Topology optimization with optimality criteria and transmissible loads

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A B S T R A C T

The paper describes how to take into consideration the presence of transmissible loads in a topology optimization method based on optimality criteria. The optimization problem has been defined as a total potential energy maximization problem with stress, displacement or stiffness constraints. The final volume of the optimal structural configuration has not to be specified a priori and is a consequence of the imposed structural constraints. The implementation of the proposed method is quite simple and leads to the identification of well defined optimal structures. The results obtained by solving several benchmark problems are shown.

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1. Introduction

The shape of the boundaries and the number of internal holes of an admissible design domain are considered, in the optimization of the topology of continuum structures, concurrently with respect to a predefined objective function, usually the compliance minimization or a natural frequency maximization, and at least one constraint, usually concerning the volume of the optimal structural configuration. Various families of structural topology optimization algorithms for generalized shape optimization problems have been developed [1,2]. The first family of methods is that based on the homogenization theory [3,4]. It includes the so-called microstructure or ‘homogenization’ method based on the modelling of a perforated material and on the description of the structure by using the distribution of the material density [5–7] and the so-called ‘power-law approach’ or SIMP (Solid Isotropic Material with Penalization) method based on the utilization of constant material properties within each element, and element relative densities as design variables [8,9]. Due to the large number of design variables proportional to the number of elements of the discretized design domain, the use of mathematical programming methods for realistic topology optimization problems is somewhat impractical and the distribution of material throughout the structure is usually optimized by using an optimality criteria procedure. A second family of methods is that based on an evolutionary approach [10] such as the Evolutionary Structural Optimization (ESO) method [11–14], the soft kill and hard kill methods [15–17] and the biological growth method [18]. These methods have their origin in fully stressed design techniques and generate structural topologies by eliminating at each iteration elements having a low value of some ‘criterion function’, such as stress, energy density (compliance) or some other response parameter. Other methods for topology optimization of continuum structures have been proposed like the simulated annealing method [19], genetic algorithms and the bubble method described in [20]. A topology optimization method belonging to the evolutionary approach family has been presented in [21]. The optimization problem is set up as the maximization of the total potential energy with a volume constraint and is solved by taking advantage of optimality criteria [22]. The final volume of the optimal structural configuration unknown a priori has not to be specified and is directly controlled by the stress, displacement and/or
stiffness constraints defined at the problem layout phase. The optimization process does not require the implementation of any other adjunctive control method to converge and naturally leads to a bulk or void structure. Loading conditions applied in a fixed point of the design domain are usually taken into consideration in classic topology optimization problems. The presence of design independent loading conditions represents a constraint for the identification of an optimal solution. In order to leave the optimization procedure able to explore a wider set of possible solutions, it is necessary to include the possibility of introducing design dependent loads. Three different design dependent loading conditions are usually taken into consideration: the structural self-weight, movable or transmissible loads and pressure loads. In transmissible load problems the load applying position is movable along a prescribed action line. In pressure load problems the fluid/material interface is not fully defined until the convergence of the solution. The introduction of transmissible loads is usually carried out by using a uniform displacement optimality criterion. The implementation of such an optimality criterion has been obtained by introducing a series of fictitious rigid bars preserving a uniform displacement or by imposing constraint equations on the degrees of freedom at the action line of the applied force in order to simulate a series of fictitious rigid bars preserving a uniform displacement.

The present paper shows how is possible to take into consideration the presence of transmissible loads in the topology optimization method belonging to the evolutionary approach family, described in, following a strict mathematical approach without introducing any further element in the discrete model and any constraint equation. The qualitative analysis takes advantage of mathematical methods in homogenization theory and shape optimization.

