \iff \) \( i = k \).

Proof. 1 \( \iff \) 2. By Theorem 1, \( A \) is weakly consistent if and only if the set \( \tilde{A} \) is completely ordered by the strict order \( \triangleright \); as a consequence, each one of the inequalities \( a_{ih} > a_{kh} \) and \( a_{ik} > 1 \) means that \( a_i \triangleright a_k \). Conversely, by choosing \( h = k \) in item 2, we get the equivalence (16) defining \( A \) as a priority matrix and so as a weakly consistent matrix.

2 \( \iff \) 3. Because of the reciprocity of the matrix \( A = (a_{ij}) \).

1 \( \Rightarrow \) 4. Because of the equivalence 1 \( \iff \) 2 and Condition 5.

Theorem 2. Let \( A = (a_{ij}) \) be a priority matrix and \( F \) a functional (7). Then \( F \) is an ordinal evaluation operator if and only if its restriction to \( \tilde{A} \) is strictly increasing with respect to the relation \( \triangleright \), that is
\[
a_i \triangleright a_j \Rightarrow F(a_i) > F(a_j)
\]

The class \( \mathcal{F} \) of the ordinal evaluation functionals includes

- the quasilinear mean operators,\(^{11,12}\)
\[
F_{\phi p}(a_1, a_2, \ldots, a_n) = \phi^{-1} \left( \sum_{i=1}^{n} p_i \phi(a_i) \right)
\] (17)

built up starting from a nonnegative weighting vector \( p = (p_1, p_2, \ldots, p_n) \) verifying the condition \( \sum p_i = 1 \) and a strict monotonic function \( \phi \) on \( \mathbb{R}_+^n \).
the ordered quasilinear means,

\[ O_{\Phi}(a_1, a_2, \ldots, a_n) = F_{\Phi}(a_1', a_2', \ldots, a_n') = \phi^{-1}\left(\sum_{i=1}^{n} p_i \phi(a_i')\right) \]  

obtained by applying the quasilinear mean (17) to the decreasing rearrangement \((a_1', a_2', \ldots, a_n')\) of \(a = (a_1, a_2, \ldots, a_n)\) (see also Ref. 13).

3. PROPERTY OF GENERALIZED CONSISTENCY

In this section we investigate the following properties lying between the condition of consistency \(\mathbf{c}\) and the condition of weak consistency \(\mathbf{wc}\):

\(\mathbf{qc}\)  \[ a_{ij} a_{jk} > a_{ij} a_{js} \iff a_{ik} > a_{rs} \] (quasi consistency)

\(\mathbf{qc}'\)  \[(i \neq r, k \neq s) \Rightarrow (a_{ij} a_{jk} > a_{ij} a_{js} > 1 \iff a_{ik} > a_{rs} > 1)\]

\(\mathbf{qc}_0\)  \[ a_{ij} a_{jk} = a_{ij} a_{js} \iff a_{ik} = a_{rs} \]

\(\delta\)  \[ a_{ik} > a_{rs} \Rightarrow a_{ir} > a_{ks} \] (index-exchangeability)

\(\delta'\)  \[ (\{i, k\} \cap \{r, s\} = \emptyset) \Rightarrow (a_{ik} > a_{rs} > 1 \Rightarrow a_{ir} > a_{ks}) \]

\(\delta_0\)  \[ a_{ik} = a_{rs} \iff a_{ir} = a_{ks} \]

\(\epsilon_1\)  \[ (a_{ij} > a_{ir} \text{ and } a_{jk} \geq a_{is}) \Rightarrow a_{ik} > a_{rs} \]

\(\epsilon_2\)  \[ (a_{ij} \geq a_{ir} \text{ and } a_{jk} > a_{is}) \Rightarrow a_{ik} > a_{rs} \]

\(\epsilon_0\)  \[ (a_{ij} = a_{ir} \text{ and } a_{jk} = a_{is}) \Rightarrow a_{ik} = a_{rs} \]

Proposition 3. The following statements hold:

\[ \mathbf{qc} \Rightarrow \mathbf{qc}_0, \quad \delta \Rightarrow \delta_0, \quad \epsilon_1 \iff \epsilon_2 \iff \epsilon_0 \]

Proof. The first two implications are straightforward and \(\epsilon_1 \iff \epsilon_2\) because of condition \(\mathbf{r}\) for which \(a_{jk} > a_{is} \iff a_{st} > a_{kj}\) and \(a_{ij} \geq a_{ir} \iff a_{rt} \geq a_{ji}\).

