Reducible chains in several types of 2-connected graphs*

Fuji Zhang and Xiaofeng Guo

Department of Mathematics, Xinjiang University, Wulumuqi, Xinjiang, China

Received 30 March 1988
Revised 30 October 1990

Abstract

Let $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ denote the sets of all 2-connected graphs, minimally 2-connected graphs, critically 2-connected graphs, and critically and minimally 2-connected graphs, respectively. We introduce the concept of $\mathcal{G}_i$-reducible chains of a graph $G$ in $\mathcal{G}_i$, $i = 0, 1, 2, 3$, and give the upper bound and the lower bound of a number of $\mathcal{G}_i$-reducible chains of $G$ which are both sharp. Furthermore, a construction method of $\mathcal{G}_3$ is obtained.

Let $G = (V(G), E(G))$ be a finite simple graph, and let $\kappa(G)$ be the connectivity of $G$. $G$ is 2-connected if $\kappa(G) \geq 2$, $G$ is minimally 2-connected if $\kappa(G) = 2$ but $\kappa(G - e) < 2$ for any $e \in E(G)$, and $G$ is critically 2-connected if $\kappa(G) = 2$ but $\kappa(G - v) < 2$ for any $v \in V(G)$.

We denote by $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ the sets of all 2-connected graphs, minimally 2-connected graphs, critically 2-connected graphs, and critically and minimally 2-connected graphs, respectively. We call a vertex $v$ critical if $\kappa(G) > 2$ but $\kappa(G - v) < 2$. The cyclomatic number of $G$ and the degree of a vertex $v$ in $G$ are denoted by $\gamma(G)$ and $d_v$, respectively.

A satisfactory construction method of $\mathcal{G}_0$ can be found in Tutte's book [2]. Dirac gave a construction method of $\mathcal{G}_1$. In this paper, by using the concept of $\mathcal{G}_3$-reducible chain, we obtain a method for constructing $\mathcal{G}_i$, $i = 0, 1, 2, 3$, and give the sharp upper and lower bounds of the number of $\mathcal{G}_i$-reducible chains.

Definition 1. Let $H$ be a subgraph of $G$. The graph induced by $E(G) - E(H)$ is denoted by $G - H$ (i.e., $E(G - H) = E(G) - E(H)$, and $V(G - H) = \{v \mid v$ is incident with an edge in $E(G) - E(H)\}$).

* The project was supported by NSFC.
**Definition 2.** A block in $G$ is a maximal 2-connected subgraph of $G$. A block $B$ of $G$ is said to be extremal if $B$ and $G - B$ have exactly one common vertex.

**Definition 3.** Let $P$ be a path in $G$ of length greater than or equal to 1. If both the degrees of the origin and terminus of $P$ are not equal to 2 and the degree of any other vertex of $P$ is equal to 2 in $G$, then $P$ is said to be a maximal chain (note that if both the degrees of the origin and terminus of $P$ are greater than 2, then $P$ may also be called 'handle', as in topology).

**Definition 4.** Let $G \in \mathcal{K}_i$, $i = 0, 2, 3$, and let $P$ be a path of $G$. If $G - P \in \mathcal{K}_i$, we call $P$ a $\mathcal{K}_i$-reducible chain, otherwise a $\mathcal{K}_i$-irreducible chain. The number of $\mathcal{K}_i$-reducible chains of $G$ is denoted by $\rho_i(G)$.

**Lemma 5.** Let $G \in \mathcal{K}_0$. Then:

(i) $G \in \mathcal{K}_0$ iff the length of any $\mathcal{K}_0$-reducible chain of $G$ is greater than 1;
(ii) if $G \in \mathcal{K}_1$ then the length of any $\mathcal{K}_1$-reducible chain of $G$ is not equal to 2;
(iii) $G \in \mathcal{K}_0$ iff the length of any $\mathcal{K}_0$-reducible chain of $G$ is greater than 2.

**Proof.** (i) and (ii) are straightforward, so we only prove (iii). By (i) and (ii), we only need to prove the sufficiency.

By (i), we have that $G \in \mathcal{K}_0$. If $G \notin \mathcal{K}_0$, then there exists a noncritical vertex $v$ in $V(G)$. If $d_C(v) = 2$, there is a $\mathcal{K}_0$-reducible chain of length 2 in $G$. If $d_C(v) > 2$, any edge incident with $v$ is a $\mathcal{K}_i$-reducible chain of length 1 in $G$. Both cases contradict the assumption.