2. Variational approach

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded region and $\mathcal{P}$ a finite partition, composed by $N_e$ disjoint elements $\Omega_i$ with non-zero misure $V_i$. Considering a linear isotropic elastic material $\mathcal{M}$ with coefficient tensor $A$, let’s define $S_{M, \eta}$ as follows:

**Definition.** With $S_{M, \eta}$ we define the set of all the structures on $\Omega$ characterized by the elastic tensors

$$A_i = \eta_i A \quad 0 < \eta_0 \leq \eta_i < +\infty, \quad i = 1, \ldots, N_e.$$  

The elements of the class $S_{M, \eta}$ are then the structures on $\Omega$ with constant material in every element of the partition $\mathcal{P}$. The effect of $\eta_i$, then, is to locally vary the mechanical properties of the material. The topology optimization problem to solve is the following:

**Statement of the problem.** Let $\Omega \subseteq \mathbb{R}^3$ be a bounded region with regular boundary $\partial \Omega$. Fixed a finite partition $\mathcal{P}$, boundary conditions and a volumic force $f$, find the vector $\eta = \{\eta_i\}$ solving:

$$\begin{align*}
\max_{\eta} \min_{v \in V} \Pi(\eta, v), \\
\sum_{i=1}^{N_e} \eta_i V_i = V_{\text{tot}}, \\
0 < \eta_0 \leq \eta_i < +\infty \quad i = 1, \ldots, N_e,
\end{align*}$$

(1)

where $V$ is the set of the admissible displacements and $\Pi(\eta, v)$ is the potential energy of the system.

We observe that the constraint on $\eta_i V_i$ implies

$$\eta_i \leq \max_{\omega \in \mathcal{P}} \frac{V_{\text{tot}}}{V_i} \quad \frac{V_{\text{tot}}}{V_{\text{min}}} = k \quad i = 1, \ldots, N_e.$$

(2)

In order to express the potential energy $\Pi$ and to define the set $V$, let’s consider the variational formulation of the problem. The parameters $\eta_i$ transform the classical elasticity problem in

$$\begin{align*}
\sum_{i=1}^{N_e} \eta_i \int_{\Omega_i} a_{ijkh} \frac{\partial u_k}{\partial x_h} \frac{\partial v_l}{\partial x_j} \, dx = \int_{\Omega} f_i v_i \, dx \quad \forall v \in V, \\
u = 0 \quad \text{in} \; \Gamma_D, \\
\frac{\partial u_k}{\partial x_h} n_j = 0,
\end{align*}$$

(3)

where $a_{ijkh}$ are the terms of the tensor $A$. The potential energy of the system, corresponding to an admissible displacement $v$, can be expressed by

$$\Pi(\eta, v) = \frac{1}{2} a_{\eta}(v, v) - l(v),$$
with
\[ a_\eta(u, v) = \sum_{i=1}^{N_\eta} \eta_i \int_{\Omega_i} A \varepsilon(u) \varepsilon(v) \, dx, \]
and
\[ l(v) = \int_{\Omega} f \cdot v \, dx, \]
where \( \varepsilon \) is the elastic strain tensor. So far, the max–min problem (1) can be stated as a max problem in \( \eta \), with the additional constraint that \( u \) is the solution of the elastic variational problem (see [26–29,31]):
\[
\max \min_{\eta, v} I(\eta, v) \iff \begin{cases} 
\max_{\eta} I(\eta, v), \\
\eta \geq \eta_0 > 0 
\end{cases} \quad \forall v \in \mathcal{V}.
\]

For the principle of virtual work, furthermore, the potential energy at the elastic equilibrium can be expressed as:
\[ I(\eta, u) = \frac{1}{2} a_\eta(u, u) - l(u) = \frac{1}{2} a_\eta(u, u) = -\frac{1}{2} l(u), \]
while the problem (1) is stated as follows:
\[
\begin{cases} 
\min_{\eta} a_\eta(u, u), \\
\eta \geq \eta_0 > 0 
\end{cases} \quad \forall v \in \mathcal{V},
\]
\[
\sum_{i=1}^{N_\eta} \eta_i V_i = V_{\text{tot}},
\]
\[
\eta_i \geq \eta_0 > 0 \quad i = 1, \ldots, N_\eta.
\]

This transformation requires the solution of the variational problem (3).