To prove that \(\epsilon_1\) implies \(\epsilon_0\), let us assume \(\epsilon_1\) and

\[ a_{ij} = a_{rt}, \quad a_{jk} = a_{is} \]  

Suppose, ab absurdo, that \(a_{ik} > a_{rs}\); then, by \(\epsilon_1\) and the equivalence \(a_{kj} = a_{st} \iff a_{jk} = a_{is}\), we get the inequality \(a_{ij} > a_{jr}\), which contradicts the assumption (19). If we suppose, ab absurdo, \(a_{rs} > a_{ik}\), then, by \(\epsilon_1\) and the equality \(a_{kj} = a_{st}\), we get \(a_{rt} > a_{ij}\); this inequality also contradicts (19). So \(a_{rt} = a_{ij}\) and the implication \(\epsilon_1 \Rightarrow \epsilon_0\) is proved.

Proposition 4. \(\mathbf{c} \Rightarrow \mathbf{qc} \Rightarrow \delta \Rightarrow \epsilon_1 \Rightarrow \mathbf{wc}\).

Proof. \(\mathbf{c} \Rightarrow \mathbf{qc}\). It is straightforward.
\(q_c \Rightarrow \delta\). By \(r\): \(1/a_{rj} = a_{jr}\). So by applying \(q_c\) and \(r\) we get \(\delta\) as follows:

\[
\begin{align*}
  a_{jk} > a_{rs} &\iff a_{ij} a_{jk} > a_{rj} a_{js} \iff a_{ij} a_{jr} > a_{kj} a_{js} \iff a_{ir} > a_{ks} \\
  \delta &\Rightarrow \epsilon_1.\end{align*}
\]

Applying condition \(\delta\) two times we get \(\epsilon_1\) as follows:

\[
\begin{align*}
  (a_{ij} > a_{rr} \text{ and } a_{jk} \geq a_{rs}) &\Rightarrow (a_{ir} > a_{jr} \geq a_{ks}) \Rightarrow a_{ik} > a_{rs} \\
  \epsilon_1 &= wc.\end{align*}
\]

Assume \(a_{ij} > 1\) and \(a_{jk} > 1\). Then, by \(\epsilon_2\) and \(\epsilon_1\), we get

\[
\begin{align*}
  (a_{ij} \geq a_{ij} \text{ and } a_{jk} > a_{jj}) &\Rightarrow a_{ik} > a_{ij}, \quad (a_{ij} > a_{jj} \text{ and } a_{jk} \geq a_{jk}) \Rightarrow a_{ik} > a_{jk}
\end{align*}
\]

As a consequence, \(a_{ik} > a_{ij} \lor a_{jk}\).

The reverse implications do not hold (see examples in the Appendix).

**Proposition 5.** \(q_c \Leftrightarrow (wc \text{ and } q_c')\), \(\delta \Leftrightarrow (wc \text{ and } \delta')\).

**Proof.** The direct implications follow from Proposition 4, for which each one of the conditions \(q_c\) and \(\delta\) implies \(wc\) and, in regard to the first implication, from Corollary 1, item 2, by which, if \(wc\), then \(a_{rj} a_{js} > 1 \iff a_{rs} > 1\). Let us show the inverse implications.

