Strong Tractability of Quasi-Monte Carlo Quadrature Using Nets for Certain Banach Spaces

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Abstract

We consider multivariate integration in the weighted spaces of functions with mixed first derivatives bounded in $L_p$ norms and the weighted coefficients introduced via $\ell_q$ norms, where $p, q \in [1, \infty]$. The integration domain may be bounded or unbounded. The worst-case error and randomized error are investigated for quasi-Monte Carlo quadrature rules. For the worst-case setting the quadrature rule uses deterministic Niederreiter sequences, and for the randomized setting the quadrature rule uses randomly scrambled Niederreiter digital nets. Sufficient conditions are found under which multivariate integration is strongly tractable in the worst-case and randomized settings, respectively. Results presented in this article extend and improve upon those found previously.

Running Title: Strong Tractability of Quasi-Monte Carlo Quadrature

1. Introduction

In many applications, one needs to approximate the weighted multivariate integral

$$I_\rho(f) = \int_D f(x)\rho(x)dx,$$

where $D$ is a bounded or unbounded subset of the Euclidean space $R^s$, the dimension $s$ can be large, and the weight function $\rho(x)$ is nonnegative. Such problems arise in mathematical physics [Kei96], mathematical finance [PT96], and statistical analysis [FW94, Gen92]. For example, the problem in [Kei96] has $D = R^s$ and $\rho(x) = \prod_{k=1}^s \tilde{\rho}(x_k)$ with $\tilde{\rho}(-x_k) = \tilde{\rho}(x_k)$

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for \( x_k \in \mathbb{R} \). In statistics, many Bayesian problems require the computation of the integral \( I_\rho(f) \) where \( D = \mathbb{R}^s \) and \( \rho(x) \) is the product of the likelihood function and the prior density function.

In this article we assume that \( D \) is an \( s \)-dimensional box of the form

\[
D = (a, b) := (a_1, b_1) \times \cdots \times (a_s, b_s) \subseteq \mathbb{R}^s,
\]

where each of the \((a_k, b_k)\) may possibly be finite, semi-infinite or infinite interval. In addition, we assume that the weight function \( \rho(x) \) has product form:

\[
\rho(x) = \prod_{k=1}^{s} \rho_k(x_k)
\]

for nonnegative functions \( \rho_k \in L_1((a_k, b_k)) \). For simplicity, it is assumed that

\[
\int_{a_k}^{b_k} \rho_k(x_k) dx_k = 1.
\]

An approximation to the integral \( I_\rho(f) \) is defined by quasi-Monte Carlo rules of the form

\[
Q_n(f) = \frac{1}{n} \sum_{j=0}^{n-1} f(x^j)
\]

where \( x^0, x^1, \ldots, x^{n-1} \) are all the points in \( D \). In the case with \( D = [0, 1]^s \) and \( \rho(x) \equiv 1 \), integration is called here the classical problem. In this case we denote the integral in (1) by \( I(f) \) for simplicity.

For the classical problem, quasi-Monte Carlo methods based on deterministic or randomly scrambled \((T, m, s)\)-nets or \((T, s)\)-sequences in base \( b \) are widely used. It is known that the deterministic net quadrature rules have superior asymptotic accuracy \([Nie92]\), while randomly scrambled net quadrature rules remove bias, provide a basis for error estimation and can even improve the convergence rate, see, e.g., \([Owe95, Owe97a, Owe97b, Owe98, YM99, HY00, HY04]\).

For the net quadrature rules there have been a few investigations in recent years concentrating on the tractability problem of multivariate integration of different spaces of integrands using \((T, m, s)\)-nets or \((T, s)\)-sequences. Yue and Hickernell \([YH01]\) use a non-constructive approach to consider the tractability of scrambled net rules for the weighted Hilbert spaces spanned by Haar wavelets in base 2. Wang \([Wan03]\) considers deterministic \((T, s)\) sequence rules and weighted Sobolev spaces. Yue and Hickernell \([YH04]\) consider weighted Hilbert spaces spanned by multidimensional Haar-like wavelets in base \( b \) and weighted Sobolev-Hilbert spaces. Dick and Pillichshammer \([DP05]\) consider shifted net rules for weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces. Most analyses for the problem of multivariate integration has focused on reproducing kernel Hilbert spaces of integrands defined on the \( s \)-dimensional unit cube.

However, as pointed out in \([Slo02]\), it is necessary to consider integration problems for Banach space of functions since the reproducing kernel Hilbert space methods are too restrictive for classes of problems. For example, integrals from mathematical finance are typically
with respect to probability measures over unbounded domains. After mapping to the unit cube most problems of this kind yield integrands whose derivatives are integrable, but not square integrable. Therefore, a fundamental difficulty arises in applying the Hilbert space results to such problems. Recently, there have been some studies of the tractability problem for weighted integration based on general quadrature rules and lattice rules for weighted Banach spaces of functions whose mixed partial derivatives are bounded in $L_p$ norms for $p \in [1, \infty]$, see [HSW04a, HSW04b, HSW04c]. We think that it is also interesting to consider the quadrature rules that use $(T, m, s)$-nets and $(T, s)$-sequences for these spaces.