In the next section, the conditions for the existence and uniqueness of the solution are studied using the Lax–Milgram theorem. Let’s define the class of tensors
\[
\mathcal{C}(c_1, c_2, \Omega) : \begin{cases} 
A_{i j k h} \in L^\infty(\Omega), \\
A_{i j k h} = A_{j i h k}, \\
|A M| \leq c_1 |M| \quad \forall \text{ squared matrix } M, \\
AS S \geq c_2 |S|^2 \quad \forall \text{ symmetric matrix } S,
\end{cases}
\]
with \( c_1 \) and \( c_2 \) positive constants. In the topology optimization problem, with the hypothesis \( A \in \mathcal{C} \), it is necessary to study the effect of \( \eta_i \) in \( a_\eta(u, v) \). The upper bound (2) grants the continuity of \( a_\eta(u, v) \):
\[
|a_\eta(u, v)| \leq \sum_{i=1}^{N_\eta} \eta_i \int_{\Omega_i} A \varepsilon(u) \varepsilon(v) \, dx \leq \int_{\Omega} k |A \varepsilon(u) \varepsilon(v)| \, dx \leq \int_{\Omega} kc_1 |\varepsilon(u)||\varepsilon(v)| \, dx.
\]
The lower bound \( \eta_i \geq \eta_0 > 0 \) grants the coercivity:
\[
\eta_i \geq \eta_0 > 0 \implies a_\eta(u, u) = \sum_{i=1}^{N_\eta} \eta_i \int_{\Omega_i} A \varepsilon(u) \varepsilon(u) \, dx \geq c_2 \eta_0 \int_{\Omega} |\varepsilon(u)|^2 \, dx.
\]
The functional framework is then the following
\[
\begin{cases} 
A \in \mathcal{C}(c_1, c_2, \Omega), \\
f \in (L^2(\Omega))^3, \\
\mathcal{V} = (H_0^1(\Omega))^3,
\end{cases}
\]
while the optimization problem is stated as follows:

**Problem 2.1.** Given data satisfying (5) and a finite partition \( \mathcal{P} \) of \( \Omega \), find \( \eta = \{ \eta_i \} \) solving:
\[
\begin{cases} 
\min_{\eta} a_\eta(u, u), \\
a_\eta(u, v) = l(v) \quad \forall v \in \mathcal{V}, \\
\sum_{i=1}^{N_\eta} \eta_i V_i = V_{\text{tot}}, \\
\eta_i \geq \eta_0 > 0 \quad i = 1, \ldots, N_\eta.
\end{cases}
\]

For Lax–Milgram for every \( \eta \) there is a unique solution \( u_\eta \), taking into account that \( \eta_0 \leq \eta \leq k \) and that \( a_\eta(u, v) \) is continuous respect to \( \eta \), there is the existence and uniqueness of a minimum point of the problem (6).
3. Optimality criteria

To obtain the optimality criteria for the topological optimization problem in exam, all the disequality constraints are transformed into equivalent ones using slack variables $t_i$:

$$\eta_i \geq \eta_0 > 0 \rightarrow \eta_i - \eta_0 - t_i^2 = 0 \quad i = 1, \ldots, N_e.$$

The lagrangian function associated to this problem is:

$$\mathcal{L}(\eta, \lambda_1, \lambda_2, \theta_i, t_i) = a_\eta(u, u) - \lambda_1 [a_\eta(u, v) - l(v)] - \lambda_2 \left[ \sum_{i=1}^{N_e} \eta_i V_i - V_{\text{tot}} \right] - \theta_i [\eta_i - \eta_0 - t_i^2].$$

The Kuhn–Tucker conditions for optimality are:

$$\begin{align*}
\frac{\partial \mathcal{L}}{\partial \eta_i} &= 0 \quad i = 1, \ldots, N_e, \\
\frac{\partial \mathcal{L}}{\partial \lambda_j} &= 0 \quad j = 1, 2, \\
\frac{\partial \mathcal{L}}{\partial \theta_i} &= 0 \quad i = 1, \ldots, N_e, \\
\frac{\partial \mathcal{L}}{\partial t_i} &= 0 \quad i = 1, \ldots, N_e, \\
\lambda_{1,2}, \theta_i &\geq 0.
\end{align*}$$

The condition on the derivative of $\mathcal{L}$ respect to $\lambda_1$ impose that $u$ is the displacement field at the elastic equilibrium:

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = 0 \rightarrow a_\eta(u, v) = l(v).$$

The condition on $\lambda_2$ implies the satisfaction of the constraint

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = 0 \rightarrow \sum_{i=1}^{N_e} \eta_i V_i = V_{\text{tot}}.$$