**Lemma 6.** Let $G \in \mathcal{K}_i$. Let $G_1$ be a 2-connected subgraph of $G$ and $A_1 = V(G_1) \cap V(G - G_1)$.

(i) If $G \in \mathcal{K}_2$, then each vertex in $V(G_1) - A_1$ is a critical vertex of $G_1$.
(ii) If $G \in \mathcal{K}_1$, then $G_1 \in \mathcal{K}_1$ [1, Corollary 3.3].
(iii) If $G \in \mathcal{K}_0$, then $G_1 \in \mathcal{K}_0$.

**Proof.** (i) Suppose that there is a vertex $v \in V(G_1) - A_1$ such that $G_1 - v$ is 2-connected. Since $G \in \mathcal{K}_2$, there exists a vertex $u \in V(G) - v$ which is a cut vertex of $G - v$. By $G_1 - v \in \mathcal{K}_2$, $G_1 - \{u, v\}$ is contained in one component $D$ of $G - \{u, v\}$. Furthermore, by $v \in V(G_1) - A_1$, $v$ is not adjacent to any vertex in $V(G) - V(D) - u - v$, and $u$ is also a cut vertex of $G$, contradicting that $G \in \mathcal{K}_0$.

(iii) By (ii), we have $G \in \mathcal{K}_0$. If $G \notin \mathcal{K}_0$, there is a noncritical vertex $u$ of $G_1$. By (i), $u$ is a vertex in $A_1$. Moreover, by $G_1 - u \in \mathcal{K}_0$ and $G_1 \in \mathcal{K}_0$, $d_{G_1}(u) = 2$, namely, in $G_1$ there are exactly two vertices, say $v_1$ and $v_2$, adjacent to $u$. Let $e_1 = uv_1$, $e_2 = uv_2$. From [1, Theorem 3.1], $G - e_1$ has exactly two extremal blocks, say $B_1$ and $B_2$, containing $v_1$ and $u$ respectively. Clearly, $B_1$ contains $G_1 - u$ and $B_2$ contains $e_2$, since $G_1 - u \in \mathcal{K}_0$ and $e_2 = uv_2$. Thus $v_2 \in V(B_1) \cap V(B_2)$ and $G - e_1 = B_1 \cup B_2$. By the same reason, $B_2 - e_2$ has two extremal
blocks, containing \( v_2 \) and \( u \) respectively. But, then \( G - e_2 = B_1 \cup B_2 - e_2 + e_1 \) would be 2-connected. This contradicts that \( G \in \mathcal{G}_i \).

Now Lemma 6 is proved. \( \square \)

**Theorem 7.** Let \( G \in \mathcal{G}_i \), \( i = 0, 1, 3 \), and \( v(G) \geq 2 \). Then \( \rho_i(G) \geq v(G) + 1 \).

**Proof.** We first prove that \( \rho_0 \geq v(G) + 1 \).

If \( v(G) = 2 \), the theorem is obviously true.

Suppose it holds for \( 2 \leq v(G) < v \), we consider the case \( v(G) = v \).

Let \( P \) be a chain of \( G \) with origin \( x \) and terminus \( y \).

**Case 1:** \( P \) is a \( \mathcal{G}_r \)-reducible chain.

Then \( G - P \in \mathcal{G}_i \). By the induction hypothesis, we have that

\[
\rho_0(G - P) \geq v(G - P) + 1 = v(G) - 1 + 1 = v(G).
\]

Take a \( \mathcal{G}_r \)-reducible chain \( P' \) of \( G - P \) with origin \( x' \) and terminus \( y' \). If \( x \) and \( y \) are not internal vertices of \( P' \), then \( P' \) is a \( \mathcal{G}_r \)-reducible chain of \( G \). If both \( x \) and \( y \) are internal vertices of \( P' \), then the path from \( x \) to \( y \) contained in \( P' \) is a \( \mathcal{G}_r \)-reducible chain of \( G \). In the other cases, without loss of generality, we may assume that \( x \) is an internal vertex of \( P' \) but \( y \) is not, and \( y \neq x' \). Then the path from \( x \) to \( y' \) contained in \( P' \) is a \( \mathcal{G}_r \)-reducible chain of \( G \). Note that \( P \) is a \( \mathcal{G}_r \)-reducible chain of \( G \). We have that \( \rho_0(G) \geq v(G) + 1 \).