\((wc \text{ and } q_c') \Rightarrow q_c\). If \(wc\), then the condition \(q_c\) is verified in each one of the cases \(i = r\) and \(k = s\). Indeed, for \(k = s\), the equivalence \(q_c\) becomes \(a_{ij} > a_{rj} \iff a_{is} > a_{rs}\), which is true because of item 2 of Corollary 1; item 3 of the same corollary supplies the validity of \(q_c\) in the case \(i = r\). Thus, to complete the proof, we have to show the equivalence \(q_c\) for \(i \neq r, k \neq s\), in the following cases:

**Case 1.** \(a_{ij} a_{jk} > 1 \geq a_{rj} a_{js}\).

**Case 2.** \(1 \geq a_{ij} a_{jk} > a_{rj} a_{js}\).

By \(wc\) and Corollary 1, we get in case 1

\[
\begin{align*}
  a_{ij} a_{jk} > 1 \geq a_{rj} a_{js} &\iff a_{ij} > a_{k_j} \quad \text{and} \quad a_{rj} \leq a_{sj} \iff a_{ik} > 1 \geq a_{rs}
\end{align*}
\]

Accordingly with this equivalence and \(q_c'\), we also get

\[
\begin{align*}
  a_{ij} a_{jk} > a_{rj} a_{js} \geq 1 \iff a_{ik} > a_{rs} \geq 1
\end{align*}
\]

So, in case 2, the assertion is proved by applying the condition \(r\) and (20).

\((wc \text{ and } \delta') \Rightarrow \delta\). The condition \(\delta\) is trivially verified in the case \(k = r\), whereas in the case \(i = s\) it is a consequence of the property \(r\). Let us prove the implication \(\delta\) in each one of the cases, \(i = r\) and \(k = s\). If \(k = s\), then \(\delta\) becomes \(a_{is} > a_{rs} \Rightarrow a_{ir} > 1\) and this implication is true because of item 2 of Corollary 1. Analogously, by applying item 3 in Corollary 1 we get the validity \(\delta\) in the case \(i = r\). So it remains to verify that \(a_{ir} > a_{ks}\) in each one of the following cases:

\[\text{International Journal of Intelligent Systems} \quad \text{DOI} 10.1002/int\]
Case 3. \( a_{ik} > a_{rs} = 1 \).

Case 4. \( 1 \geq a_{ik} > a_{rs} \).

Case 5. \( a_{ik} > 1 > a_{rs} \).

In case 3, \( r = s \) and by Corollary 1,
\[
a_{ik} > 1 \iff a_i > a_k \iff a_{kr} = a_{ks}
\]
By this result and \( \delta' \) we also get
\[
a_{ik} > a_{rs} \geq 1 \Rightarrow a_{ir} > a_{ks}
\] (21)

In case 4, \( a_{sr} > a_{ki} \geq 1 \) because of the condition of reciprocity \( r \), and, by (21), \( a_{sk} > a_{ri} \), that is, \( a_{ir} > a_{ks} \). Hence
\[
1 \geq a_{ik} > a_{rs} \Rightarrow a_{ir} > a_{ks}
\] (22)

Finally, in case 5, \( a_{ik} > a_{rr} = 1 \) and \( a_{kk} = 1 > a_{rs} \); so applying (21) and (22) we get
\[
a_{ir} > a_{kr} > a_{ks}
\]

3.1. Generalized Consistency and Intensity Vectors

**Proposition 6.** The existence of an intensity vector implies each one of the conditions \( \delta, \varepsilon_1, \) and \( \varepsilon_2 \).

**Proof.** Let \( \boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in R^n_+ \) be an intensity vector. Then
\[
a_{ik} > a_{rs} \iff \frac{w_i}{w_k} > \frac{w_r}{w_s} \iff \frac{w_i}{w_r} > \frac{w_k}{w_s} \iff a_{ir} > a_{ks}
\]
So \( \delta \) is verified and, because of Proposition 4, the assertion is proved.

**Corollary 2.** The existence of an intensity vector for the matrix \( A \) implies that \( A \) is a priority matrix verifying the condition \( \delta \).

**Proof.** This follows from Proposition 6 and Theorem 1.

Example 2 in the Appendix shows that a weakly consistent matrix does not ensure the existence of an intensity vector.

**Proposition 7.** A vector \( \boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in R^n_+ \) is an intensity vector if and only if it is a coherent priority vector and
\[
a_{ik} > a_{rs} > 1 \iff \frac{w_i}{w_k} > \frac{w_r}{w_s} > 1
\] (23)
Proof. It is enough to show that if $w$ is a coherent priority vector and (23) holds, then (4) is verified in each one of the following cases:

Case 6. $a_{ik} > 1 \geq a_{rs}$.