From the derivatives respect to $\theta_i$ and $t_i$ we obtain:

$$\begin{align*}
\frac{\partial \mathcal{L}}{\partial \theta_i} &= \eta_i - \eta_0 - t_i^2 = 0 \quad i = 1, \ldots, N_e, \\
\frac{\partial \mathcal{L}}{\partial t_i} &= -2t_i \theta_i = 0 \quad i = 1, \ldots, N_e,
\end{align*}$$

leading to

$$\eta_i > \eta_0 \rightarrow \theta_i = 0 \quad i = 1, \ldots, N_e.$$

The condition on the derivative of $\mathcal{L}$ respect to $\eta_i$ imposes

$$\frac{\partial \mathcal{L}}{\partial \eta_i} = \frac{\partial}{\partial \eta_i} a_\eta(u, u) - \lambda_2 V_i - \theta_i = 0,$$

that gives:

$$\frac{\partial}{\partial \eta_i} a_\eta(u, u) = \lambda_2 \quad \forall i | \eta_i > \eta_0. \quad (7)$$

In order to obtain the term $\partial/\partial \eta_i a_\eta(u, u)$ we consider $\eta(x)$ as a function defined on $\Omega$ and $\eta(x) \in L^\infty(\Omega)$. Actually $\eta(x) = \eta_i$ if $x \in \Omega_i$. With this statement, we emphasize the dependence of $u$ from $\eta$ and we write:

$$a_\eta(u, v) = \int_\Omega \eta(x) \mathcal{A} \varepsilon(u_\eta) \varepsilon(v) \, dx = \int_\Omega f(x) v \, dx,$$

where $u_\eta, v \in \mathcal{V}$ and

$$0 < \eta_0 \leq \eta(x) \quad \text{and} \quad \int_\Omega \eta(x) \, dx \leq c < +\infty.$$

Moreover

$$a_\eta(u_\eta, u_\eta) = \int_\Omega \eta(x) \mathcal{A} \varepsilon(u_\eta) \varepsilon(u_\eta) \, dx = \int_\Omega f(x) u_\eta \, dx.$$
Our purpose is to obtain
\[
\frac{\partial}{\partial \eta} \int_{\Omega} \eta(x) A \varepsilon(u_{\eta}) \varepsilon(u_{\eta}) \, dx,
\]
or equivalently
\[
\frac{\partial}{\partial \eta} \int_{\Omega} f(x) u_{\eta} \, dx.
\]
For this objective we need to obtain
\[
\frac{\partial}{\partial \eta} u_{\eta} = \lim_{h \to 0} \frac{u_{\eta-h} - u_{\eta}}{h},
\]
and \(u_{\eta-h}\) satisfies the equation:
\[
\int_{\Omega} (\eta(x) - h) A \varepsilon(u_{\eta-h}) \varepsilon(v) \, dx = \int_{\Omega} f(x) v \, dx,
\]
and then
\[
\int_{\Omega} \eta(x) A \varepsilon(u_{\eta-h}) \varepsilon(v) \, dx - \int_{\Omega} h A \varepsilon(u_{\eta-h}) \varepsilon(v) \, dx = \int_{\Omega} f(x) v \, dx.
\] (10)
Subtracting (10) and (8) and dividing by \(h\) we get:
\[
\int_{\Omega} \eta(x) A \varepsilon\left(\frac{u_{\eta-h} - u_{\eta}}{h}\right) \varepsilon(v) \, dx - \frac{h}{h} \int_{\Omega} A \varepsilon(u_{\eta-h}) \varepsilon(v) \, dx = 0.
\] (11)
By Eq. (11) and standard inequalities we get:
\[
\left\| \frac{u_{\eta-h} - u_{\eta}}{h} \right\|_{\mathcal{V}} \leq K,
\]
and then
\[
\frac{u_{\eta-h} - u_{\eta}}{h} \rightharpoonup w \quad \text{weakly in } \mathcal{V}.
\]
In the limit the Eq. (11) becomes:
\[
\int_{\Omega} \eta(x) A \varepsilon(w) \varepsilon(v) \, dx - \int_{\Omega} A \varepsilon(u_{\eta}) \varepsilon(v) \, dx = 0.
\] (12)
On the other hand, by (9), we see that:
\[
\frac{\partial}{\partial \eta} \int_{\Omega} \eta(x) A \varepsilon(u_{\eta}) \varepsilon(u_{\eta}) \, dx = \lim_{h \to 0} \int_{\Omega} f(x) \frac{u_{\eta-h} - u_{\eta}}{h} \, dx = \int_{\Omega} f(x) w \, dx.
\] (13)
But, by Eq. (8) with \(v = w\), we get
\[
\int_{\Omega} \eta(x) A \varepsilon(u_{\eta}) \varepsilon(w) \, dx = \int_{\Omega} f(x) w \, dx.
\] (14)
Now, by Eq. (12) with \(v = u_{\eta}\), we get
\[
\int_{\Omega} \eta(x) A \varepsilon(u_{\eta}) \varepsilon(u_{\eta}) \, dx = \int_{\Omega} A \varepsilon(u_{\eta}) \varepsilon(u_{\eta}) \, dx.
\] (15)
Taking Eq. (15) together with (14) and (13) we get:
\[
\frac{\partial}{\partial \eta} \int_{\Omega} \eta(x) A \varepsilon(u_{\eta}) \varepsilon(u_{\eta}) \, dx = \int_{\Omega} A \varepsilon(u_{\eta}) \varepsilon(u_{\eta}) \, dx.
\] (16)
So far, the Eq. (7) can be explicit as:
\[
\frac{1}{\mathcal{V}_{i}} \int_{\Omega_{i}} A \varepsilon(u) \varepsilon(u) \, dx = \lambda_{2} \quad \forall i | \eta_{i} > \eta_{0}.
\] (17)

