ASYMPTOTIC TRAFFIC FLOW IN AN HYPERBOLIC NETWORK II: NON-UNIFORM TRAFFIC

YULIY BARYSHNIKOV AND GABRIEL H. TUCCI

Abstract. In this work we study the asymptotic traffic behaviour in Gromov’s hyperbolic spaces when the traffic decays exponentially with the distance. We prove that under general conditions, there exist a phase transition between local and global traffic. More specifically, assume that the traffic rate between two nodes $u$ and $v$ is given by $R(u,v) = \beta^{-d(u,v)}$ where $d(u,v)$ is the distance between the nodes, then there exists a constant $D$ that depends on the geometry of the network such that if $1 < \beta < D$ the traffic is global and there is a small set of highly congested nodes called the core. However, if $\beta > D$ then the traffic is essentially local and the core is empty.

1. Introduction

The structure of networks has been mainly the domain of a branch of discrete mathematics known as graph theory. Some basic ideas, used later by physicists, were proposed in 1959 by the Hungarian mathematician Paul Erdős and his collaborator Rényi. Graph theory has witnessed many exciting developments and has provided answers to a series of practical questions such as: what is the maximum flow per unit time from source to sink in a network of pipes or how to color the regions of a map using the minimum number of colours so that neighbouring regions receive different colours. In addition to the developments in mathematical graph theory, the study of networks has seen important achievements in some specialized contexts, as for instance in the social sciences. Most of the results of graph theory relevant to large complex networks, are related to the simplest models of random graphs.

Recent years however have witnessed a substantial new movement in network research, with the focus shifting away from the analysis of single small graphs and the properties of individual vertices or edges to considerations of “large scale” statistical properties. The great majority of real world networks, including the World Wide Web, the Internet, basic cellular networks, social networks and many others have a more complex architecture than classical random graphs. Abstracting the network details away allows one to concentrate on the phenomena intrinsically connected with the underlying geometry, and discover connections between the metric properties and the network characteristics. Over the past few years, there has been growing
evidence that many communication networks have characteristics of negatively curved spaces [10, 9, 11, 12, 13]. From the large scale point of view, it has been experimentally observed that, on the Internet and other networks, traffic seems to concentrate quite heavily on some very small subsets.

We believe that many of the complex real world networks have characteristics of negatively curved or more generally Gromov’s hyperbolic spaces. In Figure 1 and Figure 2, we see a picture of the World Wide Web and the Internet network that suggest an hyperbolic structure. In this project we continue the analysis and approach done in [1]. We study the traffic behaviour for large Gromov hyperbolic spaces when the traffic rate decays exponentially with the metric distance between the nodes.

In Section 2 we review the concept of Gromov’s hyperbolic space and present some of the important examples and properties. We also recall the construction of the boundary of an hyperbolic space, its visual metric and harmonic measure. In Section 3 we study the traffic phenomena in a general locally finite tree when the rate decays exponentially with the distance. Finally, in Section 4 we study the asymptotic traffic behaviour in general Gromov’s hyperbolic graphs when the traffic decays exponentially with the distance. We prove that under general conditions there exist a phase transition between local and global traffic. More specifically, assume that the traffic rate between two nodes $u$ and $v$ is given by $R(u, v) = \beta^{-d(u,v)}$ where $d(u,v)$ is the distance between the nodes. We show that there exists a constant $D$ such that if $1 < \beta < D$ the traffic is global and there is a small set of
highly congested nodes called the core. However, if $\beta > D$ then the traffic is essentially local and the core is empty. This implies in particular, that polynomially decaying rate functions do not affect the locality of the traffic, and the existence or non–existence of a core.

Acknowledgement: We would like to thank Iraj Saniiee for many helpful discussions and comments. This work was supported by AFOSR Grant No. FA9550-08-1-0064.

2. Preliminaries

In this Section we review the notion of Gromov $\delta$–hyperbolic space as well as some of the basic properties, Theorems and constructions.

2.1. $\delta$–Hyperbolic Spaces. There are many equivalent definitions of Gromov’s $\delta$–hyperbolicity but the one we will take as our definition is the property that triangles are slim.

Definition 2.1. Let $\delta > 0$. A geodesic triangle in a metric space $X$ is said to be $\delta$–slim if each of its sides is contained in the $\delta$–neighbourhood of the union of the other two sides. A geodesic space $X$ is said to be $\delta$–hyperbolic if every triangle in $X$ is $\delta$–slim.

