Meshless Methods Coupled with Other Numerical Methods

Y. T. GU** , G. R. LIU

Centre for Advanced Computations in Engineering Science (ACES),
Department of Mechanical Engineering, National University of Singapore,
10 Kent Ridge Crescent, Singapore 119260, Singapore

Abstract: Meshless or mesh-free (or shorten as MFree) methods have been proposed and achieved remarkable progress over the past few years. The idea of combining MFree methods with other existing numerical techniques such as the finite element method (FEM) and the boundary element method (BEM), is naturally of great interest in many practical applications. However, the shape functions used in some MFree methods do not have the Kronecker delta function property. In order to satisfy the combined conditions of displacement compatibility, two numerical techniques, using the hybrid displacement shape function and the modified variational form, are developed and discussed in this paper. In the first technique, the original MFree shape functions are modified to the hybrid forms that possess the Kronecker delta function property. In the second technique, the displacement compatibility is satisfied via a modified variational form based on the Lagrange multiplier method. Formulations of several coupled methods are presented. Numerical examples are presented to demonstrate the effectiveness of the present coupling methods.

Key words: meshless method; mesh-free method; finite element method; boundary element method; coupling method; stress analysis

Introduction

To avoid the shortcomings of numerical methods based on mesh, e.g., the finite element method (FEM), the boundary element method (BEM), and the finite difference method (FDM), many mesh-free (Mfree) methods have been proposed and achieved remarkable progress over the past few years[1-8]. These MFree methods have been successfully applied to a large variety of problems. Many details about MFree methods have been presented[9,10].

However, because the MFree method is still in its infancy, there are still some technical problems that need to be solved. The major inconveniences or disadvantages in using MFree methods include:

1) MFree methods are usually computationally expensive because of the complex MFree interpolations and the complex implementation algorithm. In some MFree methods, numerical integrations are also very computationally expensive.

2) In some MFree methods that are based on the moving least squares (MLS) shape functions, it is difficult to implement essential boundary conditions.

3) MFree methods are inefficient for some special problems, such as problems with infinite or semi-infinite domains.

Some strategies have been developed to alleviate the above mentioned problems. Alternatively, following the idea of the coupling of FEM with BEM, these problems can also be overcome if the use of the MFree methods is limited to the sub-domain where their unique advantages are beneficial. In the remaining part of the domain, other established numerical methods, e.g., FEM or BEM, are employed.
It is often desirable and beneficial to combine two established numerical methods to exploit their advantages while avoiding their disadvantages. The idea of combining MFree methods with other numerical techniques (FEM and BEM) is naturally of great interest in many practical applications. Liu and Gu have done much research in this direction. A number of combined methods including EFG/BEM, EFG/HBEM, MLPG/FEM/BEM, HBPIIM/EFG, etc. have been developed. Here, EFG represents the element-free Galerkin, HBEM represents Hybrid BEM, MLPG represents the meshless local Petrov-Galerkin, and HBPIIM represents the hybrid boundary point interpolation method.

In these coupled methods, the condition of displacement compatibility must be satisfied. To satisfy this condition, two numerical techniques are developed and discussed in this paper. Formulations of several coupled methods are presented.

1 Combination Techniques

Consider a problem of two-dimensional solids. The problem domain consists of two sub-domains Ω₁ and Ω₂, joined by an interface boundary Γ₁. Numerical method 1 is used in Ω₁ and numerical method 2 is used in Ω₂ as shown in Fig. 1. In the coupling method, the displacement compatibility and the force equilibrium conditions on Γ₁ must be satisfied. Thus,

1) The nodal displacements $U^{(1)}_i$ and $U^{(2)}_i$ at Γ₁ for Ω₁ and Ω₂ should be equal, i.e.,

$$U^{(1)}_i = U^{(2)}_i = U_i$$ (1)

2) The summation of forces $F^{(1)}_i$ and $F^{(2)}_i$ at Γ₁ for Ω₁ and Ω₂ should be zero, i.e.,

$$F^{(1)}_i + F^{(2)}_i = 0$$ (2)

In a coupled method, it is ideal to satisfy both the displacement compatibility and the force equilibrium conditions, Eqs. (1) and (2). The displacement compatibility Eq. (1) is the most important in these two requirements and must be satisfied in a coupled method although the force equilibrium condition Eq. (2) cannot be exactly satisfied in some coupled methods.