Case 7. $1 \geq a_{ik} > a_{rs}$.

In case 6, as $w$ is a coherent priority vector, we get

$$a_{ik} > 1 \geq a_{rs} \iff w_i > w_k \quad \text{and} \quad w_r \leq w_s \iff \frac{w_i}{w_k} > 1 \geq \frac{w_r}{w_s}$$

In case 7, the equivalence (4) is trivial for $a_{ik} = 1$, that is, $i = k$, and, for $a_{ik} < 1$, it follows by applying the condition $r$ and the equivalence (23).

**Corollary 3.** Let $A$ be weakly consistent and $w = (w_1, w_2, \ldots, w_n)$ a coherent priority vector for $A$. Then $w$ is an intensity vector if and only if the equivalence (23) is true for $i \neq r$ and $k \neq s$.

**Proof.** By Proposition 7 it is enough to show that the equivalence (4) defining an intensity vector holds in the cases $i = r$ and $k = s$. By item 3 in Corollary 1 and condition $r$, if $i = r$, then (4) becomes $a_{ks} < 1 \iff w_k < w_s$ and is true because $w$ is a coherent priority vector; if $k = s$ the validity of (4) follows by item 2 in Corollary 1.

By definition, $A = (a_{ij})$ is an intensity matrix if and only if each column $a^j$ is an intensity vector.

**Proposition 8.** $A$ verifies $qc$ if and only if $A$ is an intensity matrix.

**Proof.** The condition $qc$ can be written

$$\left( a_{ik} > a_{rs} \iff \frac{a_{ij}}{a_{kj}} > \frac{a_{rj}}{a_{sj}} \quad \forall j = 1, \ldots, n \right)$$

and this proves that each column $a^j$ of $A$ is an intensity vector.

\section*{4. CHARACTERIZATION OF THE INTENSITY MATRICES}

**Lemma 1.** The condition $qc$ is equivalent to each one of the following conditions:

1) $a_{ik} > a_{rs} \iff a_i / a_k > a_r / a_s$

2) $w^c$ and “$(i \neq r, k \neq s) \Rightarrow (a_{ik} > a_{rs} > 1 \iff a_i / a_k > a_r / a_s > 1)$”

Moreover if $qc$, then $a_{ik} = a_{rs} \iff a_i / a_k = a_r / a_s$.
Proof. \( qc \iff j \). The assertion follows by the equivalence (24) expressing the condition \( qc \) (see proof of Proposition 8) and the definition of \( \gg \).

As \( qc' \) can be written

\[
(i \neq r, k \neq s) \Rightarrow \left( a_{ik} > a_{rs} > 1 \iff \frac{a_{ij}}{a_{kj}} > \frac{a_{ij}}{a_{sj}} > 1 \ \forall j = 1, \ldots, n \right)
\]

the assertion follows by the equivalence \( qc \leftrightarrow (wc \text{ and } qc') \) (see Proposition 5) and the definition of \( \gg \).

The last assertion follows from the implication \( qc \Rightarrow qc'_0 \) (see Proposition 3) for which if \( qc \), then \( a_{ik} = a_{rs} \Rightarrow a_{ij}/a_{kj} = a_{ij}/a_{sj} \ \forall j = 1, \ldots, n \). \( \blacksquare \)

The next theorem improves a result stated in Ref. 10.

**Theorem 3.** The following statements are equivalent:

\[(k) A \text{ is an intensity matrix}

(kk) the set \( \tilde{A}_+ = \{a_i/a_j : a_i, a_j \in \tilde{A}\} \) is totally ordered by \( \gg \)

(kkk) \( A \) is a priority matrix and the set \( A_+ = \{a_i/a_j \in \tilde{A}_+ : a_i/a_j \gg 1\} \) is totally ordered by \( \gg \).

**Proof.** \( A \) is an intensity matrix if and only if it is quasi consistent (see Proposition 8) and the rows of a weakly consistent matrix are completely ordered by \( \gg \) (see Theorem 1). Then the assertion follows by Lemma 1.

