New existence and multiplicity theorems of periodic solutions for non-autonomous second order Hamiltonian systems

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Abstract

In the present paper, the non-autonomous second order Hamiltonian systems

\[
\begin{align*}
\ddot{u}(t) &= \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T] \\
u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0,
\end{align*}
\]

are studied and a new existence theorem and a new multiplicity theorem of periodic solutions are obtained. © 2007 Elsevier Ltd. All rights reserved.

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1. Introduction and main results

Consider the non-autonomous second order Hamiltonian systems

\[
\begin{align*}
\ddot{u}(t) &= \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T] \\
u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0,
\end{align*}
\]

where $T > 0$ and $F: [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in $t$ for every $x \in \mathbb{R}^N$ and continuously differentiable in $x$ for a.e. $t \in [0, T]$, and there exist $a \in C(R^+, R^+)$, $b \in L^1(0, T; R^+)$, such that

\[
|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)
\]

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

Then the corresponding functional $\varphi$ on $H^1_T$ given by

\[
\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T F(t, u(t)) \, dt
\]
is continuously differentiable and weakly lower semicontinuous on $H^1_T$, where
\[ H^1_T = \{ u : [0, T] \to \mathbb{R}^N \mid u \text{ is absolute continuous}, u(0) = u(T) \text{ and } \dot{u} \in L^2(0, T; \mathbb{R}^N) \} \]
is a Hilbert space with the norm defined by
\[ \| u \| = \left( \int_0^T |u(t)|^2 \, dt + \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} \]
for $u \in H^1_T$ (see [4]). Moreover,
\[ \langle \varphi'(u), v \rangle = \int_0^T \dot{u}(t) \cdot \dot{v}(t) \, dt + \int_0^T \nabla F(t, u(t)) \cdot v(t) \, dt \]
for all $u, v \in H^1_T$. It is well known that the weak solutions of the problem (1.1) correspond to the critical points of $\varphi$.

When $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$, Mawhin–Willem [4] studied the existence of solutions for the problem (1.1). For non-convex potential cases, the existence and multiplicity of solutions has been also researched by many people, for example, see [1,3,5,7–16] and their references. Particularly, Antonacci–Magrone [1] studied symmetrical potential changing sign case, Tang–Wu [7–13] studied $\gamma$-Quasisubadditive, subadditive, coercive potential cases and the cases of the nonlinearity grow sublinearly and subquadratically. In [14], first author studied the case of the nonlinearity grow linearly, and in [3,5,15,16], homoclinic solutions and infinitely many periodic solutions problems are studied, too.

In the present paper, we give a existence theorem and a multiplicity theorem under some new conditions for the problem (1.1). Our main results are the following two theorems.

**Theorem 1.** If following hold:

(F0) there exist a $\delta > 0$ and an integer $k \geq 0$ such that:
\[
-\frac{1}{2}(k + 1)^2 \omega^2 |x|^2 \leq F(t, x) \leq -\frac{1}{2} k^2 \omega^2 |x|^2
\]
for all $|x| \leq \delta$ and a.e. $t \in [0, T]$, where $\omega = \frac{2\pi}{T}$, 

(F1) $\liminf_{|x| \to \infty} \frac{F(t, x)}{|x|^2} \geq 0$, a.e. $t \in [0, T]$, 

and 

(F2) whenever $\{u_n\} \subset H^1_T$ is such that: $\|u_n\| \to \infty$ and $\frac{|u_n|\sqrt{T}}{\|u_n\|} \to 1$ ($\overline{u_n} = \frac{1}{T} \int_0^T u_n(t) \, dt$),
\[ \limsup_{n \to \infty} \int_0^T F(t, u_n(t)) \, dt = +\infty, \]

then the problem (1.1) has at least three distinct solutions in $H^1_T$.

**Theorem 2.** If the following hold:

(F3) whenever $\{u_n\} \subset H^1_T$ is such that $\|u_n\| \to \infty$ and $\frac{|u_n|\sqrt{T}}{\|u_n\|} \to 1$,
\[ \liminf_{n \to \infty} \int_0^T \nabla F(t, u_n(t)) \cdot \frac{u_n}{|u_n|} \, dt < 0, \]

and 

(F4) there exists a bounded measurable function $g : [0, T] \to \mathbb{R}$ such that
\[ g(t) \leq \liminf_{|x| \to \infty} \frac{\nabla F(t, x) \cdot x}{|x|}, \quad \text{a.e. } t \in [0, T], \]

then the problem (1.1) has at least one solution in $H^1_T$. 