If shape functions along the interface boundary, Γ₁, in both domain Ω₁ and domain Ω₂, satisfy the Kronecker delta function properties, these two numerical methods can be combined directly along the interface boundary. The coupled methods developed based on the direct method include: FEM/BEM, PIM/FEM/BEM, LPIM/FEM/BEM, BPIM/FEM/BEM, HBPIIM/FEM/BEM, PIM/LPIM, etc.

In some MFree methods (e.g., EFG, MLPG), MLS shape functions are used. Because the MLS shape functions lack the delta function properties, $u_h$ in MLS approximation differs from the nodal displacement value $u$ at point $x$. It is impossible to couple these MFree methods and other numerical methods directly along Γ₁. Some techniques are needed to satisfy combination conditions, Eqs. (1) and (2). Two combination techniques will be discussed in the following sections.

2 Combination Technique 1

2.1 Combination technique 1: Using hybrid displacement shape function

This method introduces interface elements in the MFree domain, in which the MLS shape functions are used near the interface boundary Γ₁ (see Fig. 1). In these
interface elements, a hybrid displacement approximation is defined so that the MLS shape functions in the MFree domain along \( I_1 \) possess the delta function property.

A detailed figure of the interface domain is shown in Fig. 1. \( \Omega_\iota \) is a layer of sub-domain along the interface boundary \( I_1 \) within the MFree domain \( \Omega_\chi \). \( \Omega_\chi \) is called the interface domain and it is divided into several interface elements (finite elements). The new displacement approximation in the MFree domain \( \Omega_\chi \) can be rewritten as

\[
 u^h_i(x) = \sum_{i=1}^{n} \hat{\phi}_i(x)u_i
\]

where the hybrid shape functions of the interface element are defined as

\[
 \hat{\phi}_i(x) = \begin{cases} 
  (1 - R(x))\phi_i(x) + R(x)N_i(x), & x \in \Omega_\iota; \\
  \phi_i(x), & x \in \Omega_\chi - \Omega_\iota
\end{cases}
\]

The derivatives of the interface shape functions are:

\[
 \hat{\phi}_i(x) = \begin{cases} 
  (1 - R(x))\phi_i(x) - R(x)\phi_i(x) + R(x)N_i(x) + R(x)N_i(x), & x \in \Omega_\iota; \\
  \phi_i(x), & x \in \Omega_\chi - \Omega_\iota
\end{cases}
\]

Hence, the modified displacement approximation in domain \( \Omega_\chi \) becomes

\[
 u^h_i(x) = \begin{cases} 
  u_{(\text{MFree})}(x) + R(x)(u_{(\text{FE})}(x) - u_{(\text{MFree})}(x)), & x \in \Omega_\iota; \\
  u_{(\text{MFree})}(x), & x \in \Omega_\chi - \Omega_\iota
\end{cases}
\]

where \( u^h_i \) is the displacement of a point in \( \Omega_\chi \) and \( u_{(\text{MFree})} \) is the MFree displacement given by the MLS approximation,

\[
 u_{(\text{MFree})}(x) = \sum_{i=1}^{n} \phi_i(x)u_i
\]

where \( \phi_i \) is the MLS shape function and \( n \) is the number of field nodes selected in the support domain of the sampling point \( x \).

\( u_{(\text{FE})} \) is the finite element (FE) displacement given by the following standard FE interpolation

\[
 u_{(\text{FE})}(x) = \sum_{i=1}^{n} N_i(x)u_i, \quad n_e = 3, 4, 5, \ldots
\]

where \( N_i(x) \) is the finite element shape function and \( n_e \) is the number of nodes in an FE interface element.

The ramp function \( R \) is equal to the sum of the FE shape functions of an interface element associated with interface element nodes that are located on the interface boundary \( I_1 \), i.e.,

\[
 R(x) = \sum_{i=1}^{k} N_i(x), \quad x \in I_1
\]

where \( k \) is the number of nodes located on the interface boundary \( I_1 \) for an interface element. According to the property of FE shape functions, \( R \) will be unity along \( I_1 \) and vanish outside of the interface domain:

\[
 R(x) = \begin{cases} 
  1, & x \in I_1; \\
  0, & x \in \Omega_\chi - \Omega_\iota
\end{cases}
\]

Both the FE interpolations and the MLS approximation satisfy consistency, which means that the modified interface shape functions satisfy consistency and interpolate a linear field exactly.