**Case 2:** \( P \) is a \( \mathcal{G}_r \)-irreducible chain of \( G \).

Then \( G - P \notin \mathcal{G}_i \). From [1, Theorem 3.1], the block-cut-vertex graph \( bc(G - P) \) is a nontrivial path and \( G - P \) has exactly two extremal blocks \( B_1 \) and \( B_2 \) with cyclomatic number greater than zero, and, for \( i = 1, 2 \), \( B_i \) and \( G - B_i \) have exactly two common vertices.

If \( v(B_i) = 1 \), then \( B_i \) consists of two chains of \( G \), each of them is obviously a \( \mathcal{G}_r \)-reducible chain of \( G \). If \( v(B_i) \geq 2 \), by the induction hypothesis, \( \rho_0(B_i) \geq v(B_i) + 1 \geq 3 \). Let \( Q \) be a \( \mathcal{G}_r \)-reducible chain of \( B_i \). By a similar argument as in Case 1, we can see that there is at least one \( \mathcal{G}_r \)-reducible chain of \( G \) in \( B_i \). We take a \( \mathcal{G}_r \)-reducible chain of \( G \) as \( P \). Then this case is reduced to Case 1. \( \square \)

From Lemma 6, we can assert that if \( G \in \mathcal{G}_1 \), then \( \rho_1(G) = \rho_0(G) \geq v(G) + 1 \), and if \( G \in \mathcal{G}_3 \), then \( \rho_3(G) = \rho_1(G) \geq v(G) + 1 \). Fig. 1 gives a graph \( G \in \mathcal{G}_3 \), which has exactly \( v(G) + 1 \) \( \mathcal{G}_3 \)-reducible chains. It shows that the lower bound given in the theorem is sharp.

![Fig. 1.](image-url)
When \( G \in \mathcal{G}_2 \), a 2-connected subgraph of \( G \) needn’t be critical. It is natural to look for a sharp lower bound of the number of \( \mathcal{G}_2 \)-reducible chains of \( G \). In fact, we have the following.

**Theorem 8.** If \( G \in \mathcal{G}_2 \) and \( v(G) \geq 2 \), then \( \rho_2(G) \geq \lceil \frac{1}{2}(v(G) + 1) \rceil \).

**Proof.** We will find \( \mathcal{G}_2 \)-reducible chains of \( G \) from \( \mathcal{G}_0 \)-reducible chains of \( G \).

Let

\[
R = \{ P \mid P \text{ is a } \mathcal{G}_0 \text{-reducible chain but not a } \mathcal{G}_2 \text{-reducible chain of } G \},
\]

\[
R_1 = \{ P \mid P \in R, \text{ and the origin and terminus of } P \text{ are adjacent} \},
\]

\[
R_2 = \{ P \mid P \in R, \text{ and the origin and terminus of } P \text{ aren’t adjacent} \}.
\]

If \( R \) is empty then \( \rho_2(G) = \rho_0(G) \), and the desired conclusion follows from Theorem 7. We may thus assume that \( R \neq \emptyset \). Let \( P \) be any element of \( R \). Then, by Lemmas 5 and 6, the length of \( P \) is greater than 2 and at least one of its end vertices \( x \) and \( y \), say \( x \), is noncritical in \( G - P \). Now we consider the following two cases.

**Case 1:** \( P \in R_1 \), namely, \( e = xy \in E(G) \).

Evidently, \( e \) is a \( \mathcal{G}_0 \)-reducible chain of \( G \) (see Fig. 2), and also a \( \mathcal{G}_2 \)-reducible chain of \( G \). For another \( P' \in R_1 \) with end vertices \( x' \) and \( y' \) such that \( G - P' - x' \in \mathcal{G}_0 \) and \( e' = x'y' \in E(G) \), obviously, \( e' \) is also a \( \mathcal{G}_2 \)-reducible chain of \( G \). Since the length of \( P' \) is greater than 2, \( x \neq x' \) or \( y' \). Otherwise, \( G - P - x \notin \mathcal{G}_0 \), a contradiction. Thus \( e \neq e' \).

Now we can conclude that \( |R_1| \) \( \mathcal{G}_0 \)-reducible chains in \( R_1 \) correspond to at least \( |R_1| \) \( \mathcal{G}_2 \)-reducible chains of length 1 in \( G \).