The first member of (17) represents the ratio between the strain energy on \(\Omega_{i}\) and \(\eta_{i}\), while the second member is a constant. So far, this gives the following result:

**Optimality criteria:** The optimal solution of problem (6) is such that the ratio between the local strain energy and \(\eta_{i}\) is constant in every subdomain \(\Omega_{i}\) with \(\eta_{i} > \eta_{0}\).
4. Trasmissible loads

In the framework of optimization, the case of transmissible loads represents an extension of the problem previously studied. The volumic force indeed is not defined a priori, but it’s defined a total force \( \mathbf{F} \), to be distributed inside a subdomain \( \Omega_F \). The force distribution is then inserted in the model as a design variable. In this paper, we use a concentrated distribution of the form:

\[
\mathbf{F} = (0, 0, f_{\text{tot}}),
\]

with

\[
f(x) = \sum_{j \in I} f_j \delta(x - P_j),
\]

and the constraints

\[
\begin{align*}
\int_{\Omega_F} f \, dx &= \sum_{j \in I} f_j = f_{\text{tot}}, \\
f_j f_{\text{tot}} &\geq 0 \quad \forall j \in I.
\end{align*}
\]

The topology optimization problem with transmissible loads is then the following:

**Problem 4.1.** Given data satisfying (5) and a finite partition \( \mathcal{P} \) of \( \Omega \) with a finite point subset \( \mathcal{P}_j \subset \mathcal{P} \subset \Omega \), find \( \eta = \{\eta_i\} \) and \( f = \{f_j\} \) solving:

\[
\begin{align*}
\min_{\eta} & \quad a_{\eta}(\mathbf{u}, \mathbf{u}), \\
a_{\eta}(\mathbf{u}, \mathbf{v}) &= l_F(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{V}, \\
\sum_{i=1}^{N_e} \eta_i V_i - V_{\text{tot}} &= 0, \\
\sum_{j \in I} f_j - f_{\text{tot}} &= 0, \\
f_j f_{\text{tot}} &\geq 0 \quad \forall j \in I, \\
\theta_i - \eta_0 - t_i^2 &= 0 \quad i = 1, \ldots, N_e.
\end{align*}
\]

The introduction of a transmissible force modifies the expression of the functional associated to the internal forces \( l_F \)

\[
l_F(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx = \sum_{j \in I} f_j v_3(\mathcal{P}_j),
\]

while the bilinear form \( a_{\eta} \) remains the same. The Lax–Milgram conditions are satisfied under the hypothesis

\[
\begin{align*}
\mathcal{A} &\in C(c_1, c_2, \Omega), \\
f_j &\in L^\infty(\Omega), \\
\mathcal{V} &= (H_0^1(\Gamma_D, \Omega))^3,
\end{align*}
\]

and, with \( \eta \) and \( f \) fixed, there is a unique solution in terms of displacements. The domain is compact respect to \( \eta \) and \( f, a_{\eta} \) and \( l_F \) are continuous, so there is a unique solution of problem (18).