The modified interface shape functions and their first order derivatives in a one-dimensional domain are shown in Fig. 2. It can be seen that the displacement approximation is continuous from the purely MFree domain \( (\Omega_\chi - \Omega_\iota) \) passing to the interface domain \( \Omega_\iota \). The derivative is, however, discontinuous across the boundary. These discontinuities do not remarkably affect the overall results since they only affect a small number of nodes.

Using the above mentioned approximation, the shape functions of the MFree domain along \( I_1 \) possess the Kronecker delta function properties. The MFree (e.g., EFG, MLPG) domain and the domain used by other numerical methods (e.g., FEM, BEM, HBEM, PIM, LPIM, BPIM, HBPIM), in which the shape functions satisfy the delta function properties, can be coupled directly in the standard manner.
Example method 1: The coupled EFG/BEM method

1) EFG formulation

The function of EFG is presented as follows:

$$\Pi^{(EFG)} = \frac{1}{2} e^T \cdot \sigma d\Omega - \int u^T \cdot b d\Gamma - \int u^T \cdot \bar{r} d\Gamma \quad (11)$$

Substituting Eqs. (3), (7), and (9) into Eq. (11), and using the stationary condition, the following modified EFG equations can be obtained:

$$K^{(EFG)} U^{(EFG)} = F^{(EFG)} \quad (12)$$

where $K^{(EFG)}$ and $F^{(EFG)}$ are stiffness matrix and force vector of EFG, respectively.

It should be noted here that in computing EFG system equations, the modified shape function presented in Eq. (4) are used to ensure that the shape function along the interface boundary satisfies the Kronecker delta function properties.

2) Equivalent BEM formulation

The formulation of the conventional boundary integral equation (BIE) can be written as

$$c^T u + \int u^T \bar{r} d\Gamma = \int u^T \bar{r} d\Gamma + \int b u^T d\Omega \quad (13)$$

Consider the case that the boundary displacement and traction of $u$ and $t$ are given by interpolation functions and the values at the boundary nodes. The boundary integral equation, Eq. (13), can be written in matrix form as

$$H^{(BE)} U^{(BE)} = G^{(BE)} T^{(BE)} + B^{(BE)} \quad (14)$$

In order to combine the BE region with the EFG region, the BEM formulation Eq.(14) is converted to an equivalent EFG (or FEM) formulation. Let us transform Eq. (14) by inverting $G$ and multiplying the result by the distribution matrix $M$,

$$(M G^{-1}_{(BE)} H_{(BE)}) U^{(BE)} - (M G^{-1}_{(BE)} B_{(BE)}) = M T^{(BE)} \quad (15)$$

where distribution matrix $M$ is obtained from BEM interpolants and defined as

$$M = \int \Phi \Psi^T d\Gamma \quad (16)$$

We can now define:

$$K'_{(BE)} = M G^{-1}_{(BE)} H_{(BE)} \quad (17)$$

$$F_{(BE)} = M T_{(BE)} + M G^{-1}_{(BE)} B_{(BE)} \quad (18)$$

Hence, Eq. (15) has the following equivalent BEM form:

$$K'_{(BE)} U = F_{(BE)} \quad (19)$$

The equivalent BE stiffness matrix $K'_{(BE)}$ is generally asymmetric. The asymmetry arises from the approximations involved in the discretization process and the choice of the assumed solution. If Eq. (19) directly combined with the system equation of EFG, Eq. (12), the symmetry of EFG stiffness matrix would be destroyed. The symmetrization should be done for $K'_{(BE)}$.