It is easy to see that any tree is 0-hyperbolic. Other examples of hyperbolic spaces include, any finite graph, the fundamental group of a surface of genus greater or equal than 2, the classical hyperbolic space, and any regular tessellation of the hyperbolic space (i.e. infinite planar graphs with uniform degree $q$ and $p$–gons as faces with $(p−2)(q−2) > 4$).

Definition 2.2. (Hyperbolic Group) A finitely generated group $\Gamma$ is said to be word–hyperbolic if there is a finite generating set $S$ such that the Cayley graph $C(\Gamma, S)$ is $\delta$–hyperbolic with respect to the word metric for some $\delta$. 
It turns out that if $\Gamma$ is a word hyperbolic group then for any finite generating set $S$ of $\Gamma$ the corresponding Cayley graph is hyperbolic, although the hyperbolicity constant depends on the choice of $S$.

**Definition 2.3. (Gromov Product)** Let $(X, d)$ be a metric space. For $x, y$ and $z \in X$ we define

$$(y, z)_x := \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

We will call $(y, z)_x$ the Gromov product of $y$ and $z$ with respect to $x$.

In hyperbolic metric spaces the Gromov product measures how long two geodesics travel close together. Namely if $x, y$ and $z$ are three points in a $\delta$ hyperbolic metric space $(X, d)$, then the initial segments of length $(y, z)_x$ of any two geodesics $[x, y]$ and $[x, z]$ are $2\delta$ Hausdorff close. Moreover, in the case of Gromov product $(y, z)_x$ approximates within $2\delta$ the distance from $x$ to a geodesic $[y, z]$.

### 2.2. Boundary of Hyperbolic Spaces.

We will say that two geodesic rays $\gamma_1 : [0, \infty) \to X$ and $\gamma_2 : [0, \infty) \to X$ are equivalent and write $\gamma_1 \sim \gamma_2$ if there is $K > 0$ such that for any $t \geq 0$

$$d(\gamma_1(t), \gamma_2(t)) \leq K.$$  

It is easy to see that $\sim$ is indeed an equivalence relation on the set of geodesic rays. Moreover, two geodesic rays $\gamma_1, \gamma_2$ are equivalent if and only if their images have finite Hausdorff distance. The Hausdorff distance is defined as the infimum of all the numbers $H$ such that the images of $\gamma_1$ is contained in the $H$–neighbourhood of the image of $\gamma_2$ and vice versa.

The boundary is usually defined as the set of equivalence classes of geodesic rays starting at the base–point, equipped with the compact–open topology. That is to say, two rays are “close at infinity” if they stay close for a long time. We will make this notion precise.

**Definition 2.4. (Geodesic Boundary)** Let $(X, d)$ be a $\delta$–hyperbolic metric space and let $x_0 \in X$ be a base–point. We will define the relative geodesic boundary of $X$ with respect to the base–point $x$ as

$$\partial_x X := \{[\gamma] : \gamma : [0, \infty) \to X \text{ is a geodesic ray with } \gamma(0) = x\}. \quad (2.1)$$

It turns out that the boundary has a natural metric.

**Definition 2.5.** Let $(X, d)$ be a $\delta$–hyperbolic metric space. Let $a > 1$ and let $x_0 \in X$ be a base–point. We will say that a metric $d_a$ on $\partial X$ is a visual metric with respect to the base point $x_0$ and the visual parameter $a$ if there is a constant $C > 0$ such that the following holds:

1. The metric $d_a$ induces the canonical boundary topology on $\partial X$. 

For any two distinct points \( p, q \in \partial X \), for any bi-infinite geodesic \( \gamma \) connecting \( p, q \) in \( X \) and any \( y \in \gamma \) with \( d(x_0, \gamma) = d(x_0, y) \) we have:

\[
\frac{1}{C} a^{-d(x_0, y)} \leq d_a(p, q) \leq C a^{-d(x_0, y)}.
\]

**Theorem 2.6.** ([6], [7]) Let \((X, d)\) be a \( \delta \)-hyperbolic metric space. Then:

1. There is \( a_0 > 1 \) such that for any base point \( x_0 \in X \) and any \( a \in (1, a_0) \) the boundary \( \partial X \) admits a visual metric \( d_a \) with respect to \( x_0 \).
2. Suppose \( d' \) and \( d'' \) are visual metrics on \( \partial X \) with respect to the same visual parameter \( a \) and the base points \( x'_0 \) and \( x''_0 \) accordingly. Then \( d' \) and \( d'' \) are Lipschitz equivalent, that is there is \( L > 0 \) such that

\[
\frac{d'(p, q)}{L} \leq d''(p, q) \leq L d'(p, q)
\]

for any \( p, q \in \partial X \).