**Proposition 9.** Let \( A = (a_{ij}) \) be an intensity matrix and \( p = (p_1, p_2, \ldots, p_n) \) a weighting vector verifying the conditions \( p_i \geq 0 \ \forall i \in \{1, 2, \ldots, n\} \) and \( \sum_{i=1}^{n} p_i = 1 \). Then the weighted geometric mean operator

\[
F_{\ln p}(a_1, a_2, \ldots, a_n) = a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}
\]

is an intensity operator.

**Proof.** Assume that \( F \) is the functional (25). By Proposition 8 \( A \) verifies \( qc \) and by Lemma 1 \( a_{ik} > a_{rs} \iff a_i/a_k \gg a_r/a_s \). Hence by the definition of \( \gg \) and the equalities

\[
\frac{F(a_i)}{F(a_k)} = \left( \frac{a_{i1}}{a_{k1}} \right)^{p_1} \left( \frac{a_{i2}}{a_{k2}} \right)^{p_2} \cdots \left( \frac{a_{in}}{a_{kn}} \right)^{p_n}
\]

and

\[
\frac{F(a_r)}{F(a_s)} = \left( \frac{a_{r1}}{a_{s1}} \right)^{p_1} \left( \frac{a_{r2}}{a_{s2}} \right)^{p_2} \cdots \left( \frac{a_{rn}}{a_{sn}} \right)^{p_n}
\]
we get
\[ a_{ik} > a_{rs} \Rightarrow \frac{F(a_i)}{F(a_k)} > \frac{F(a_r)}{F(a_s)} \quad a_{ik} = a_{rs} \Rightarrow \frac{F(a_i)}{F(a_k)} = \frac{F(a_r)}{F(a_s)} \quad (26) \]

The implications (26) are actually equivalences. Indeed just one of the relations
\[ a_{ik} > a_{rs}, \quad a_{ik} = a_{rs}, \quad \text{and} \quad a_{ik} < a_{rs} \]
is true and, by the implications (26), if
\[ \frac{F(a_i)}{F(a_k)} > \frac{F(a_r)}{F(a_s)} \]
then necessarily \( a_{ik} > a_{rs} \) and, if \( \frac{F(a_i)}{F(a_k)} = \frac{F(a_r)}{F(a_s)} \), then \( a_{ik} = a_{rs} \).

**Proposition 10.** Let \( A = (a_{ij}) \) be an intensity matrix. Then the min operator \( F = F_\wedge \) and the max operator \( F = F_\vee \) are intensity operators.

**Proof.** It is enough to observe that the matrix \( A \) is a priority matrix, and, by Theorem 1, if \( x_{i_1} > x_{i_2} > \cdots > x_{i_n} \) then \( w_{F_\wedge} = a_{i_1}^{i_n} \) and \( w_{F_\vee} = a_{i_1} \).

5. **CONCLUSIONS**

The condition \( wc \) and the stronger conditions of Section 3 are weaker than the condition of consistency \( c \). The condition \( wc \) characterizes the Rts matrices whose columns are coherent priority vectors, and item ii of Theorem 1 provides a tool to recognize this kind of matrices; \( qc \) characterizes the matrices whose columns are intensity vectors (see Proposition 8): these matrices are particular weakly consistent matrices and item kkk of Theorem 3 provides a simple tool to verify if a weakly consistent matrix is an intensity matrix. Of course \( qc \) is a sufficient condition for the existence of an intensity vector but it is not a necessary condition (see Example 5). On the other hand the condition \( wc \) and the stronger conditions \( \delta, \epsilon_1 \) are necessary conditions for the existence of an intensity vector but they are not sufficient. So it remains to better analyze the role played by the conditions introduced in Section 3 and to look for a condition weaker than the quasi consistency and equivalent to the existence of intensity vectors.

**References**

In this Appendix we construct comparison matrices in order to clarify some
links among the properties investigated in the article and related to the existence
of intensity vectors.

