ON THE TENSOR CONVOLUTION AND THE QUANTUM SEP ARABILITY PROBLEM

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Abstract. We consider the problem of separability: decide whether a Hermitian operator on a finite dimensional Hilbert tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ is separable or entangled. We show that the tensor convolution $(\phi^1 \otimes \cdots \otimes \phi^m) : G \to \mathcal{H}^m$ for mappings $\phi^\mu : G \to \mathcal{H}_\mu$ on an almost arbitrary locally compact abelian group $G$, give rise to formulation of an equivalent problem to the separability one.

1. Introduction

The problem of separability for a given Hilbert tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ is to determine whether a given positive semi-definite operator $\rho \in \mathcal{P}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m)$ is separable or not, i.e. if it can be written as a finite sum

$$\rho = \sum_p \lambda_p |v^1_p \otimes \cdots \otimes v^m_p \rangle \langle v^1_p \otimes \cdots \otimes v^m_p|,$$

where $\lambda_p > 0$ and $v^\mu_p \in \mathcal{H}_\mu$. Note that in quantum mechanics the density operators $\mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m)$ are considered instead of positive semi-definite ones, and then the separability problem is to recognize if $\rho \in \mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m)$ is of the form (1) where additionally $\sum_p \lambda_p = 1$, and $v^\mu_p \in \mathcal{H}_\mu$ have norm 1. Clearly, both problems are equivalent – we consider the first one only for technical reasons.

The main obstacle to efficient solution of this problem is that each separable mixed state (except projectors on simple tensors in a Hilbert tensor product – separable projectors) is ambiguously decomposable into a convex combination of separable projectors (1). Therefore, if we take, for example, a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and arbitrary $v^1, w^1 \in \mathcal{H}_1$, $v^2, w^2 \in \mathcal{H}_2$, then it is not so simple to recognize that

$$|v_0 \otimes w_0 + v_1 \otimes w_1 \rangle \langle v_0 \otimes w_0 + v_1 \otimes w_1| + |v_0 \otimes w_1 + v_1 \otimes w_0 \rangle \langle v_0 \otimes w_1 + v_1 \otimes w_0|$$

is always a separable operator (of course one can use the Horodecki’s criterion [1] in order to check this). But even if we know that it is separable, it is not immediate to see that it can be written in the form

$$\frac{1}{2} \mathcal{P} ((v_0 + v_1) \otimes (w_0 + w_1)) + \frac{1}{2} \mathcal{P} ((v_0 - v_1) \otimes (w_0 - w_1)),$$

where $\mathcal{P} (v) = |v \rangle \langle v|$ for $v \in \mathcal{H}$. In this article we propose a method of transforming operators on an $m$-fold Hilbert tensor product written in a special form – in analogy with (2) – into the positive combination of separable projectors – like in (3). Moreover, we prove that each separable operator and none of entangled one can be written in such form (see Theorem 2). In other words, we restate the problem of separability (see Remark 2.2 after Theorem 2).

Entanglement has been considered since 1935 when the EPR paradox was formulated [3] by the famous triple Einstein, Podolsky and Rosen. Nevertheless the history of the separability problem begun when Werner defined the notion of separability of bipartite mixed states in
his celebrated article \[2\]. Years of research convinced scientists that the problem is not so easy to solve. Horodecki’s showed connection of the problem of separability for bipartite states with the one of classifying the so called positive maps [1], and then generalized this result to the multipartite case [3]. They also formulated an efficient criterion for the problem in case \(\mathcal{H}^1 \otimes \mathcal{H}^2\) is such that \(\dim \mathcal{H}^1 \cdot \dim \mathcal{H}^2 \leq 6\), when all positive maps can be expressed in terms of completely positive maps and a partial transpose map (in a certain basis) \[6, 7\]. The intuitive assertion that the problem of separability is very hard to solve, has been formally proven by Gurvits who showed NP-hardness of the problem \[8\] and recently Gharibian strengthen this result [9] proving its strong NP-hardness (see also \[10\]). A compact group theoretical approach to the problem has been proposed by Korbicz, Wehr and Lewenstein in \[11, 12\] and recently they formulated a similar quantum group approach \[13\]. Also geometry of the cone of separable operators (or the convex body of separable mixed states), strictly related to the separability problem, was studied for example by Kuś and Życzkowski \[14\], Grabowski, Kuś and Marmo \[15\]. In practice the most useful, in low dimensions, is a numerical test discovered by Doherty, Parrilo and Spedalieri \[16, 17, 18\].