The metric on the boundary is particularly easy to understand when \((X, d)\) is a tree. In this case \( \partial X \) is the space of ends of \( X \). The parameter \( a_0 \) from the above proposition is \( a_0 = \infty \) here and for some base point \( x_0 \in X \) and \( a > 1 \) the visual metric \( d_a \) can be given by an explicit formula:

\[
d_a(p, q) = a^{-d(x_0, y)}
\]

for any \( p, q \in \partial X \) where \([x_0, y] = [x_0, p] \cap [x_0, q]\) so that \( y \) is the bifurcation point for the geodesic rays \([x_0, p]\) and \([x_0, q]\).

Here are some more examples of boundaries of hyperbolic spaces (for more on this topic see [6] [7] [8].)

**Example 2.7.**

1. If \( \Gamma \) is a finite graph then \( \partial \Gamma = \emptyset \).
2. If \( \Gamma = \mathbb{Z} \), the infinite cyclic group, then \( \partial \Gamma \) is homeomorphic to the set \( \{0, 1\} \) with the discrete topology.
3. If \( n \geq 2 \) and \( \Gamma = \mathbb{F}_n \), the free group of rank \( n \), then \( \partial \Gamma \) is homeomorphic to the space of ends of a regular \( 2n \)-valent tree, that is to a Cantor set.
4. Let \( S_g \) be a closed oriented surface of genus \( g \geq 2 \) and let \( \Gamma = \pi_1(S_g) \). Then \( \Gamma \) acts geometrically on the hyperbolic plane \( \mathbb{H}^2 \) and therefore the boundary is homeomorphic to the circle \( S^1 \).
5. Let \( M \) be a closed \( n \)-dimensional Riemannian manifold of constant negative sectional curvature and let \( \Gamma = \pi_1(M) \). Then \( \Gamma \) is word hyperbolic and \( \partial \Gamma \) is homeomorphic to the sphere \( S^{n-1} \).
6. The boundary of the classical \( n \) dimensional hyperbolic space \( \mathbb{H}^n \) is \( S^{n-1} \).

**2.3. Harmonic Measures.** Characterizing special subclasses of hyperbolic groups such as co-compact Kleinian groups often requires the construction of special metrics and measures on the boundary which carry some geometrical information. For example, Bonk and Kleiner proved that a group admits a co-compact Kleinian action on the hyperbolic space \( \mathbb{H}^n \), \( n \geq 3 \), if and only if its boundary has topological dimension \( n - 1 \) and carries an Ahlfors-regular metric of dimension \( n - 1 \) (for more details on this see [3]).
There are two main constructions of measures on the boundary of a Gro-mov’s hyperbolic space: quasi–conformal measures and harmonic measures. In this paper we will only use the notion of harmonic measure that we will describe below.

2.3.1. Harmonic Measures. Let \( X \) be an infinite graph that we will assume is simple (no loops, or multiple edges) and locally finite (every node in the graph has finite degree). Let \( x_0 \) be a fixed base-point in the graph. Given a probability measure \( \mu \) on \( X \), consider the random walk \( (Z_n)_n \) starting from a fixed base-point \( x_0 \) with initial probability distribution \( \mu \). Under some mild assumptions on \( \mu \), the random walk \( (Z_n)_n \) almost surely converges to a point \( Z_\infty \in \partial X \). The law of \( Z_\infty \) is by definition the harmonic measure \( \nu \) associated with the law \( \mu \).

Many work has been done in trying to understand the properties of these measures. In particular, it has been proved that for general hyperbolic groups, the Hausdorff dimension of the harmonic measure can be explicitly computed and satisfies a “dimension–entropy–rate of escape” formula [2].

2.3.2. Dimension of the Harmonic Measure.

Definition 2.8. The sequence \( (Z_n)_n \) of random variables previously defined determine two asymptotic quantities. The \textbf{asymptotic entropy}

\[
h := \lim_{n} - \frac{\sum_{\gamma \in \Gamma} \mu^n(\gamma) \log \mu^n(\gamma)}{n} = \lim_{n} - \frac{\sum_{\gamma \in \Gamma} P[Z_n = \gamma] \log P[Z_n = \gamma]}{n}
\]

which measures the way the law \( (Z_n)_n \) is spread in different directions, and the \textbf{rate of escape or drift}

\[
l := \lim_{n} \frac{d(1, Z_n)}{n}
\]

which estimates how far \( Z_n \) is from its initial point. The above limits for \( h \) and \( l \) are almost surely and in \( L^1 \) and they are finite as soon as the law \( \mu \) has finite first moment (see [2]).