One simple method is minimizing the squares of the errors in the asymmetric off-diagonal terms of $K'_{(BE)}$. Hence, a new symmetric equivalent BE stiffness matrix $K_{(BE)}$ can be obtained,

$$K_{(BE)} = \frac{1}{2} (K'_{(BE)} + K'_{(BE),r}) \quad (20)$$

Equation (19) can be rewritten as

$$K_{(BE)} U_{(BE)} = F_{(BE)} \quad (21)$$

3) Combination EFG with BEM

After using the modified shape function, the nodal parameters of field nodes along the interface boundary in the EFG domain become the nodal displacements, i.e.,

$$u_i^h(x) = u_i(x), \text{ on } \Gamma^i \quad (22)$$

Hence, the EFG domain and the BEM domain can be combined as

$$\begin{bmatrix}
K^{(EFG)}_{11} & K^{(EFG)}_{12} & 0 \\
K^{(EFG)}_{21} & K^{(EFG)}_{22} + K^{(BEM)}_{11} & K^{(BEM)}_{21} \\
0 & K^{(BEM)}_{11} & K^{(BEM)}_{22}
\end{bmatrix} \cdot \begin{bmatrix}
U^{(EFG)}_1 \\
U^{(EFG)}_2 \\
U^{(BEM)}_1 \\
U^{(BEM)}_2
\end{bmatrix} = \begin{bmatrix}
F^{(EFG)}_1 \\
F^{(EFG)}_2 \\
F^{(BEM)}_1 \\
F^{(BEM)}_2
\end{bmatrix} \quad (23)$$

Solving Eq. (23), the results of the problem for the coupled EFG/BEM method can be obtained.

The coupled method using the combination technique 1 (using interface elements) has the following advantages:

- It is easy to use and can obtain satisfactory results for most problems considered.
- It is suitable for most coupled methods.
- It can also be used to enforce essential boundary conditions in an MFree method in which the MLS
where the last term of Eq. (25) is to enforce the essential boundary conditions using the Lagrange multiplier method. The Lagrange multiplier $\zeta$ is given by the following approximation:

$$\zeta = N^T \zeta_0$$

where $N$ is an interpolation function and $\zeta_0$ is the vector of the unknown parameter.

The functional of the HBPIIM method is given as follows:\(^{11}\):

$$\Pi_{\text{(HBPIIM)}} = \int_{\Gamma} \left[ \sigma \cdot \partial \Omega - \int_{\Gamma} u^T \cdot b d\Omega - \int_{\Gamma} u^T \cdot \tau d\Gamma + \int_{\Gamma} \lambda^T_{\text{(HBPIIM)}} \cdot (\bar{u} - u) d\Gamma \right]$$

(27)

where the last term of Eq. (27) is to enforce the essential compatibility conditions between domain displacements and boundary displacements.

Introducing $\Pi_{\text{(1)}}$ and $\Pi_{\text{(2)}}$ of Eq. (24) separately into functions Eqs. (25) and (27), generalized functional forms of EFG and HBPIIM can be written as:

$$\Pi_{\text{(EFG)}} = \int_{\Gamma} \left[ \frac{1}{2} \sigma^T \cdot \sigma d\Omega - \int_{\Gamma} u^T \cdot b d\Omega - \int_{\Gamma} u^T \cdot \tau d\Gamma \right.$$ \nonumber

$$\int_{\Gamma} \left[ \sigma \cdot \partial \Omega - \int_{\Gamma} u^T \cdot b d\Omega - \int_{\Gamma} u^T \cdot \tau d\Gamma + \int_{\Gamma} \lambda^T_{\text{(EFG)}} \cdot (\bar{u} - u) d\Gamma \right]$$

(28)

$$\Pi_{\text{(HBPIIM)}} = \int_{\Gamma} \left[ \frac{1}{2} \sigma^T \cdot \sigma d\Omega - \int_{\Gamma} u^T \cdot b d\Omega - \int_{\Gamma} u^T \cdot \tau d\Gamma \right.$$ \nonumber

$$\int_{\Gamma} \left[ \sigma \cdot \partial \Omega - \int_{\Gamma} u^T \cdot b d\Omega - \int_{\Gamma} u^T \cdot \tau d\Gamma + \int_{\Gamma} \lambda^T_{\text{(HBPIIM)}} \cdot (\bar{u} - u) d\Gamma \right]$$

(29)

In these variational formulations, the domains of EFG and HBPIIM are connected via Lagrange multipliers, $\gamma$.