2. Results

Let \(\mathcal{H} = \mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^m\) be a Hilbert tensor product of finite dimension. Consider an arbitrary locally compact abelian (LCA) group \(G\) with the Haar measure \(dg\) on it. A tensor convolution \((\phi^1 \otimes \cdots \otimes \phi^m) : G \to \mathcal{H}\) of mappings \(\phi^\mu : G \to \mathcal{H}^\mu\) is defined inductively as follows:

\[
\phi^1 \otimes \phi^2(g) = \int_G \phi^1(g-h) \otimes \phi^2(h) \, dh \in \mathcal{H}^1 \otimes \mathcal{H}^2, \\
\vdots \\
(\phi^1 \otimes \cdots \otimes \phi^m)(g) = \int_G (\phi^1 \otimes \cdots \otimes \phi^{m-1})(g-h) \otimes \phi^m(h) \, dh \in \mathcal{H}.
\]

Equivalently, we can write

\[
\int \cdots \int_{G^{m-1}} \phi^1(g-g_2-\cdots-g_m) \otimes \phi^2(g_2) \otimes \cdots \otimes \phi^m(g_m) \, dg_2 \cdots dg_m.
\]

The concept of the tensor convolution is based on that of usual convolution of functions. However, it is highly asymmetric. For example, assume that for \(\mu = 1, \ldots, m\) there exist \(f^\mu : G \to \mathbb{C}\) and a vector \(v^\mu \in \mathcal{H}^\mu\) such that \(\phi^\mu = f^\mu \cdot v^\mu\). Then the tensor convolution of \(f^\mu\)'s equals the convolution of \(f^\mu\)'s multiplied by the tensor product of \(v^\mu\)'s, that is

\[
(\phi^1 \otimes \cdots \otimes \phi^m) = f^1 \star \cdots \star f^m \cdot v^1 \otimes \cdots \otimes v^m,
\]

where \(\star\) denotes the standard convolution of functions on the group.

In section \[4\] we prove the following theorem.

\textbf{Theorem 1.} Let \(G\) be a locally compact abelian group with the Haar measure \(dg\) on it. Assume that \(\phi^\mu : G \to \mathcal{H}^\mu\) is absolutely and square integrable, for \(\mu = 1, \ldots, m\). Then the Hermitian operator

\[
(\bigotimes) \quad \rho(\phi) = \int_G |(\phi^1 \otimes \cdots \otimes \phi^m)(g) \rangle \langle (\phi^1 \otimes \cdots \otimes \phi^m)(g)| \, dg,
\]

acting on \(\mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^m\), is separable.
Let us consider the example from the introduction once again. Take $G = \mathbb{Z}_2$ with the counting measure as the Haar measure. Consider a Hilbert space $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$. Let $\phi^1(g) = v_g \in \mathcal{H}^1$ and $\phi^2(g) = w_g \in \mathcal{H}^2$ for $g \in \mathbb{Z}_2$. Then the tensor convolution $\phi^1 \otimes \phi^2$ takes the values
\[
\phi^1 \otimes \phi^2(0) = v_0 \otimes w_0 + v_1 \otimes w_1,
\]
\[
\phi^1 \otimes \phi^2(1) = v_0 \otimes w_1 + v_1 \otimes w_0.
\]
The theorem asserts that
\[
(5) \quad |v_0 \otimes w_0 + v_1 \otimes w_1 \rangle \langle v_0 \otimes w_0 + v_1 \otimes w_1| + |v_0 \otimes w_1 + v_1 \otimes w_0 \rangle \langle v_0 \otimes w_1 + v_1 \otimes w_0|
\]
is a separable operator, that is what we pointed out in [2]. To see that it equals [3], we need to recall that for each LCA group $G$ there exists the dual locally compact abelian group $G^\ast$. Each element $\gamma \in G^\ast$, which is called a character, represents the continuous function $\chi_\gamma : G \to \mathbb{C}$ (in fact it is a continuous homomorphism of the groups $G$ and the unit circle $\mathbb{T}$ in $\mathbb{C}$ – see section [3.2]). Then for an absolutely integrable functions $f : G \to \mathbb{C}$ we define its Fourier transform $\mathcal{F}(f) : G^\ast \to \mathbb{C}$ by
\[
\mathcal{F}(f)(\gamma) = \int_G \chi_\gamma(-g) \cdot f(g) \, dg.
\]
Actually, by the same formula we can define the Fourier transform for an absolutely integrable mapping $\phi^\mu : G \to \mathcal{H}^\mu$ as well. Finally, since $G^\ast$ is an LCA group it possesses the Haar measure $d\gamma$, which additionally can by normalized so that the Parseval equality (7) holds. We say that $d\gamma$ is conjugated with $dg$. The main part of the proof of Theorem [1] is the following proposition (proved in section [4]).