Let \( (X, d) \) be a complete metric space. One defines the \( \alpha \)–Hausdorff measure of a set \( Z \subset X \) as

\[
m_H(Z, \alpha) := \liminf_{\epsilon \to 0} \sum_{U \in G_\epsilon} (\text{diam}(U))^\alpha,
\]

the infimum being taken over all the covers \( G_\epsilon \) of \( Z \) by open sets of diameter at most \( \epsilon \). The usual Hausdorff dimension of \( Z \) is taken

\[
\dim_H(Z) = \inf\{\alpha : m_H(Z, \alpha) = 0\} = \sup\{\alpha : m_H(Z, \alpha) = +\infty\}.
\]

When \( m_H(X, \dim_H(X)) \) is finite and non zero, the function

\[
Z \to m_H(Z, \dim_H(X))
\]

is after normalization a probability measure on \( X \), called the Hausdorff measure.
2.3.3. Dimension of Measures. Let $\nu$ be a probability measure on $(X, d)$. We define the Hausdorff of $\nu$ as:

$$\dim_H(\nu) := \inf\{\dim_H(Z) : \nu(Z) = 1\}.$$ 

**Remark 2.9.** The dimension of a measure $\nu$ characterizes to some extent its type. Indeed if $\lambda$ is absolutely continuous wrt $\nu$, so $\dim_H(\lambda) \leq \dim_H(\nu)$.

To estimate the dimension of a set, and a fortiori of a measure, is usually not easy. There is however a more direct way to estimate the dimension of a measure, introducing the (lower and upper) pointwise dimensions at a point $x$ as

$$\underline{\dim}_P \nu(x) := \liminf_{r \to 0} \frac{\log \nu(B(x,r))}{\log r},$$

and

$$\overline{\dim}_P \nu(x) := \limsup_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}.$$ 

Moreover, this notion allows a more intuitive vision of the dimension of a measure. To relate the Hausdorff and pointwise dimensions, we will need a condition on the space (see [17], appendix 1 for more on this). A probability measure with a constant $d$ such that $\underline{\dim}_P \nu(x) = \overline{\dim}_P \nu(x), \nu$–almost surely is said to be an exact dimensional. This is the case for a big class of hyperbolic groups.

2.3.4. Case of an Hyperbolic Group. Given an hyperbolic group $\Gamma$, it is natural to ask about the Hausdorff dimension of the limit set $\Lambda_\Gamma$. We define the critical exponent of base $a$ as

$$\delta_a(\Gamma) := \limsup_{R \to \infty} \frac{\log |\{\gamma \in \Gamma : d(1, \gamma) \leq R\}|}{R}. \quad (2.4)$$

**Theorem 2.10.** (See [2]) Let $\Gamma$ be a non–elementary hyperbolic group acting on its Cayley graph $(X, d)$. Let $d_a$ be a visual metric on the boundary $\partial X$, and let $B(x,r)$ be the ball of center $x \in \partial X$ and radius $r$ for the distance $d_a$. Let $\nu$ be the harmonic measure of a random walk $(Z_n)_n$ whose increments are given by a symmetric law $\mu$ with finite first moment. Then the pointwise Hausdorff dimension

$$\lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$$

exists for $\nu$–almost every $x \in \partial X$, and is independent from the choice of $x$. Moreover, for $\nu$–almost every $x \in \partial X$.

$$0 < \dim_H(\nu) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} = \frac{h}{\log(a)}l \quad (2.5)$$

where $l > 0$ denotes the rate of escape of the walk wrt $d$ and $h$ the asymptotic entropy of the walk.
Remark 2.11. In the case $X$ is the Cayley graph of $\Gamma$, the support of the harmonic measure is $\partial X$. The Hausdorff dimension $\partial X$ is equal to $\delta_a(\Gamma)$ (see [5]). We have a fundamental relation between $h$, $l$, and $v$ (see [2]):
\[
h \leq \log(a)\delta_a(\Gamma)l
\]
In view of this inequality and Theorem 2.10 we see that
\[
0 < \dim_H(\nu) \leq \delta_a(\Gamma).
\]