**Example 1.** Let us consider the matrix

\[
A = \begin{pmatrix}
1 & 4 & 7 & 15/2 & 9 \\
1/4 & 1 & 5 & 6 & 8 \\
1/7 & 1/5 & 1 & 3 & 4 \\
2/15 & 1/6 & 1/3 & 1 & 2 \\
1/9 & 1/8 & 1/4 & 1/2 & 1
\end{pmatrix}
\]

It is \( n(a_1) = 4 > n(a_2) = 3 > n(a_3) = 2 > n(a_4) = 1 > n(a_5) = 0 \). So \( A \) is Rts
and the actual ranking is represented by the chain \( x_1 > x_2 > x_3 > x_4 > x_5 \) (see
Proposition 1). \( A \) is also weakly consistent because \( a_1 \triangleright a_2 \triangleright a_3 \triangleright a_4 \triangleright a_5 \) (see
Theorem 1). Nevertheless \( A \) does not verify \( e_1 \) because \( a_{13} \not\geq a_{24} \) and \( a_{34} > a_{45} \) but \( a_{14} < a_{25} \). So an intensity vector does not exist.

**Example 2.** The matrix

\[
A = \begin{pmatrix}
1 & 4 & 7 & 8 \\
1/4 & 1 & 5 & 6 \\
1/7 & 1/5 & 1 & 4 \\
1/8 & 1/6 & 1/4 & 1
\end{pmatrix}
\]

is Rts, and the actual ranking is represented by the chain \( x_1 > x_2 > x_3 > x_4 \) because
\( n(a_1) = 3 > n(a_2) = 2 > n(a_3) = 1 > n(a_4) = 0 \) (see Proposition 1). \( A \) is also
weakly consistent because \( a_1 \triangleright a_2 \triangleright a_3 \triangleright a_4 \) (see Theorem 1). We stress that \( A \)
does not verify \( \delta \) because \( a_{13} \not\geq a_{24} \), but \( a_{12} = a_{34} \). Nevertheless \( A \) verifies \( e_1 \).
Hence, we get that the condition $\varepsilon_1$ does not imply the condition $\delta$ and, because of Proposition 6, $\varepsilon_1$ does not ensure the existence of an intensity vector. This proves also that a priority matrix may not be an intensity matrix.

**Example 3.** The matrix

$$
A = \begin{pmatrix}
1 & 4 & 7 & 8 & 9 \\
1/4 & 1 & 5 & 6 & 13/2 \\
1/7 & 1/5 & 1 & 3 & 5/2 \\
1/8 & 1/6 & 1/3 & 1 & 2 \\
1/9 & 2/13 & 2/5 & 1/2 & 1
\end{pmatrix}
$$

is a priority matrix because $a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5$ (see Theorem 1) and verifies the condition $\delta$. $A$ does not verify $q_c$ because $a_{12} > a_{34}$, but $a_{13}/a_{23} = 7/5 < a_{33}/a_{43} = 3$.

**Example 4.** The weakly consistent matrix

$$
A = \begin{pmatrix}
1 & 9/7 & 3 & 7 \\
7/9 & 1 & 2 & 4 \\
1/3 & 1/2 & 1 & 2 \\
1/7 & 1/4 & 1/2 & 1
\end{pmatrix}
$$

verifies the condition $q_c'$. So $A$ is a quasi consistent matrix (see Proposition 5) or an intensity matrix. Nevertheless $A$ is not consistent because $a_{12}a_{23} = 18/7 < a_{13} = 3$ (then the second column of the matrix is not a consistent vector).

**Example 5.** The matrix

$$
A = \begin{pmatrix}
1 & 9/8 & 3 & 8 \\
8/9 & 1 & 2 & 4 \\
1/3 & 1/2 & 1 & 2 \\
1/8 & 1/4 & 1/2 & 1
\end{pmatrix}
$$

induces the ranking $x_1 > x_2 > x_3 > x_4$ on the set of alternatives $X = \{x_1, x_2, x_3, x_4\}$. $A$ is a priority matrix because each column represents the actual ranking, but it is not an intensity matrix because it does not verify the condition $q_c$: indeed it is $a_{14}a_{42} = 2 = a_{24}a_{43}$, but $a_{12} = 9/8 < a_{23} = 2$. Nevertheless $a^3$ is an intensity vector.