**Proposition 2.1.** Under the assumptions of Theorem [1] let $G^\ast$ be the dual group of $G$ and $d\gamma$ the Haar measure conjugated with $dg$. Then
\[
(6) \quad \rho(\phi) = \int_{G^\ast} \mathcal{P}(\mathcal{F}(\phi^1)(\gamma) \otimes \cdots \otimes \mathcal{F}(\phi^m)(\gamma)) \, d\gamma,
\]
where $\rho(\phi) \in \mathcal{S}(\mathcal{H})$ is given by [2], and $\mathcal{P}(v) = |v\rangle \langle v|$ for $v \in \mathcal{H}$.

Coming back to the example, for $G = \mathbb{Z}_2$ the dual group is $G^\ast = \mathbb{Z}_2$, and a character $\gamma \in G^\ast$ acts by $\chi_\gamma(g) = e^{ig\gamma}$. Therefore,
\[
\mathcal{F}(\phi^1)(\gamma) = v_0 + (-1)^\gamma \cdot v_1,
\]
\[
\mathcal{F}(\phi^2)(\gamma) = w_0 + (-1)^\gamma \cdot w_1.
\]
Since the counting measure divided by 2 (on $G^\ast$) is the Haar measure conjugated with the counting measure on $G$, we get from the above proposition that (9) equals
\[
\frac{1}{2} \mathcal{P}((v_0 + v_1) \otimes (w_0 + w_1)) + \frac{1}{2} \mathcal{P}((v_0 - v_1) \otimes (w_0 - w_1))
\]
as we mentioned in the introduction.

The above example is representative in the sense that for a random choice of functions $\phi^1, \ldots, \phi^m$, on an arbitrary LCA group, their tensor convolution takes entangled values almost everywhere. Therefore, we represent separable operators by "sums" of entangled projectors.

Having got Theorem [1], a natural question occurs: does for each separable operator $\rho \in \mathcal{S}(\mathcal{H})$ there exist mappings $\phi^\mu : G \to \mathcal{H}^\mu$ such that $\rho = \rho(\phi)$ given by [2]? The answer is affirmative provided that the cardinality of the group $G$ is at least $(\dim \mathcal{H})^2$.

**Theorem 2.** Let $\mathcal{H} = \mathcal{H}^1 \otimes \cdots \mathcal{H}^m$ be an $m$-fold tensor product of finite dimensional Hilbert spaces $\mathcal{H}^\mu$. Let $G$ be a locally compact abelian group of cardinality $\#G \geq (\dim \mathcal{H})^2$ with the
Proposition 3.2. A Hermitian operator $\rho$ is a separable operator iff there exist absolutely and square integrable mappings $\phi^\mu : G \to \mathcal{H}^\mu$ such that

$$\rho = \int_G \langle (\phi^1 \otimes \ldots \otimes \phi^m) (g) \rangle \langle (\phi^1 \otimes \ldots \otimes \phi^m) (g) \rangle \, dg.$$

This theorem is an immediate consequence of Theorems [1] and Proposition 5.1, which is stated in section 5.

Remark 2.2. Consider a Hilbert tensor product $\mathcal{H} = \mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^m$. Let $G$ be a locally compact abelian group of cardinality $\# G \geq (\dim \mathcal{H})^2$ with the Haar measure $dg$ on it. By virtue of the above theorem the separability problem for $\mathcal{H}$ can be written as follows:

Determine whether for a given operator $\rho \in B(\mathcal{H})$ there exist absolutely and square integrable mappings $\phi^\mu : G \to \mathcal{H}^\mu$ such that

$$\rho = \int_G \langle (\phi^1 \otimes \ldots \otimes \phi^m) (g) \rangle \langle (\phi^1 \otimes \ldots \otimes \phi^m) (g) \rangle \, dg.$$

3. Preliminaries

3.1. Separable operators. Let $m \geq 2$ be a natural number. For $\mu = 1, \ldots, m$, let $\mathcal{H}^\mu$ be a finite $N_{\mu}$-dimensional Hilbert space with a Hermitian product $\langle \cdot \, | \, \cdot \rangle_\mu$ (and according norm $\cdot \, | \, \cdot \rangle_\mu$) $C$-linear with respect to the second argument. Let $\mathcal{H} = \mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^m$ be the Hilbert tensor product with Hermitian product $\langle \cdot \, | \, \cdot \rangle$ given by the linear extension of

$$\langle v^1 \otimes \cdots \otimes v^m \, | \, w^1 \otimes \cdots \otimes w^m \rangle = \langle v^1 \, | \, w^1 \rangle_1 \cdots \langle v^m \, | \, w^m \rangle_m$$

for $v^1 \otimes \cdots \otimes v^m$ and $w^1 \otimes \cdots \otimes w^m$ in $\mathcal{H}$.