3. Asymptotic Traffic Flow in a Tree

In this Section, we study the asymptotic traffic behaviour in a locally finite tree when the traffic decays exponentially with the distance. More specifically, let $\{k_l\}_{l=0}^\infty$ be a sequence of positive integers with $k_0 = 1$. For each sequence like this we consider the infinite tree $T$ with the property that each element at depth $l$ has $k_{l+1}$ descendants. In other words, the root has $k_1$ descendants, each node in the first generation has $k_2$ descendants and so on. The root is considered the 0 generation. Let us denote by $T_n$ the finite tree generated by the first $n$ generations of $T$. Let $M = M(n)$ be the number of elements in $T_n$. It is clear that
\[
M(n) = 1 + k_1 + k_1k_2 + \ldots + k_1k_2\ldots k_n = \sum_{l=0}^n \prod_{i=0}^l k_i.
\]
For each fixed $n \geq 1$, assume that there is traffic between $\partial T_n$, the leaves of the truncated tree $T_n$. We will assume also that the traffic rate between $x_i$ and $x_j$ in $\partial T_n$ depends only on the distance between these two leaves and decays exponentially. More specifically,
\[
R(i, j) = \beta^{-d(x_i, x_j)} \quad \text{where } \beta > 1.
\]
Denote by $x_0$ the root of the tree. For simplicity let us first assume that the tree $T$ is $k$–regular which is equivalent to assume that $k_l = k$ for all $l \geq 1$. It is an easy observation to see that the number of elements of $\partial T_n$ is equal to $N(n) := |\partial T_n| = k^n$. Let us denote these points as $x_1, \ldots, x_N$. Let $n_p = |\{x_i : d(x_1, x_i) = p\}|$. Then
\[
n_p = \begin{cases} 
(k-1)k^{r-1} & \text{if } p = 2r \text{ for } 0 \leq r \leq n \\
0 & \text{otherwise}
\end{cases}
\]
The total traffic between the points $x_1, x_2, \ldots, x_N$ is
\[
T(n) = N \cdot \left( \sum_{p=0}^\infty n_p \beta^{-p} \right) = N \cdot \left( 1 + (k-1) \cdot \sum_{i=0}^{n-1} k^i \beta^{-2(i+1)} \right)
\]
\[
= N \cdot \left( 1 + \frac{k-1}{\beta^2} \cdot \frac{(k/\beta^2)^n - 1}{(k/\beta^2) - 1} \right)
\]
The total traffic passing through the root of the tree is \( N(k-1)^{k^{n-1} \beta^{-2n}} \). Hence the proportion of the traffic passing through the root of the tree is equal to

\[
P(n) = \frac{(k-1)^{k^{n-1} \beta^{-2n}}}{1 + \beta^{-2}(k-1)^{(k\beta-2)^{n-1}}}.\]

Here we can distinguish two cases. The first case is \( \beta \geq \sqrt{k} \). In this case

\[
\lim_{n \to \infty} P(n) = 0.
\]

The other case is \( 1 < \beta < \sqrt{k} \) in which

\[
\lim_{n \to \infty} P(n) = 1 - \frac{\beta^2}{k}.
\]

Note that this shows in particular that if the traffic decay is sub-exponential then the asymptotic proportion of the traffic through the root is \( 1 - \frac{1}{k} \). A similar analysis and conclusion can be carried out for the general tree \( T \) as long as there is an upper bound on the coefficients \( k_l \). We will deduce a more general Theorem in the next Section which includes this result as a particular case.

4. Asymptotic Traffic Flow in a \( \delta \)-Gromov Hyperbolic Graph

In this Section, we study the asymptotic traffic flow in a \( \delta \)-hyperbolic graph. Throughout this Section we will assume that \( X \) is an infinite, locally finite (every node has finite degree), simple (no loops or multiple edges) graph. Assume that there exists \( \delta > 0 \) such that \( X \) is Gromov \( \delta \)-hyperbolic. Let \( x_0 \in X \) be a fixed basepoint and let

\[
\{x_0\} = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots \subset X
\]

be a sequence of finite subsets with the properties that:

- \( \cup_{n \geq 1} X_n = X \),
- for every \( x \in X_n \) and for every geodesic segment \([0, x]\) connecting 0 and \( x \) then every intermediate point belongs to \( X_n \).

Denote as usual by \( \partial X_n \) the boundary set of \( X_n \) in \( X \) and recall that a point \( y \) belongs to \( \partial X_n \) if \( y \in X_n \) and there exists \( z \in X \setminus X_n \) such that \( z \sim y \) (\( z \) and \( y \) are adjacent).

We will assume that for each fixed \( n \geq 1 \), we have traffic going from \( \partial X_n \) to \( \partial X_n \). We will also assume that there is a non-increasing, and continuous function \( f : [0, \infty) \to [0, \infty) \) such that the traffic rate between \( x \) and \( y \) in \( \partial X_n \) is equal to

\[
R(x, y) = f(d(x, y)) \tag{4.1}
\]

where \( d(x, y) \) is the distance between these two points. The traffic flow will go through the geodesic connecting \( x \) and \( y \), and if there are more than one geodesic connecting these points we assume that the load is divided equally...
between the different paths. We will pay special attention to the case where \( f(t) = \beta^{-t} \) for \( \beta > 1 \).