2. Proof of theorems

We need the following theorem, which was due to Brezis–Nirenberg [2].

**Theorem A.** Let $X$ be a Banach space with a direct sum decomposition

$$X = X_1 igoplus X_2$$

with $k = \dim X_2 < \infty$. Let $\varphi$ be a $C^1$ function on $X$ satisfying (P.S.) condition. Assume that, for some $r > 0$,

$$\varphi(u) \geq 0, \quad \text{for } u \in X_1, \quad \|u\| \leq r$$

and

$$\varphi(u) \leq 0, \quad \text{for } u \in X_2, \quad \|u\| \leq r.$$ 

Assume also that $\varphi$ is bounded below and $\inf X \varphi < 0$. Then $\varphi$ has at least two nonzero critical points.

**Proof of Theorem 1.** By $(F_0)$ we know that $F(t, 0) = 0$ and $\nabla F(t, 0) = 0$ for a.e. $t \in [0, T]$, and hence $u(t) \equiv 0$ is a solution of (1.1).

To complete the proof we prove the following lemma.

**Lemma 1.** If conditions $(F_1)$ and $(F_2)$ hold, then $\lim_{\|u\| \to \infty} \varphi(u) = +\infty$, $\varphi$ satisfies the (P.S.) condition and is bounded below.

**Proof.** If there are a sequence $\{u_n\}$ and a constant $c$ such that $\|u_n\| \to \infty$ as $n \to \infty$ and $\varphi(u_n) \leq c, n = 1, 2, \ldots$, then let $v_n = \frac{\|u_n\|}{\|u_n\|} u_n$, $\overline{v}_n = \frac{1}{n} \int_0^T u_n(t)dt$ and $\overline{u}_n = u_n - \overline{v}_n$. Since $H^1_T$ is a Hilbert space, there is a point $v_0 \in H^1_T$ and a sub-sequence of $\{v_n\}$, we still note by $\{v_n\}$, such that

$$v_n \to v_0 \quad \text{in } H^1_T,$$

and

$$v_n \to v_0 \quad \text{in } L^2[0, T].$$

For any $\varepsilon > 0$, by $(F_1)$ there is a $M > 0$ such that $F(t, x) > -\frac{\varepsilon}{2} |x|^2$ for all $x \in \mathbb{R}^N$ with $|x| > M$ and a.e. $t \in [0, T]$. Let $a_M = \max_{|x| \leq M} a(|x|)$. Then by the assumption (A), one has $|F(t, x)| \leq a_M b(t)$ for all $x \in \mathbb{R}^N$ with $|x| \leq M$ and a.e. $t \in [0, T]$. Hence

$$F(t, x) \geq -\frac{\varepsilon}{2} |x|^2 - a_M b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Consequently,

$$\frac{c}{\|u_n\|^2} \geq \frac{\varphi(u_n)}{\|u_n\|^2} \geq \frac{1}{2} \int_0^T |\dot{v}_n(t)|^2 dt + \frac{1}{\|u_n\|^2} \int_0^T F(t, u_n(t)) dt \geq \frac{1}{2} \int_0^T |\dot{v}_n(t)|^2 dt - \frac{1}{\|u_n\|^2} \int_0^T \left[ \frac{\varepsilon}{2} |u_n(t)|^2 + a_M b(t) \right] dt \geq \frac{1}{2} \int_0^T |\dot{v}_n(t)|^2 dt - \frac{\varepsilon}{2} \int_0^T |v_n(t)|^2 dt - \frac{a_M}{\|u_n\|^2} \int_0^T b(t) dt = \frac{1}{2} - \frac{1}{2} (1 + \varepsilon) \int_0^T |v_n(t)|^2 dt - \frac{a_M}{\|u_n\|^2} \int_0^T b(t) dt.$$ 