For each vector $|v\rangle \in \mathcal{H}$ denote by $\langle v \rangle \in \mathcal{H}^*$ the functional acting on $\mathcal{H}$ associated with $|v\rangle$ by the Hermitian product $\langle \cdot \, | \, \cdot \rangle$ in the standard way. Denote by $P(v) : \mathcal{H} \to \mathcal{H}$ the unnormalized Hermitian projector operator (for abbreviation later called projector operator or projector) on $|v\rangle \in \mathcal{H}$, that is

$$P(v) |w\rangle = \langle v \, | \, w \rangle |v\rangle.$$

Note that in the first two sections we were using the Dirac notation $|v\rangle \langle v|$ for such operators. We denote by $A(\mathcal{H})$ the real linear space of Hermitian operators acting on $\mathcal{H}$. Simple tensors $v = v^1 \otimes \cdots \otimes v^m \in \mathcal{H}$ are called separable and the projector $P(v^1 \otimes \cdots \otimes v^m)$ on a separable vector is called a separable projector. Vectors that are not separable are called non-separable or entangled.

In $A(\mathcal{H})$ we distinguish the cone $S(\mathcal{H})$ generated by the positive combinations of separable projectors. If $\rho \in S(\mathcal{H})$ then we call it separable. Operators that are not separable are called non-separable or entangled. The following proposition is a consequence of [20, Lemma 2] or [19, Corollary 1].

Proposition 3.1. The set of separable operators $S(\mathcal{H})$ is a closed convex cone with nonempty interior in $A(\mathcal{H})$.

Denote by $A(\mathcal{H})^*$ the dual space to $A(\mathcal{H})$ of linear functionals acting on $A(\mathcal{H})$. For every closed convex cone $C \subset A(\mathcal{H})$ the dual cone $C^* \subset A(\mathcal{H})^*$ is defined by

$$C^* = \{ \omega \in A(\mathcal{H})^* \mid \forall \rho \in C \, \omega(\rho) \geq 0 \}.$$

By the Hahn-Banach theorem, the second dual cone equals the initial one, that is $C^{**} = C$. Therefore $\rho \in C$ iff $\omega(\rho) \geq 0$ for all $\omega \in C^*$. Taking $C = S(\mathcal{H})$ we get the following proposition.

Proposition 3.2. A Hermitian operator $\rho$ is separable iff $\omega(\rho) \geq 0$ for all $\omega \in S(\mathcal{H})^*$. 
3.2. LCA groups. In this section we gather some useful facts on Fourier analysis on locally compact abelian groups (we refer to [22] for a more comprehensive view). Let $G$ be an LCA-group. These are, for example, all abelian groups (e.g. $\mathbb{Z}_n$, $\mathbb{Z}^d$, $\mathbb{Q}^d$, $\mathbb{R}^d$, $\mathbb{R}^d/\Lambda$, where $\Lambda \subset \mathbb{R}^d$ is a discrete subgroup) with discrete topology as well as $\mathbb{R}^n$ and $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with standard Euclidean topology ($\mathbb{Q}^d$ with standard topology is not an LCA group since it is not locally compact). Consider all continuous homomorphisms $\chi$ from $G$ to the unit circle in $\mathbb{C}$. Denote by $G^*$ the set of such homomorphisms; we let an element $\gamma \in G^*$ correspond to the homomorphism denoted by $\chi_{\gamma} : G \to \mathbb{C}$. Then $G^*$ is an abelian group with the neutral element $\epsilon$ s.t. $\chi_{\epsilon} \equiv 1$ and the operation “$+$” s.t. $\chi_{\gamma + \gamma'}(g) = \chi_{\gamma}(g) \cdot \chi_{\gamma'}(g)$ for all $g \in G$. Now, $G^*$ embedded with the weak topology (the weakest topology for which $\chi_{\gamma}(g) : G^* \to \mathbb{C}$ is continuous for all $g \in G$) is an LCA group. Moreover, by the Pontryagin duality the double dual group $G^{**}$ is canonically isomorphic with $G$. Some examples are in order. The groups $\mathbb{Z}_n$ and $\mathbb{R}^d$ (with Euclidean topology) are self-dual, i.e. $\mathbb{Z}_n^* = \mathbb{Z}_n$ and $\mathbb{R}^{d*} = \mathbb{R}^d$. The action of the element $\gamma \in \mathbb{Z}_n$ is $\chi_{\gamma}(g) = e^{2\pi i \gamma g/n}$, and the action of the element $\gamma \in \mathbb{R}^{d*}$ is $\chi_{\gamma} = e^{2\pi i \gamma g}$, where $(\cdot | \cdot)$ is any scalar product. The dual group of $\mathbb{Z}^d$ is $\mathbb{T}^d$ (with topology induced from the Euclidean one) and vice versa. The action of $\gamma \in \mathbb{T}^d$ on $\mathbb{Z}^d$ (as well as the action of $\gamma \in \mathbb{Z}^d$ on $\mathbb{T}^d$) is given by the same formula as in the $\mathbb{R}^d$ case.