The lagrangian function associated to this problem is

\[
\mathcal{L}(\eta, f, \lambda_1, \lambda_2, \lambda_3, \theta_i, t_i) = a_{\eta}(\mathbf{u}, \mathbf{u}) - \lambda_1 [a_{\eta}(\mathbf{u}, \mathbf{v}) - l_F(\mathbf{v})] - \lambda_2 \left[ \sum_{i=1}^{N_e} \eta_i V_i - V_{\text{tot}} \right] - \lambda_3 \left[ \sum_{j \in I} f_j - f_{\text{tot}} \right] - \theta_i [\eta_i - \eta_0 - t_i^2],
\]

while the Kuhn–Tucker conditions for optimality are:

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial \eta_i} &= 0 \quad i = 1, \ldots, N_e, \\
\frac{\partial \mathcal{L}}{\partial f_j} &= 0 \quad j \in I, \\
\frac{\partial \mathcal{L}}{\partial \lambda_j} &= 0 \quad j = 1, 2, 3, \\
\frac{\partial \mathcal{L}}{\partial \theta_i} &= 0 \quad i = 1, \ldots, N_e, \\
\frac{\partial \mathcal{L}}{\partial t_i} &= 0 \quad i = 1, \ldots, N_e, \\
\lambda_{1,2,3}, \theta_i &\geq 0.
\end{align*}
\]
The new properties are inside the derivatives respect to $\lambda_3$ and $f_j$: the first imposes to satisfy the constraint

$$\frac{\partial L}{\partial \lambda_3} = 0 \implies \sum_{j \in I} f_j = f_{\text{tot}},$$

while the condition on $f_j$ leads to a new optimality criteria:

$$\frac{\partial L}{\partial f_j} f(u) - \lambda_3 = 0 \implies u_3(P_j) = \lambda_3 \quad \forall P_j \in I \subset \Omega.$$

**Optimality criteria:** The optimal solution of problem (18) is such that:
- the ratio between the local strain energy and $\eta_i$ is constant in every subdomain $\Omega_i$ with $\eta_i > \eta_0$,
- displacements are uniform in $I$ in the direction of the volumic force.

**5. Method implementation: Benchmarks and examples**

The optimization method is characterized by an algorithm that iteratively updates both the material properties and the applied force distribution.

Once the region $\Omega$, the material properties, the displacements constraints, the total load and its application region have been defined, an initialization routine generates an input file for the FEM code.

The iterative optimization routine carries out the following steps at every iteration $k$:

1. evaluation of the displacement field $u_k$, the stress field $\sigma_k$ and the potential energy $\Pi(u^{k-1}, f^{k-1}, u^k)$,
2. material properties distribution update:

$$E_i^{k+1} = \begin{cases} E_i^k \frac{e_i^k}{e_{\text{avg}}^k} & \forall \Omega_i \subset \mathcal{P}, \\ E_{M} & \text{if } E_i^k \frac{e_i^k}{e_{\text{avg}}^k} > E_{M}, \\ E_i^k & \text{if } E_i^k \frac{e_i^k}{e_{\text{avg}}^k} < \xi, \end{cases}$$

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3. average displacements evaluation in each subdomain $\Omega_F$.
4. force distribution update:

$$f_j^{k+1} = f_j^k \frac{u_{\text{avg}}^j}{u_j^k} \quad \forall j \in J_h.$$  \hspace{1cm} (20)

5.1. Benchmark 1

The first benchmark concerns a square design domain (Figs. 1a and 1b). The results obtained by applying fixed and transmissible loads are shown.
Comparing Fig. 1c towards Fig. 1d it is evident how much the application of the loading conditions on fixed points constrains the results of the optimization process. The potential energy of the solution obtained with fixed point loading condition is 33% larger than if the transmissibility is taken into consideration. The result shown in Fig. 1d, with truss members characterized by an inclination angle of 45 deg, fully corresponds to the theoretical results [30]. In both cases the solution converges to a 0–1 (void-bulk) configuration (a structure with $\eta = 0$ or $\eta = 1$ only) in just 10–15 iterations (Figs. 1e and 1f). Finally, Figs. 1g and 1h show that the effect of the volume constraint is to vary the final truss thickness.