Of central importance in this work will be the case where the sets \( \{X_n\}_n \) are balls. More precisely, assume that

\[
X_n := \{x \in X : d(x_0, x) \leq n\}. \tag{4.2}
\]

In this case it is clear that \( \partial X_n = \{x \in X : d(x_0, x) = n\} \). Recall from Section 2 that for any \( x \) and \( y \) in \( X \),

\[
(x, y)_{x_0} = \frac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y)),
\]

and

\[
h(x, y) = d(x_0, \gamma_{x,y}),
\]

where \( \gamma_{x,y} \) is the geodesic connecting \( x \) and \( y \) (if there is more than one geodesic connecting \( x \) and \( y \) then we consider the minimum).

It is not difficult to see that

\[
h(x, y) - 2\delta \leq (x, y)_{x_0} \leq h(x, y)_{x_0} + 2\delta.
\]

In particular, we see that for every pair of points \( x \) and \( y \) in \( \partial X_n \)

\[
n - \frac{d(x, y)}{2} - \frac{\delta}{2} \leq h(x, y) \leq n - \frac{d(x, y)}{2} + \frac{\delta}{2}. \tag{4.3}
\]

By Theorem 2.6 we know that exists \( a_0 > 1 \) such that for all \( a \in (1, a_0) \) the boundary \( \partial X \) admits a visual metric \( d_a \) with base point \( x_0 \). Hence, there exists \( C > 0 \) such that:

(1) The metric \( d_a \) induces the canonical boundary topology on \( \partial X \).
(2) For all $p \neq q \in \partial X$

$$
\frac{1}{C} a^{-h(p,q)} \leq d_a(p, q) \leq C a^{-h(p,q)}. \tag{4.4}
$$

Let $r > 0$ be fixed, and let $p$ and $q \in \partial X$ with $h(p,q) \leq r$ then

$$
\frac{1}{C} a^{-r} \leq d_a(p, q) \leq C a^{-r}. \tag{4.5}
$$

On the other hand, if $d_a(p, q) \geq C a^{-r}$ then $h(p,q) \leq r$. Note that $C$ is a positive fixed constant that only depends on $a$.

Let $p$ and $q \in \partial X$ and $x_n$ and $y_n \in \partial X_n$ such that $x_n \to p$ and $y_n \to q$ as $n$ goes to infinity. Then by equation (4.3)

$$
n - \frac{d(x_n, y_n) - \delta}{2} \leq h(x_n, y_n) \leq n - \frac{d(x_n, y_n) + \delta}{2}.
$$

Since $\lim_{n \to \infty} h(x_n, y_n) = h(p,q)$ we conclude that for $n$ sufficiently large

$$
2(t - h(p,q)) - \delta \leq d(x_n, y_n) \leq 2(t - h(p,q)) + \delta. \tag{4.6}
$$

Consider the random walk $\{Z_n\}_{n=0}^\infty$ such that $Z_0 = x_0$ and at every stage all the adjacent nodes are equally likely. In other words,

$$
\mathbb{P}(Z_{n+1} = y | Z_n = x) = \begin{cases} \frac{1}{\deg(x)} & \text{if } y \sim x \\ 0 & \text{otherwise} \end{cases} \tag{4.7}
$$

We know that almost surely the random walk will converge to a point in the boundary $\lim_{n \to \infty} Z_n = Z_\infty \in \partial X$, and the law of $Z_\infty$ defines an harmonic measure $\nu$ on $\partial X$. We will assume that

$$
0 < \dim_H(\nu) = \lim_{r \to 0} \frac{\log \nu(B(p, r))}{\log r} \tag{4.8}
$$

for $\nu$–almost every $p$ in $\partial X$. In the case where $X$ is a $\delta$–hyperbolic group this is true by Theorem 2.10, and moreover this measure satisfies

$$
0 < \dim_H(\nu) = \lim_{r \to 0} \frac{\log \nu(B(p, r))}{\log r} = \frac{h}{\log(a)l} \leq \delta_a(\Gamma)
$$

for $\nu$–almost every $p \in \partial \Gamma$ where $h$ and $l$ are the entropy and drift of the random walk, and $\delta(\Gamma)$ is the growth exponent of the group. For every integer $n \geq 1$, and every $x \in \partial X_n$ define

$$
L_x = |\{\text{geodesics connecting } x_0 \text{ and } x\}|,
$$

and

$$
R_n = |\{\text{geodesics starting at } x_0 \text{ of length } n\}|.
$$

Define the measure $\mu_n$ by

$$
\mu_n = \frac{1}{R_n} \sum_{x \in \partial X_n} L_x \delta_x, \tag{4.9}
$$
This measure is supported in $\partial X_n$. Each measure $\mu_n$ defines a visual Borel measure $\mu_v^n$ in the boundary $\partial X$. The way the measure $\mu_v^n$ is defined is the following.