For each LCA group $G$ there exists the Haar measure, i.e. the only (up to a positive multiplicative constant) regular non-negative Borel measure invariant with respect to all translations. We denote such measure by $dg$. Now, for measurable $f, f' : G \to \mathbb{C}$ such that $\int_G |f(g - h) \cdot f'(h)| dh < \infty$ we define their convolution $f \ast f' : G \to \mathbb{C}$ in the standard way: $f \ast f'(g) = \int_G f(g - h) \cdot f'(h) dh$. For our purposes it is important that for $p = 1, 2$, if $f \in L_1(G, \mathbb{C})$ (i.e. it is absolutely integrable) and $f' \in L^p(G, \mathbb{C})$ (i.e. $\int_G |f'(g)|^p dg < \infty$), then $\|f \ast f'\|_p \leq \|f\|_{L^1} \cdot \|f'\|_{L^p}$. Hence $L_1(G, \mathbb{C}) \cap L_2(G, \mathbb{C})$ is closed under the convolution operation.

Given an LCA group $G$, the Fourier transform $\mathcal{F}$ is the mapping from $L_2(G, \mathbb{C})$ to $L_2(G^*, \mathbb{C})$ (recall that $G^*$ is an LCA group so it has the well defined Haar measure). Formally, we first define $\mathcal{F}(f) : G^* \to \mathbb{C}$ for each $f \in L_1(G, \mathbb{C})$ by

$$
\mathcal{F}(f)(\gamma) = \int_G \chi_{\gamma}(g) \cdot f(g) \, dg.
$$

Then we distinguish a dense class of functions $f$ in $L_1(G, \mathbb{C}) \cap L_2(G, \mathbb{C})$ for which $\mathcal{F}(f) \in L_1(G^*, \mathbb{C})$ and finally extend the Fourier transform to all functions in $L_2(G, \mathbb{C})$ with the image $L_2(G^*, \mathbb{C})$. The point is that for the Haar measure $dg$ on $G$ there exists the Haar measure $d\gamma$ on $G^*$ such that for all $f, f' \in L_2(G, \mathbb{C})$ the Parseval equality

$$
\int_G \overline{f}(g) \cdot f'(g) \, dg = \int_{G^*} \mathcal{F}(f) (\gamma) \cdot \mathcal{F}(f') (\gamma) \, d\gamma
$$

holds true. Moreover, $\mathcal{F}$ is an isomorphism of the Hilbert spaces, so there exists the inverse Fourier transform $\mathcal{F}^{-1} : L_2(G^*, \mathbb{C}) \to L_2(G, \mathbb{C})$ given by

$$
\mathcal{F}^{-1}(\hat{f})(g) = \int_{G^*} \chi_{\gamma}(g) \cdot \hat{f}(\gamma) \, d\gamma
$$

for $\hat{f} : G^* \to \mathbb{C}$.

Finally, we need to know that the convolution of functions on $G$ Fourier-transforms to the multiplication of the transformed functions, i.e.

$$
\mathcal{F}(f \ast f')(\gamma) = \mathcal{F}(f)(\gamma) \cdot \mathcal{F}(f')(\gamma)
$$

for all $f, f' \in L_1(G, \mathbb{C}) \cap L_2(G, \mathbb{C})$ and $\gamma \in G^*$. 
4. Proof of Theorem 1

Recall that we want to prove that if \( G \) is an LCA group with the Haar measure \( dg \), and if \( \phi = (\phi^1, \ldots, \phi^m) \in \prod_{\mu} (L_1(G, \mathcal{H}^\mu) \cap L_2(G, \mathcal{H}^\mu)) \), then the Hermitian operator

\[
(\star) \quad \rho(\phi) = \int_G \mathcal{P}((\phi^1 \otimes \cdots \otimes \phi^m)(g)) \, dg,
\]

acting on \( \mathcal{H} = \mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^m \), is separable.