It implies $\int_0^T |v_0(t)|^2 dt \geq 1$. On the other hand, by weak lower semi-continuity of the norm, one has $\|v_0\| \leq \lim \inf \|v_n\| = 1.$
Hence $|\dot{v}_0(t)| = 0$ a.e. $t \in [0, T]$, so that $|v_0(t)| = \text{constant}$ for all $t \in [0, T]$, and hence $|v_0|^2 = \frac{1}{T}$. Consequently,

$$\frac{|u_n|^2}{\|u_n\|^2} \to \frac{1}{T},$$

and hence

$$\frac{|u_n|}{\|u_n\|} \to 1.$$  

By (F2), $\limsup_{n \to \infty} \int_0^T F(t, u_n(t))dt = +\infty$. Hence

$$c \geq \limsup_{n \to \infty} \varphi(u_n) \geq \limsup_{n \to \infty} \int_0^T F(t, u_n(t))dt \to +\infty.$$  

This is a contradiction. Hence $\varphi$ is coercive on $H^1_\Psi$. By weak lower semi-continuity of $\varphi$, $\varphi$ satisfies the (P.S.) condition and is bounded below.

By Lemma 1 we know that $\lim_{\|u\| \to \infty} \varphi(u) = +\infty$, $\varphi$ satisfies the (P.S.) condition and is bounded below. Take $\rho = \frac{\delta}{c}$, where $c$ is a positive constant such that $\|u\| \leq c\|u\|$ for all $u \in H^1_\Psi$. Let $X_2 = \{\sum_{j=0}^k (a_j \cos j\omega t + b_j \sin j\omega t): a_j, b_j \in \mathbb{R}^N, j = 1, 2, \ldots, k\}$ and $X_1$ is the orthogonal complement of $X_2$ in $H^1_\Psi$. By (F0) we have

$$\varphi(u) \leq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2} k^2 \omega^2 \int_0^T |u(t)|^2 dt \leq 0$$

for all $u \in X_2$ with $\|u\| \leq \rho$ and

$$\varphi(u) \geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2} (k + 1)^2 \omega^2 \int_0^T |u(t)|^2 dt \geq 0$$

for all $u \in X_1$ with $\|u\| \leq \rho$.

If $\inf\{\varphi(u): u \in H^1_\Psi\} = 0$, then all $u \in X_2$ with $\|u\| \leq \rho$ are minima of $\varphi$, which implies that $\varphi$ has infinite critical points. If $\inf\{\varphi(u): u \in H^1_\Psi\} < 0$, then by Theorem 4 in [2] $\varphi$ has at least two nonzero critical points. Hence problem (1.1) has at least two nontrivial solutions in $H^1_\Psi$. Therefore, problem (1.1) has at three distinct solutions in $H^1_\Psi$. This completes the proof.

**Proof of Theorem 2.** We divide the proof into several lemmas.

**Lemma 2.** If (F3) condition hold, then $\varphi$ is anti-coercive on $\mathbb{R}^N$.

**Proof.** First, we prove that there exist $\delta > 0$, $\rho > 0$ such that $\int_0^T \nabla F(t, x) \cdot x dt \leq -\delta |x|$ for all $x \in \mathbb{R}^N$ with $|x| \geq \rho$.

If not, there is a sequence $\{x_n\} \subset \mathbb{R}^N$ with $|x_n| \to \infty$ and

$$\int_0^T \nabla F(t, x_n) \cdot \frac{x_n}{|x_n|} > -\frac{1}{n}, \quad \forall n \geq 1.$$  

This contradicts (F3). Therefore, for $u \in \mathbb{R}^N$ with $|u| > \rho$, one has

$$\varphi(u) = \int_0^T F(t, u)dt$$

$$= \int_0^T \left[ \int_0^1 \nabla F(t, su) \cdot u ds \right] dt + \int_0^T F(t, 0)dt$$

$$= \int_0^T \left[ \int_0^\rho \nabla F(t, su) \cdot u ds + \int_0^1 \nabla F(t, su) \cdot u ds \right] dt + c_1.$$  

Since

$$\int_0^T \left[ \int_0^\rho \nabla F(t, su) \cdot u ds \right] dt \leq \int_0^T \left[ \int_0^\rho |\nabla F(t, su)||u| ds \right] dt$$

...
Lemma 3

There exists $c > 0$ such that, if $u \in W_T^{1,p}$, then:

$$\|u\|_\infty := \max_{t \in [0, T]} |u(t)| \leq c \|u\|_{W_T^{1,p}}.$$ 

Moreover, if $\int_0^T u(t) \, dt = 0$, then

$$\|u\|_\infty \leq c \|\dot{u}\|_{L^p}.$$ 

Lemma 3. Let $\tilde{H}_T^1 := \{ u \in H_T^1 : \int_0^T u(t) \, dt = 0 \}$. If condition (F4) holds, then $\varphi$ is coercive on $\tilde{H}_T^1$.