In the EFG domain, $u$ is given by the MLS approximation. $\gamma$ is given by interpolation functions $A$ and nodal value $\gamma_1$ of the interface boundary is
\[ \gamma = A^T \gamma_1 \]  
(30)

\( A \) is a matrix formed by the PIM shape functions. Substituting Eqs. (26) and (30) into Eqs. (28), and using the stationary condition, the following modified EFG equations can be obtained:

\[
\begin{bmatrix}
K_{(EFG)} & G_{(EFG)} & C_{(EFG)} & \mathbf{U}_{(EFG)} & \mathbf{F}_{(EFG)} \\
C_{(EFG)}^T & 0 & 0 \\
C_{(EFG)}^T & 0 & 0 & \zeta_{(EFG)} & \mathbf{Q}_{(EFG)} \\
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\
\end{bmatrix}
= \begin{bmatrix}
\gamma_1 \\
\end{bmatrix}
\]  
(31)

where \( C_{(EFG)} \) is defined as

\[
C_{(EFG)} = \int I \phi_{(EFG)}^T d\Gamma 
\]  
(32)

Integrating the first term on the right-hand side of Eq. (29) by parts, and using fundamental solutions and the stationary condition, lead to the following modified HB PIM system equations:

\[
\begin{bmatrix}
K_{(HBPM)} & -H_{(HBPM)} \\
-H_{(HBPM)} & 0 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_{(HBPM)} \\
\gamma_1 \\
\end{bmatrix}
= \begin{bmatrix}
\mathbf{F}_{(HBPM)} \\
0 \\
\end{bmatrix}
\]  
(33)

\( H_{(HBPM)} \) is defined as

\[
H_{(HBPM)} = \int I \phi_{(HBPM)}^T d\Gamma 
\]  
(34)

Because two domains are connected along the interface boundary \( \Gamma_i \), assembling of Eqs. (31) and (33) yields a linear system of the following form:

\[
\begin{bmatrix}
K_{(EFG)} & 0 & G_{(EFG)} & C_{(EFG)} & \mathbf{U}_{(EFG)} & \mathbf{F}_{(EFG)} \\
0 & K_{(HBE)} & 0 & -H_{(HBE)} \\
G_{(EFG)}^T & 0 & 0 & 0 \\
C_{(EFG)}^T & -H_{(HBE)}^T & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_{(EFG)} \\
\mathbf{U}_{(HBE)} \\
\zeta \\
\gamma_1 \\
\end{bmatrix}
= \begin{bmatrix}
\mathbf{F}_{(EFG)} \\
\mathbf{F}_{(HBE)} \\
\mathbf{Q}_{(EFG)} \\
0 \\
\end{bmatrix}
\]  
(15)

The coupling compatibility condition is satisfied via the above technique.

The coupled method using the modified variational form method has the following advantages:

- It keeps the high continuity of MFree shape functions in the whole domain.
- It usually has better accuracy than technique 1.
- It does not require the “meshing” work in the interface domain.

However, it also has some disadvantages:

- It will enlarge the dimension of the final system equation, and increase the computational cost.
- It is only suitable for the coupled methods, in which weak forms (functional forms) are used in both numerical methods. Hence, its usage is limited. For example, it is difficult to use in the coupled methods of FDM/BPIM, EFG/MWS, MLPG/MWS, etc. This is because the strong forms are used in FDM and MWS.

3 Numerical Examples

Detailed numerical results of coupled EFG/BEM, EFG/HBEM, and MLPG/FEM/BEM have been presented in Refs.[9,12-14]. Hence, only some numerical results of the coupled EFG/HBPI/HBRPIM are presented here. Except special mentioned, the units are taken as standard international (SI) units in all examples.

Example 1: Internal pressurized hollow cylinder

A hollow cylinder under internal pressure is shown in Fig. 4. The parameters are taken as \( p = 100, G = 8000, \) and \( v = 0.25 \). The analytical solution for this problem is available. Due to the symmetry of the problem, only one quarter of the cylinder needs to be modelled. The cylinder is divided into two parts, where EFG and HBPI or HBRPIM are applied. As shown in Fig. 4b, a total of 78 nodes are used to discretize the domain and boundary in EFG/HBPI and EFG/HBRPIM.