First of all, notice that the convolution (tensor convolution) is naturally considered on the space of integrable functions (mappings). Hence we choose above space of mappings because of technical reasons. Namely, as we mentioned \( L_1(G, \mathbb{C}) \cap L_2(G, \mathbb{C}) \) is closed under the convolution operation. Hence for \( \phi^\mu \in L_1(G, \mathcal{H}^\mu) \cap L_2(G, \mathcal{H}^\mu) \), \( \phi^1 \otimes \phi^2 \in L_1(G, \mathcal{H}^1 \otimes \mathcal{H}^2) \cap L_2(G, \mathcal{H}^1 \otimes \mathcal{H}^2) \) and so on. Finally, we get that

\[
(\phi^1 \otimes \cdots \otimes \phi^m) \in L_1(G, \mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^m) \cap L_2(G, \mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^m).
\]

Therefore, the integral in \( (\star) \) is well defined.

Proof of Proposition 2.1. Since separable vectors in \( \mathcal{H} \) span all the space, a mapping \( \phi \) in \( L_2(G, \mathcal{H}) \) is unambiguously defined by functions

\[
(10) \quad \phi[v] := \langle v^1 \otimes \cdots \otimes v^m \mid \phi(\cdot) \rangle : G \to \mathbb{C},
\]

where \( v = v^1 \otimes \cdots \otimes v^m \in \mathcal{S}(\mathcal{H}) \) is arbitrary. If \( \phi \) is of the form \( (\phi^1 \otimes \cdots \otimes \phi^m) \), then \( \phi[v] \) is, in fact, a standard convolution of functions \( \phi^\mu[v^\mu] := \langle v^\mu \mid \phi^\mu(\cdot) \rangle \mu : G \to \mathbb{C} \), that is

\[
(11) \quad \phi[v] = \phi^1[v^1] \ast \cdots \ast \phi^m[v^m].
\]

Therefore, using \( (10) \) we get that

\[
\mathcal{F}(\phi[v])(\gamma) = \mathcal{F}(\phi^1[v^1])(\gamma) \cdots \mathcal{F}(\phi^m[v^m])(\gamma)
\]

\[
= \langle v^1 \mid \mathcal{F}(\phi^1)(\gamma) \rangle_1 \cdots \langle v^m \mid \mathcal{F}(\phi^m)(\gamma) \rangle_m
\]

\[
= \langle v^1 \otimes \cdots \otimes v^m \mid \mathcal{F}(\phi^1)(\gamma) \otimes \cdots \otimes \mathcal{F}(\phi^m)(\gamma) \rangle, \quad \gamma \in G^*.
\]

In the second line, we use the fact that \( \mathcal{F} \) is a linear operator. Consequently,

\[
\mathcal{F}((\phi^1 \otimes \cdots \otimes \phi^m))(\gamma) = \mathcal{F}(\phi^1)(\gamma) \otimes \cdots \otimes \mathcal{F}(\phi^m)(\gamma).
\]

Now, using the formula for \( \rho(\phi) \), we compute for a given vector \( v = v^1 \otimes \cdots \otimes v^m \),

\[
\langle v \mid \rho(\phi)|v \rangle = \int_G |(\phi^1 \otimes \cdots \otimes \phi^m)[v](g)|^2 \, dg
\]

\[
= \int_{G^*} |\phi^1[v^1] \ast \cdots \ast \phi^m[v^m]|^2 \, dg \quad \text{by } (11)
\]

\[
= \int_{G^*} |\mathcal{F}(\phi^1)[v^1](\gamma) \cdots \mathcal{F}(\phi^m[v^m])(\gamma)|^2 \, d\gamma \quad \text{by Parseval’s equality}
\]

\[
= \int_{G^*} |\mathcal{F}(\phi^1)(\gamma) \otimes \cdots \otimes \mathcal{F}(\phi^m)(\gamma)|^2 \, d\gamma \quad \text{by Parseval’s equality}
\]

\[
= \left\langle v \mid \int_{G^*} \mathcal{P}(\mathcal{F}(\phi^1)(\gamma) \otimes \cdots \otimes \mathcal{F}(\phi^m)(\gamma)) \, d\gamma \mid v \right\rangle.
\]
where \( d\gamma \) is the Haar measure on \( G^* \) conjugated with \( dg \), and Parseval’s equality is given by (2). Therefore,

\[
\rho(\phi) = \int_{G^*} \mathcal{P} \left( \mathcal{F} (\phi^1) (\gamma) \otimes \cdots \otimes \mathcal{F} (\phi^m) (\gamma) \right) \, d\gamma.
\]

□

Proof of Theorem 1. We use the fact that the set of separable operators \( S(\mathcal{H}) \) is a closed convex cone, and therefore, by Proposition 3.2, \( \rho \in S(\mathcal{H}) \) iff \( \omega(\rho) \geq 0 \) for all \( \omega \in S(\mathcal{H})^* \).