Proof. By (F4) there exist $\lambda < 0$ and $M > 0$ such that $\nabla F(t, x) \cdot x > \lambda |x|$ for all $x \in R^N$ with $|x| > M$ and a.e. $t \in [0, T]$. Moreover, Let $a_M = \max_{|x| \leq M} a(|x|)$. Then by (A) we know that:

$$\nabla F(t, x) \cdot x \geq -a_M b(t)|x|$$

for all $x \in R^N$ with $|x| \leq M$ and a.e. $t \in [0, T]$. Hence

$$\nabla F(t, x) \cdot x \geq \lambda |x| - a_M b(t)|x|$$

for all $x \in R^N$ and a.e. $t \in [0, T]$. Consequently,

$$F(t, x) = F(t, x) - F(t, 0) + F(t, 0) = \int_0^1 [\nabla F(t, sx) \cdot x] \, ds + F(t, 0) \geq \lambda |x| - a_M b(t)|x| + F(t, 0).$$

If there are a constant $c$ and a sequence $\{u_n\} \subset \tilde{H}_T^1$ such that $\|u_n\| \to \infty$ and $\varphi(u_n) \leq c$, $n = 1, 2, \ldots$, then by Proposition A, one has

$$c \geq \varphi(u_n) = \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 \, dt + \int_0^T F(t, u_n(t)) \, dt \geq \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 \, dt + \int_0^T [\lambda |u_n(t)| - a_M b(t)|u_n(t)| + F(t, 0)] \, dt \geq c_1 \|u_n\|^2 - c_2 \|u_n\| - c_3,$$

where $c_1, c_2, c_3$ are positive constants. This contradicts that $\|u_n\| \to \infty$. Hence $\varphi$ is coercive on $\tilde{H}_T^1$.

Lemma 4. If a sequence $\{u_n\} \subset H_T^1$ is such that $\varphi'(u_n) \to 0$ and $\{u_n\}$ is bounded in $H_T^1$, then $\{u_n\}$ has a convergent subsequence in $H_T^1$.

Proof. Since $H_T^1$ is a Hilbert space, passing to a subsequence if necessary, we may assume that there is a point $u_0 \in H_T^1$ such that
By the proof of Lemma 3 and hence

\[ u_n \rightarrow u_0 \quad \text{in} \quad H_T^1, \]

and

\[ u_n \rightarrow u_0 \quad \text{in} \quad L^2[0, T]. \]

By Proposition 1.2 in [4] we know that \( \{u_n\} \) converges uniformly to \( u_0 \) on \([0, T]\). Hence there is a \( M > 0 \) such that

\[ \max_{0 \leq t \leq T} |u_n(t)| \leq M, \quad n = 1, 2, \ldots. \]

Let \( a_M = \max_{|x| \leq M} a(|x|) \). Then by (A) we know that \( |\nabla F(t, u_n(t))| \leq a_M b(t) \) for a.e. \( t \in [0, T] \).

Since

\[
\langle \varphi'(u_n) - \varphi'(u_m), u_n - u_m \rangle = \int_0^T |\dot{u}_n(t) - \dot{u}_m(t)|^2 \, dt \\
+ \int_0^T \nabla F(t, u_n(t)) - \nabla F(t, u_m(t)) \cdot (u_n(t) - u_m(t)) \, dt,
\]

\[
\int_0^T |\dot{u}_n(t) - \dot{u}_m(t)|^2 \, dt \leq \|\varphi'(u_n) - \varphi'(u_m)\| \cdot \|u_n - u_m\| \\
+ 2a_M \|u_n - u_m\|_\infty \int_0^T b(t) \, dt \rightarrow 0 \quad \text{as} \ n, m \rightarrow \infty,
\]

where \( \|u_n - u_m\|_\infty = \max_{0 \leq t \leq T} |u_n(t) - u_m(t)| \). Consequence,

\[
\|u_n - u_m\|^2 = \int_0^T |\dot{u}_n(t) - \dot{u}_m(t)|^2 \, dt + \int_0^T |u_n(t) - u_m(t)|^2 \, dt \rightarrow 0, \quad \text{as} \ n, m \rightarrow \infty,
\]

and hence \( \{u_n\} \) is a Cauchy sequence in \( H_T^1 \). By the completeness of \( H_T^1 \) we know that \( \{u_n\} \) is a convergent sequence in \( H_T^1 \). This completes the proof.