5.2. Benchmark 2

The second benchmark differs from the first one in the boundary condition definition. All the parameters are equal to the first benchmark, with the exception of point B, where only the vertical displacement has been constrained (Figs. 2a and 2b).
Comparing the final structures with those of benchmark 1, it is evident that the boundary conditions affect the results of both the optimization process: the main difference is the introduction of a connecting truss between points A and B (Figs. 2c and 2d). Once more, it is evident the effect of considering transmissible loads instead of loads applied to fixed points. Also in these cases, the solution converges to a 0–1 configuration in less than 15 iterations (Figs. 2e and 2f). Figs. 1g and 1h show that the effect of the volume constraint corresponding to different final truss member thicknesses (Figs. 2g and 2h).

5.3. Benchmark 3

In the third benchmark, the load has been applied uniformly distributed over almost the entire domain, as shown in Figs. 3a and 3b.

The final structure is a bridge connecting point A with B (Fig. 3c). It’s important to notice that its thickness is not constant, but it’s larger close to the constrained points. Fig. 3d shows the results obtained by Fuchs [24], where a good agreement is found. Also in this case, the solution converges to a 0–1 configuration in less than 15 iterations (Fig. 3e).
5.4. Benchmark 4

In this benchmark, as done in benchmark 2, the boundary conditions have been changed, relaxing the horizontal displacement constraint on point B (Figs. 4a and 4b).

As shown in benchmark 2, the main change in the final structure is the introduction of a connecting truss between the two constrained points (Fig. 4c). The height of the bridge is increased, while the thickness behaviour doesn’t change. Also in this case, the solution converges to a 0–1 configuration in less than 15 iterations (Fig. 4d).
5.5. Benchmark 5

In this final benchmark, the algorithm is applied to a 3D problem (Figs. 5a and 5b). The final structure is a symmetric dome composed by four bridges connecting the constrained points (Figs. 5c, 5e and 5f). Comparing this result with those obtained by Fuchs (Fig. 5d), it’s possible to see a good agreement. The final structure has, as already said, a full 0–1 topology, whereas Fuchs has to use an interpolation scheme to obtain similar results.
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<td>Constraints at B</td>
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</table>

**Fig. 3b.** Model parameters.

**Fig. 3c.** Optimal structure with transmissible loads.

**Fig. 3d.** Fuchs result.

**Fig. 3e.** Solution convergence.
6. Conclusions

The paper describes the mathematical process required to identify the optimality criteria for transmissible loading implementation into a topology optimization method based on total potential energy maximization with a volume constraint. The identified optimality criteria have been implemented by taking advantage of a very simple recursive law, leading to an optimization method requiring a very small number of iterations in order to reach the convergence. The capability of the proposed method to take into consideration transmissible loading conditions, allows for the identification of optimal structures characterized by better performance with respect to those that can be obtained by taking into consideration loading conditions applied to fixed points of the design domain.

The transmissible loading conditions have to be defined by specifying their modulus and their direction. A point belonging to the force application line has to be specified. The final volume of the optimal structural configuration unknown a priori has not to be specified and is directly controlled by the stress, displacement or stiffness constraints at the problem layout.
Fig. 4d. Solution convergence.

Fig. 5a. Problem definition.

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<td>Constraints at A-D</td>
<td>Dofs 1-3</td>
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</tbody>
</table>

Fig. 5b. Model parameters.

Fig. 5c. Optimal structure with transmissible loads.
phase. The optimization process does not require the implementation of filter stabilization or perimeter control methods to converge and the topology of the optimal solution is usually very well defined.

The proposed method is based on the elaboration of the results generated by simple independent finite element analysis. This makes the method completely solver independent and allows us to use the numerical results coming from every finite element code. The optimal design obtained by the solution of a topology optimization problem has to be considered as a suggestion to be adjusted in order to accomplish the technological requirements.

References