Let us take absolutely and square integrable \( \phi^\mu : G \to \mathcal{H}^\mu \) and assume \( \rho(\phi) \) is given by (2). By Proposition 2.1 it is an integral of separable operators, and so for every \( \omega \in S(\mathcal{H})^* \), \( \omega(\rho(\phi)) \) can be seen as an integral of a non-negative function. Therefore, \( \omega(\rho(\phi)) \geq 0 \), so we end the proof using Proposition 3.2.

Note that Proposition 3.2 was used only for a technical purpose. Since \( S(\mathcal{H}) \) is a closed convex cone and \( \mathcal{P} \left( \mathcal{F} (\phi^1) (\gamma) \otimes \cdots \otimes \mathcal{F} (\phi^m) (\gamma) \right) \in S(\mathcal{H}) \), it is intuitive that their "infinite positive combination" is separable as well. It is because an integral of a mapping with values in a closed convex cone has value in this cone, which is a consequence of the Hahn-Banach separation theorem.

5. Proof of Theorem 2

Theorem 2 will follow immediately from Theorem 1 and underneath Proposition 5.1. Before we state it, we need one definition. A separable operator \( \rho \) is called \( P \)-separable if there exist \( P \in \mathbb{N} \) and separable vectors \( v_p \in \mathcal{H} \) (not necessarily different from \( 0 \in \mathcal{H} \)) for \( p = 1, \ldots, P \) such that

\[
\rho = \sum_{p=1}^{P} \mathcal{P} (v_p).
\]

Clearly, if \( Q \leq P \), then \( Q \)-separability of an operator implies its \( P \)-separability.

Proposition 5.1. Let \( G \) be an LCA group. If the cardinality of \( G \) is not less then \( P \), that is \( \# G \geq P \), then every \( P \)-separable operator on \( \mathcal{H} \) is representable in the form (2) in Theorem 1.

The proof will immediately follow from the ensuing lemmata.

Lemma 5.2. Let \( G \) be an LCA group and \( G^* \) its dual. For a given open set \( U \subset G^* \) there exists a continuous function \( f : G^* \to \mathbb{C} \) with compact support contained in \( U \) such that \( \|f\|_{L^2_m} > 0 \) and \( F^{-1} (f) \in L^1(G, \mathbb{C}) \) (\( F^{-1} \) is the inverse Fourier transform given by (3)).

Proof. Without loss of generality, we can consider only the case \( U \) is an open neighbourhood of the neutral element \( \eta \) in \( G^* \) since the topology of \( G^* \) is uniform with respect to the group operation (the shifts).

Take an open neighbourhood \( U' \) with the compact closure contained in \( U \). For such neighbourhood there exists an open neighbourhood \( V \) of \( \eta \) such that \( V = -V \) and \( V + V \subset U' \). Indeed, since \( \eta + \eta = \eta \) and by the continuity of the group operation there exist \( V_1, V_2 \) - open neighbourhoods of \( \eta \) such that \( V_1 + V_2 \subset U' \). Hence putting \( V = V_1 \cap V_2 \cap (-V_1) \cap (-V_2) \) we obtain the assertion. Note that the closure of \( V + V \) is compact and is contained in \( U \).

Consider the indicator function \( 1_V \) of the set \( V \). Put \( f = 1_V \ast 1_V \). By the definition, \( f \) is strictly positive on the open set \( V + V \) and \( \text{supp} \ f = \text{cl}(V+V) \), hence \( \text{supp} \ f \) is compact and is contained in \( U \). Moreover, on every LCA group open sets have strictly positive measures [22 Section 1.1.2]. Therefore, by the continuity of \( f, \|f\|_{L^2_m} > 0 \).
Finally, $f \in L_1(\mathbb{G}^*, \mathbb{C})$ as well. It is known that $f$ is a positive definite function \cite[Section 1.4.2]{22}, so by the Bochner \cite[Section 1.4.3]{22} and the inversion \cite[Section 1.5.1]{22} theorems the inverse Fourier transform $\mathcal{F}^{-1}(f)$ is integrable.

\[\square\]

**Lemma 5.3.** Let be an LCA group and $\mathbb{G}^*$ its dual. If for a given $P \in \mathbb{N}$ there exist mutually disjoint open sets $\{U_p \subset \mathbb{G}^*\}_{p=1, \ldots, P}$, then every $P$-separable operator on $\mathcal{H}$ is representable in the form $\mathbb{C}$.