**Definition 4.1.** Let $A \subseteq \partial X$ be a Borel subset. For each $a \in A$ consider all the sequences $\{x_k\}_{k=0}^{\infty}$ such that: $x_0 = 0$, the sequence is a geodesic ray in $X$ that converges to $a$. These sequences will be called rays connecting $0$ with $a$. Let $C_A$ be the set of points in $X$ that belong to some ray connecting $0$ with some $a \in A$. We define

$$\mu_v^n(A) := \mu_n(X_n \cap C_A).$$ (4.10)

It is not difficult to see that the visual components of the measure $\mu_n$ converge weakly to the harmonic measure $\mu_v$ weakly.

$$\mu_v^n \to \nu \text{ weakly.}$$ (4.11)

We will assume, as we mentioned before, that for each fixed $n \geq 1$, we have traffic in and from the boundary of $X_n$ and that the traffic rate between two points $x$ and $y$ in $\partial X_n$ is equal to is $R(x, y) = f(d(x, y))$ for some fixed function $f$. The total traffic passing through the network $X_n$ is equal to

$$T(n) = \int_{\partial X_n \times \partial X_n} R(x, y) d\mu_n(x) d\mu_n(y).$$ (4.12)

Let $r \geq 0$, and denote by $T_r(n)$ the total traffic passing through $B(x_0, r)$. Then

$$T_r(n) = \int_{\partial X_n} \left( \int_{E_x^r} R(x, y) d\mu_n(y) \right) d\mu_n(x)$$ (4.13)

where $E_x^r = \{ y \in \partial X_n : h(x, y) \leq r \}$ and $x \in \partial X_n$.

4.1. **Exponential Decay.** In what follows we will assume that the traffic rate decays exponentially with the distance, i.e. there exist $\beta > 1$ such that

$$R(x, y) = \beta^{-d(x, y)}.$$

Now we are ready to state our main Theorem.

**Theorem 4.2.** Let $X$ be an infinite $\delta$–hyperbolic graph and let $\{X_n\}_{n=0}^{\infty}$ be as in (4.2). Assume that $1 < \beta < a^{\dim_H(\nu)/2}$, then for every $\epsilon > 0$ there exist $r_0 > 0$ such that for all $r \geq r_0$

$$\lim_{n \to \infty} \frac{T_r(n)}{T(n)} \geq 1 - \epsilon.$$ (4.14)