**Lemma 5.** If \((F_3)\) condition and \((F_4)\) hold, then \( \varphi \) satisfies the (P.S.) condition.

**Proof.** By the proof of Lemma 3 we know that there exist \( \lambda < 0 \) and \( M > 0 \) such that

\[ F(t, x) \geq \lambda |x| - a_M b(t)|x| + F(t, 0). \]

If a sequence \( \{u_n\} \subset H_T^1 \) is such that \( \varphi'(u_n) \rightarrow 0 \) and there exists a constant \( c \) such that \( \varphi(u_n) \leq c, \ n = 1, 2, \ldots \), then \( \{u_n\} \) is bounded in \( H_T^1 \). Otherwise, passing to a subsequence if necessary, we may assume that \( \|u_n\| \rightarrow \infty \). Let \( v_n = \frac{u_n}{\|u_n\|} \). Since \( H_T^1 \) is a Hilbert space, there is a point \( v_0 \in H_T^1 \) and a sub-sequence of \( \{v_n\} \), we still note by \( \{v_n\} \), such that:

\[
v_n \rightharpoonup v_0 \quad \text{in} \quad H_T^1, \\
v_n \rightarrow v_0 \quad \text{in} \quad L^2[0, T], \\
v_n(t) \rightarrow v_0(t), \quad \text{a.e.} \ t \in [0, T],
\]

and there is a function \( w \in L^2([0, T]) \) such that \( |v_n(t)| \leq W(t) \) for a.e. \( t \in [0, T] \). Hence, by Proposition 1.1 in [4] one has

\[
\frac{c}{\|u_n\|^2} \geq \frac{\varphi(u_n)}{\|u_n\|^2} \\
= \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 \, dt + \frac{1}{\|u_n\|^2} \int_0^T F(t, u_n(t)) \, dt \\
\geq \frac{1}{2} \int_0^T |\dot{v}_n(t)|^2 \, dt + \frac{1}{\|u_n\|^2} \int_0^T [\lambda |u_n(t)| - a_M b(t)|u_n(t)| + F(t, 0)] \, dt.
\]
\[ \frac{1}{2} \int_0^T |\dot{v}_n(t)|^2 \, dt + \frac{\lambda}{\| u_n \|} \int_0^T |v_n(t)| \, dt - \frac{a_M}{\| u_n \|} \int_0^T b(t) |v_n(t)| \, dt + \frac{1}{\| u_n \|^2} \int_0^T F(t, 0) \, dt \]

\[ = \frac{1}{2} - \frac{1}{2} \int_0^T |v_n(t)|^2 \, dt - \frac{c_1}{\| u_n \|} + \frac{c_2}{\| u_n \|^2}. \]

It implies \( \int_0^T |v_0(t)|^2 \, dt \geq 1 \). On the other hand, by weak lower semi-continuity of the norm, one has

\[ \| v_0 \| \leq \liminf_{n \to \infty} \| v_n \| = 1. \]

Hence \( |\dot{v}_0(t)| = 0 \) a.e. \( t \in [0, T] \), so that \( |v_0(t)| = \text{constant} \) for a.e. \( t \in [0, T] \), and hence \( |v_0|^2 = \frac{1}{T} \). Consequently, \( \frac{\| u_n \| \sqrt{T}}{\| u_n \|} \to 1 \). Therefore, by (F3) one has

\[ \liminf_{n \to \infty} \int_0^T \nabla F(t, u_n(t)) \cdot \frac{u_n}{\| u_n \|} < 0. \]

On the other hand,

\[ \int_0^T \nabla F(t, u_n(t)) \cdot \frac{u_n}{\| u_n \|} = \left\langle \psi'(u_n), \frac{u_n}{\| u_n \|} \right\rangle \to 0 \quad \text{as} \quad n \to \infty. \]

This is a contradiction. Hence \( \{ u_n \} \) is bounded in \( H^1_T \).

By virtue of Lemma 2, \( \{ u_n \} \) has a convergent subsequence in \( H^1_T \), and hence \( \psi \) satisfies the (P.S.) condition. Sum up the above fact, Theorem 2 follows from Theorem 4.6 in [6].

References