**Proof.** Assume we want to represent a separable operator $\sum_{p=1}^{P} \mathcal{P}(v_{p}^1 \otimes \cdots \otimes v_{p}^m)$. Take continuous functions $f_p : \mathbb{G}^* \rightarrow \mathbb{C}$ with compact supports contained in $U_p$ such that $\|f_p\|_{L_{2m}} > 0$ and $\mathcal{F}^{-1}(f_p) \in L_1(\mathbb{G}, \mathbb{C})$, respectively. Such functions exist by Lemma \[5.2\]. Then $f_p \in L_q(\mathbb{G}^*, \mathbb{C})$ for $q = 1, 2, 2m$ (actually for all $1 \leq q \leq \infty$). Define $\Psi^\mu : \mathbb{G}^* \rightarrow \mathcal{H}^\mu$ by

$$\Psi^\mu(\gamma) = \sum_p \frac{v_{p}^\mu}{\|f_p\|_{L_{2m}}} \cdot f_p(\gamma) \quad \mu = 1, \ldots, m.$$ 

Put $\phi^\mu = \mathcal{F}^{-1}(\Psi^\mu)$. Then $(\phi^1, \ldots, \phi^m)$ is in $\prod_{\mu} (L_1(\mathbb{G}, \mathcal{H}^\mu) \cap L_2(\mathbb{G}, \mathcal{H}^\mu))$ since the Fourier transform is an isomorphism of the spaces of square integrable functions and we assumed that $\mathcal{F}^{-1}(f_p) \in L_1(\mathbb{G})$. By the Parseval equality \[7\] and formula \[6\],

$$\rho(\phi) = \sum_p \frac{1}{\|f_p\|_{L_{2m}}} \int_{\mathbb{G}^*} \mathcal{P}(v_{p}^1 \otimes \cdots \otimes v_{p}^m) \cdot |f_p(\gamma)|^{2m} d\gamma = \sum_p \mathcal{P}(v_{p}^1 \otimes \cdots \otimes v_{p}^m),$$

since $U_p$ are mutually disjoint.

\[\square\]

**Proof of Proposition \[5.7\]**. Assume that the cardinality of $\mathbb{G}$ is at least $P$. Then the same is true for its dual $\mathbb{G}^*$. Indeed, since LCA groups are, by the definition, Hausdorff the only possible topology on finite groups is the discrete one (it follows, for example, from the fact that every set composed of only one element is the intersection of all closed neighbourhoods of the element). It is well known that every finite abelian group $\mathbb{D}$ is of the form $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, which implies that they are self-dual (with respect to the discrete topology structure), that is $\mathbb{D} \simeq \mathbb{D}^*$. Now, from the duality between compact and discrete groups, we conclude that the only groups $\mathbb{G}$ with $\# \mathbb{G}^* < P$ are finite of cardinality less than $P$.

Since $\# \mathbb{G}^* \geq P$ and $\mathbb{G}^*$ is Hausdorff, there exist $P$ mutually disjoint open sets $\{U_p\}$ in it. Hence the proof is complete by the Lemma \[5.3\].

\[\square\]

**Remark 5.4.** In Theorem \[2\] the spaces $L_1(\mathbb{G}, \mathcal{H}^\mu) \cap L_2(\mathbb{G}, \mathcal{H}^\mu)$ can be replaced by smaller ones. Namely, in the proof of Proposition \[5.7\] we use, in fact, mappings $\phi^\mu : \mathbb{G} \rightarrow \mathbb{C}$ with compactly supported continuous Fourier transforms $\mathcal{F}(\phi^\mu)$.

**Proof of Theorem \[2\]**. By Caratheodory’s theorem in convex analysis \cite[Theorem 17.1]{21} all separable operators are $(\dim \mathcal{H})^2$-separable. Therefore, by Proposition \[5.7\] for each separable operator $\rho \in \mathcal{S}(\mathcal{H})$ there exist $\phi \in \prod_{\mu} (L_1(\mathbb{G}, \mathcal{H}^\mu) \cap L_2(\mathbb{G}, \mathcal{H}^\mu))$ such that $\rho = \rho(\phi)$ given by \[5.7\]. The converse is a consequence of Theorem \[1\].

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