If $\beta > a^{\dim_H(\nu)/2}$ then for every $r > 0$

$$\lim_{n \to \infty} \frac{T_r(n)}{T(n)} = 0.$$ (4.15)
Proof. Using Equations (4.6), (4.11), (4.12) and (4.13) we see that
\[
\lim_{n \to \infty} \frac{T_r(n)}{T(n)} = \frac{\int_{\partial X} \left( \int_{E^r_x} \beta^{2h(x,y)} \, d\nu(y) \right) \, d\nu(x)}{\int_{\partial X \times \partial X} \beta^{2h(x,y)} \, d\nu(x) \, d\nu(y)},
\]
where \(E^r_x = \{ y \in \partial X : h(x,y) \leq r \} \). Let \( F : \partial X \times \mathbb{R}^+ \to [0,1] \) be the function defined by
\[
F(x,r) := \frac{\int_{E^r_x} \beta^{2h(x,y)} \, d\nu(y)}{\int_{\partial X} \beta^{2h(x,y)} \, d\nu(y)}.
\]
By the compactness of the boundary \( \partial X \), to prove that for \( 1 < \beta < a^{\dim_H(\nu)/2} \) the Equation (4.14) holds it is enough to prove that
\[
\lim_{r \to \infty} F(x,r) = 1 \quad \text{for almost every} \quad x \in \partial X.
\]
Analogously, if \( \beta > a^{\dim_H(\nu)/2} \) to prove that (4.15) holds it is enough to prove that
\[
\lim_{r \to \infty} F(x,r) = 0 \quad \text{for almost every} \quad x \in \partial X.
\]
Since the function \( h : \partial X \times \partial X \to \mathbb{R}^+ \) only takes integer values (recall that \( X \) is a graph) then
\[
\int_{E^r_x} \beta^{2h(x,y)} \, d\nu(y) = \sum_{k=0}^{r} \beta^{2k} \cdot \nu(\{ y \in \partial X : h(x,y) = k \}), \quad (4.17)
\]
and
\[
\int_{\partial X} \beta^{2h(x,y)} \, d\nu(y) = \sum_{k=0}^{+\infty} \beta^{2k} \cdot \nu(\{ y \in \partial X : h(x,y) = k \}). \quad (4.18)
\]
Hence, \( \lim_{r \to \infty} F(x,r) = 1 \) if the series in Equation (4.18) converges for almost every \( x \in \partial X \), and \( \lim_{r \to \infty} F(x,r) = 0 \) if the series diverges. Note that by Equation (4.4) we have that
\[
\nu(\{ y \in \partial X : h(x,y) = k \}) \leq \nu(\{ y \in \partial X : d_a(x,y) \leq Ca^{-k} \}).
\]
Using Equation (4.8) we know that for every \( \omega > 0 \) there exists a constant \( K \) such that for \( \nu \)-almost every \( x \)
\[
\nu(\{ y \in \partial X : d_a(x,y) \leq L \}) \leq KL^{\dim_H(\nu)+\omega},
\]
for every \( L > 0 \). Therefore,
\[
\sum_{k=0}^{+\infty} \beta^{2k} \cdot \nu(\{ y \in \partial X : h(x,y) = k \}) \leq \sum_{k=0}^{+\infty} \beta^{2k} K(Ca^{-k})^{\dim_H(\nu)+\omega}.
\]
Since
\[
\sum_{k=0}^{+\infty} \beta^{2k} K(Ca^{-k})^{\dim_H(\nu)+\omega} = KC^{\dim_H(\nu)+\omega} \sum_{k=0}^{+\infty} \left( \frac{\beta^{2}}{a^{\dim_H(\nu)+\omega}} \right)^{k}, \quad (4.19)
\]
this series converges if and only if \( \beta < a^{(\dim H(\nu) + \omega)/2} \) and since \( \omega \) is arbitrary we know that the series converges for every \( 1 < \beta < a^{\dim H(\nu)/2} \). Analogously, if \( \beta > a^{\dim H(\nu)/2} \) then the series diverges and the traffic is asymptotically local.

\[ \square \]

**Remark 4.3.** Recall that \( 0 < \dim H(\nu) \leq \delta_a(\Gamma) \). For the case of the \( k \)-regular tree it is true that \( \dim H(\nu) \leq \delta_a(\Gamma) \) see [2]. Therefore,

\[ \delta_a(\Gamma) = \limsup_{n \to \infty} \frac{\log_a(|\{\gamma \in \Gamma : |\gamma| \leq n\}|)}{n} = \log_a(k). \]

Hence for the \( k \)-regular tree

\[ 1 < \beta < a^{\frac{\dim H(\nu)}{2}} = a^{\frac{\delta_a(\Gamma)}{2}} = a^{\frac{\log_a(k)}{2}} = \sqrt{k}. \]

Note that in Section 3 we proved this result directly.

**Example 4.4.** Let \( X = S_g \) be the Cayley graph of surface group of genus \( g \geq 2 \) or more generally, let \( X = H_{p,q} \) be one of the hyperbolic regular tessellations (i.e. infinite planar graphs with uniform degree \( q \) and \( p \)-gons as faces with \((p-2)(q-2) > 4\)). Then \( \partial X \simeq S^1 \) and \( \dim H(\delta S_g) = 1 \). Therefore,

\[ 1 < \beta < a^{\frac{\dim H(\nu)}{2}} \leq e. \]

**Example 4.5.** Consider \( \Gamma \) a discrete group acting isometrically in the hyperbolic space \( \mathbb{H}^n \) such that the action is convex–cocompact (exists \( U \) a convex \( \Gamma \)-invariant subset of \( \mathbb{H}^n \) such that \( U/\Gamma \) is compact). Then \( \Gamma \) is Gromov hyperbolic and \( \partial \Gamma = \text{limit set of } \Gamma \) in \( S^{n-1} \) has Hausdorff dimension \( \dim H(\delta \Gamma) \leq n-1 \).

Hence,

\[ 1 < \beta < a^{\frac{\dim H(\nu)}{2}} \leq e^{\frac{n-1}{2}}. \]
REFERENCES


Bell Laboratories Alcatel–Lucent, Murray Hill, NJ 07974, USA
E-mail address: ymb@alcatel-lucent.com
E-mail address: gabriel.tucci@alcatel-lucent.com