**Case 2:** \( P \in R_2 \), namely, \( e = xy \notin E(G) \).

Since the degree of \( x \) in \( G - P \) is greater than 1, we may assume that the set of vertices adjacent to \( x \) in \( G - P \) is \( \{a_1, a_2, \ldots, a_t\} \), \( t \geq 2 \) (see Fig. 2). Obviously, for \( 1 \leq i \leq t \), \( e_i = xa_i \) is a \( \mathcal{G}_2 \)-reducible chain of \( G \) corresponding to \( P \). Note that the number of \( \mathcal{G}_2 \)-reducible chains of \( G \) corresponding to \( P \) is at least two. For another \( P' \in R_2 \) with end vertices \( x' \) and \( y' \) such that \( G - P' - x' \in \mathcal{G}_0 \), if \( x' \in \{a_1, a_2, \ldots, a_t\} \), say \( x' = a_i \), then \( e_i = xa' \) is also a \( \mathcal{G}_2 \)-reducible chain.

![Fig. 2.](image-url)
Reducible chains in several types of 2-connected graphs

Fig. 3. (1) \(v(G) = 2m, \rho_2(G) = \left\lceil \frac{1}{2}(v(G) + 1) \right\rceil = m + 1\). (2) \(v(G) = 2m - 1, \rho_2(G) = \left\lceil \frac{1}{2}(v(G) + 1) \right\rceil = m\).

corresponding to \(P'\). Since \(x\) can not be the end vertex of a \(\mathcal{G}_r\)-reducible chain in \(R_2\) other than \(P\), each \(e_i\) corresponds to at most two \(\mathcal{G}_r\)-reducible chains in \(R_2\).

Now we conclude that there are at least \(\left| R_2 \right| \mathcal{G}_r\)-reducible chains corresponding to \(\left| R_2 \right| \mathcal{G}_r\)-reducible chains in \(R_2\).

Furthermore, for any \(P \in R_1, P' \in R_2\), their corresponding \(\mathcal{G}_r\)-reducible chains are different. Hence there are at least \(\left| R \right| \mathcal{G}_r\)-reducible chains of \(G\) corresponding to \(\left| R \right| \mathcal{G}_r\)-reducible chains of \(G\) in \(R\). Thus \(\rho_2(G) \geq \left| R \right|\). Now it follows from \(\rho_2(G) + \left| R \right| = \rho_0(G)\) that \(\rho_2(G) \geq \left\lceil \frac{1}{2}(v(G) + 1) \right\rceil \geq \frac{1}{2}(v(G) + 1)\).

Theorem 8 is thus proved. \(\square\)

Fig. 3 gives critical 2-connected graphs with \(v(G) \geq 2, \rho_0(G) = v(G) + 1,\) and \(\rho_2(G) = \left\lceil \frac{1}{2}(v(G) + 1) \right\rceil\). They show that the lower bound given in Theorem 8 is sharp.

**Corollary.** Let \(G\) be a graph with \(v(G) \geq 2\). If \(G \in \mathcal{G}_i\), then \(G\) has at least \(v(G) + 1\) vertices of degree 2. If \(G \in \mathcal{G}_i\), then \(G\) has at least \(2(v(G) + 1)\) vertices of degree 2.

Now we turn out attention to the upper bound of \(\rho_0(G)\) of \(G\).

**Theorem 9.** Let \(G \in \mathcal{G}_i, i = 0, 1, 2, 3,\) then \(\rho_i(G) \leq 3(v(G) - 1)\).

**Proof.** We denote the number of chains of \(G\) by \(\varepsilon(G)\). We substitute every chain by an edge to build a graph \(G'\) such that \(|E(G')| = \varepsilon(G), v(G') = v(G)\), and for any \(v \in V(G')\), \(d_{G'}(v) \geq 3\). Since

\[|E(G')| - v(G') + |V(G')| - 1, \quad \text{and} \quad |E(G')| - \frac{1}{2} \sum_{v \in V(G')} d_{G'}(v) > \frac{1}{2} |V(G')|,\]

we have \(\varepsilon(G) \leq 3(v(G) - 1)\). Therefore \(\rho_i(G) \leq \varepsilon(G) \leq 3(v(G) - 1)\).
In order to show that the upper bound given in Theorem 9 is sharp, we only need to give a graph $G$ such that $G \in \mathcal{G}_3$, and $\rho_3(G) = 3(v(G) - 1)$. In fact, we can construct the desired graph $G$ by subdividing every edge of a 3-regular 2-connected graph by inserting two vertices. It is evident that $G \in \mathcal{G}_3$ and $\rho_3(G) = 3(v(G) - 1)$. □

Our results can be used to construct four kinds of 2-connected graphs $\mathcal{G}_i$, $i = 0, 1, 2, 3$.