![Fig. 4 Hollow cylinder subjected to internal pressure](image-url)
EFG/HBPIM and EFG/HBRPIM results are in very good agreement with the analytical solution.

<table>
<thead>
<tr>
<th>Nodes</th>
<th>Exact.</th>
<th>EFG/HBPIM</th>
<th>EFG/HBRPIM</th>
<th>HBEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8036</td>
<td>0.8077</td>
<td>0.8106</td>
<td>0.8038</td>
</tr>
<tr>
<td>2</td>
<td>0.8036</td>
<td>0.8081</td>
<td>0.8112</td>
<td>0.8027</td>
</tr>
<tr>
<td>3</td>
<td>0.8036</td>
<td>0.8098</td>
<td>0.8109</td>
<td>0.8035</td>
</tr>
<tr>
<td>12</td>
<td>0.4464</td>
<td>0.4465</td>
<td>0.4467</td>
<td>0.4439</td>
</tr>
<tr>
<td>13</td>
<td>0.4464</td>
<td>0.4472</td>
<td>0.4471</td>
<td>0.4445</td>
</tr>
<tr>
<td>14</td>
<td>0.4464</td>
<td>0.4479</td>
<td>0.4483</td>
<td>0.4449</td>
</tr>
</tbody>
</table>

### Example 2: Semi-infinite foundation

In this example, the coupled methods are used in a semi-infinite problem, which has been solved using the coupled FEM/BEM method. A structure standing on a semi-infinite foundation is shown in Fig. 5. Loads are imposed on the structure. The infinite foundation can be treated in practice in either of the following two ways:

1) Truncating the plane at a finite distance—the approximate method;
2) Using a fundamental solution appropriate to the semi-space problem rather than a free-space Green’s function in BEM.

The first way is used in this paper because it is convenient to compare the coupled method solutions with FE solutions.

As shown in Fig. 5, Region 2 represents the semi-infinite foundation and is given a semi-circular shape of a very large diameter in relation to Region 1 that represents the structure. Boundary conditions to restrain rigid body movements are applied. Region 1 is the EFG domain and Region 2 is the HBPI or HBRPIM domain. The nodal arrangement of the coupled EFG/HBPIM and EFG/HBRPIM is shown in Fig. 5. The nodal arrangement of EFG for the structure domain is shown in Fig. 6. Consider five concentrated vertical loads on the top of the structure.

The displacements on the top of the structure are given in Table 2. The FEM results are also included in Table 2. The results obtained by the present coupled methods agree well with those obtained using FEM for the entire domain. It is interesting to note that the foundation is adequately represented using only 30 HBPI nodes in the coupled case as compared to 120 for the EFG cases. The saving is considerable.

<table>
<thead>
<tr>
<th>Node No.</th>
<th>FEM (×10⁻⁴)</th>
<th>EFG/HBPIM (×10⁻⁴)</th>
<th>EFG/HBRPIM (×10⁻⁴)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.41</td>
<td>1.44</td>
<td>1.46</td>
</tr>
<tr>
<td>2</td>
<td>1.34</td>
<td>1.35</td>
<td>1.37</td>
</tr>
<tr>
<td>3</td>
<td>1.32</td>
<td>1.33</td>
<td>1.35</td>
</tr>
<tr>
<td>4</td>
<td>1.34</td>
<td>1.35</td>
<td>1.37</td>
</tr>
<tr>
<td>5</td>
<td>1.41</td>
<td>1.44</td>
<td>1.46</td>
</tr>
</tbody>
</table>

### 4 Remarks

The coupled methods between MFree methods and established numerical methods are discussed in this paper. Two techniques are developed to satisfy the displacement compatibility condition on the interface boundary. Compared with the MFree method in the whole domain, the merits of using the coupled methods are as follows:
1) The computational cost is lower because the MFree method is only used in one part of problem domain.

2) The difficulty of imposing essential boundary conditions for the domain-type of MFree methods that use MLS shape functions can be overcome by modelling the portion of the domain with essential boundaries using methods that use shape functions with the Kronecker delta function property, such as FEM, BEM, PIM, and BPIM.

3) Coupling with boundary-type of MFree methods works particularly well for some special problems such as a problem with the infinite domain.

References