**Definition 10.** Let $G_i$ be a graph, $x, y \in V(G_i)$, $x \neq y$, and let $P$ be a path disjoint from $G_i$ and of length greater than or equal to one. Let $G = (G_i + P)_{(x,y)}$ denote the graph obtained from $G_i$ by identifying the two end vertices of $P$ with two vertices $x$ and $y$ of $G_i$, respectively.

Clearly, if $G_i \in \mathcal{G}_0$, then $G = (G_i + P)_{(x,y)} \in \mathcal{G}_0$. But, for $G_i \in \mathcal{G}_i$, $i = 1, 2, 3$, $G - (G_i + P)_{(x,y)}$ may not belong to $\mathcal{G}_i$. To ensure that $G \in \mathcal{G}_i$, we need to find some additional conditions.

Let $H$ be a graph with $\kappa(H) = 1$, and let $B$ be an extremal block of $H$. We denote by $V_f(B)$ the set of all the vertices in $B$ such that $v \in V_f(B)$ if $d_B(v) = d_H(v)$.

**Lemma 11.** Let $G = (G_i + P)_{(x,y)}$ and $G_i \in \mathcal{G}_1$. Then $G \in \mathcal{G}_1$ if and only if (i) the length of $P$ is greater than 1, (ii) for any $e \in E(G_i)$, there is an extremal block $B$ in $G_i - e$ such that $V_f(B) \cap \{x, y\} = \emptyset$.

**Proof.** Obvious. □

**Lemma 12.** Let $G = (G_i + P)_{(x,y)}$ and $G_i \in \mathcal{G}_2$. Then $G \in \mathcal{G}_2$ if and only if (i) the length of $P$ is not equal to two, (ii) for any $v \in V(G_i)$, there is an extremal block $B$ in $G_i - v$ such that $V_f(B) \cap \{x, y\} = \emptyset$.

**Proof.** Obvious. □

**Lemma 13.** Let $G = (G_i + P)_{(x,y)}$ and $G_i \in \mathcal{G}_3$. Then $G \in \mathcal{G}_3$ if and only if (i) the length of $P$ is greater than two, (ii) for any $e \in E(G_i)$, there is an extremal block $B$ in $G - e$ such that $V_f(B) \cap \{x, y\} = \emptyset$, (iii) for any $v(\neq x, y)$ of degree 2 in $G_i$, there is an extremal block $B'$ in $G_i - v$ such that $V_f(B') \cap \{x, y\} = \emptyset$.

**Proof.** The necessity is straightforward. So we only prove the sufficiency.

From (i) and (ii), we have $G \in \mathcal{G}_3$. From (i) and (iii), any vertex of degree 2 in $G_i$ or the vertices on $P$ must be critical in $G$. For any vertex of degree greater than two in $G_i$, $d_{G_i}(v)$ is equal to the number of extremal blocks of $G_i - v$. In fact, there is exactly one edge between $v$ and any extremal block of $G_i - v$, since $G_i \in \mathcal{G}_3$. Therefore, there exist extremal blocks in $G - v$ and hence $v$ is critical.

Now it follows that $G \in \mathcal{G}_3$. □
Definition 14. Let $G_1 \in \mathcal{G}_i$, $i = 0, 1, 2, 3$, and $G = (G_1 + P)_{(x,y)}$. We say that $G$ is obtained from $G_1$ by a $\mathcal{G}_i$-operation. If $(G_1 + P)_{(x,y)}$ satisfies the conditions in Lemma 11, we say that $G$ is obtained from $G_1$ by a $\mathcal{G}_i$-operation. Similarly, we can define the $\mathcal{G}_2$-operation and $\mathcal{G}_3$-operation.

Now we can obtain a construction method of $\mathcal{G}_i$, $i = 0, 1, 2, 3$.

From Theorems 7, 8, we know that if $G \in \mathcal{G}_i$, $i = 0, 1, 2, 3$, and $\nu(G) \geq 2$, then $\rho_i(G) > 0$. Let $P_1$ be a $\mathcal{G}_i$-reducible chain of $G$, and let $G_1 = G - P_1$. Then $G = (G_1 + P)_{(x,y)}$ can is said to be constructed from $G_1 \in \mathcal{G}_i$ by a $\mathcal{G}_i$-operation, by Lemmas 11–13 and Definition 14. Therefore we have the following.

Theorem 15. Let $G$ be a graph in $\mathcal{G}_i$, $i = 0, 1, 2, 3$, and $\nu(G) \geq 2$. Then $G$ can be constructed from a cycle by using $\mathcal{G}_i$-operations successively.

References