This article studies the tractability problems for weighted Banach spaces of integrands, in which two quasi-Monte Carlo rules are considered. One uses deterministic Niederreiter $(T, s)$-sequences, and another uses randomly scrambled Niederreiter digital $(T, m, s)$-nets. For deterministic Niederreiter sequence rules we assume that the integrands $f$ lie in a weighted Banach space, $F^{(1)}_{p,q,\gamma,s}$, of functions whose mixed anchored first derivatives, $\partial^{[u]} f(x_u, c_\bar{u})/\partial x_u$, are bounded in $L_p$ norms with anchor $c$ fixed in the domain $D$, and the weighted coefficients, $\gamma = \{\gamma_k\}_k$, are introduced via $\ell_q$ norms over the index $u$, where $p, q \in [1, \infty]$. This space is the same as that in [HSW04b]. For the randomly scrambled Niederreiter net rules, the class of integrands is a weighted Banach space, $F^{(2)}_{p,q,\gamma,s}$, of functions whose unanchored mixed first derivatives, $\partial^{[u]} f(x)/\partial x_u$, are bounded in $L_p$ norms and the weighted coefficients, $\gamma = \{\gamma_k\}_k$, are introduced via $\ell_q$ norms, where $p, q \in [1, \infty]$.

The reason for using the space $F^{(2)}_{p,q,\gamma,s}$ for the randomized setting is that the mean square error of the quadrature rule will be expressed in terms of Fourier coefficients of the integrand under the orthogonal system of multivariate Haar wavelets. By making use of integration by parts each of the Fourier coefficients will be expressed by an integral of the product of two functions: one is unanchored mixed first derivative of the integral, and another is defined via the Haar wavelet. See Section 3.3 for details.

The reason for using Niederreiter sequences is that the $T$ value grows more slowly with $s$ than Sobol’ sequences. On the other hand, since Niederreiter sequences are telescoping, i.e., to obtain a $(T, s+1)$-sequence one just adds another coordinate to the $(T, s)$-sequence. This makes Niederreiter sequences suitable for tractability studies, unlike Niederreiter-Xing sequences [NX96], which have smaller $T$ values but are not telescoping.

The main results of this article are Theorems 1 - 4, which provide sufficient conditions on strong tractability for $F^{(1)}_{p,q,\gamma,s}$ in the worst-case setting and $F^{(2)}_{p,q,\gamma,s}$ in the randomized setting. These results are summarized in the Table 1, where $p^*$ and $q^*$ denote the conjugates of $p$ and $q$, i.e.,

$$\frac{1}{p} + \frac{1}{p^*} = 1, \quad \frac{1}{q} + \frac{1}{q^*} = 1.$$  

The asymptotic orders of the quadrature errors are given under the assumption that the sufficient condition for strong tractability holds. The parameter $\epsilon$ is an arbitrary positive number. For comparison, related results in [HSW04b, Wan03, YH04] are also listed in the table.

The following points are worth noting about these results:

i. The spaces considered in [Wan03] are the weighted reproducing kernel Hilbert spaces and the original weights $\gamma_k$ in [Wan03] are the square of our weights. Therefore, the
space in [Wan03] in which the weights are replaced with $\gamma_k^2$ becomes $F_{2,2,\gamma,s}^{(1)}$. Our
results substantially extend and improve upon the results of [Wan03] by considering
more general spaces of integrands and deriving weaker sufficient conditions for strong
tractability.

ii. The setting in [YH04] is the randomized worst-case, and the quadrature rule is based on
a randomly scrambled Niederreiter sequence. Moreover, the weighted Sobolev-Hilbert
spaces, $H_{s,\gamma}^{SH}$, are nearly the same as $F_{2,2,\gamma,s}^{(1)}$. The sufficient condition there for strong
tractability is somewhat weaker than one in the present article. We do not know yet
whether the condition for the worst-case setting in this article can be weakened or not.

iii. Compared with the lattice rules in [HSW04b], our sufficient condition for the worst-
case setting is somewhat more stringent. In fact, we can conclude that our condition
$\sum_{k=1}^{\infty} \gamma_k k \ln k < \infty$ is roughly equivalent to $\sum_{k=1}^{\infty} \gamma_k^{a/2} < \infty$ from the following Lemma 1,
provided that $\gamma_1 \geq \gamma_2 \geq \cdots \geq 0$.

iv. As far as we are aware, there have been few studies on randomized settings in literature.
Sloan and Woźniakowski [SW01] studied the randomized error of the classical Monte
Carlo algorithm for weighted Korobov spaces. Yue and Hickernell [YH04] studied the
randomized error of the quasi-Monte Carlo algorithm based on scrambled Niederreiter
nets and sequences for weighted Sobolev-Hilbert spaces. The result of the randomized
error for $p = q = 2$ in this article is the same as that of [YH04]. Therefore, the results of
this article extend the result of [YH04] by considering more general spaces of integrands.

v. Compared with the condition with $a = 1$ in the worst-case setting for $F_{p,q,\gamma,s}^{(1)}$, the
condition in the randomized setting for $F_{p,q,\gamma,s}^{(2)}$ is weaker for $p > 1$, and the same for
$p = 1$. However, the results in the randomized setting are just for Niederreiter digital
nets, unlike Niederreiter sequences in the worst-case setting.

**Lemma 1.** Let $\{\gamma_k\}$ be a nonnegative non-increasing sequence, and let $\lambda_k$ be a sequence
satisfying

$$
\tilde{c}_{r,\delta} k^{r-\delta} \leq \lambda_k \leq c_{r,\delta} k^{r+\delta}, \quad \forall \delta \in (0,r), \quad k = 1, 2, \ldots,
$$

where $\tilde{c}_{r,\delta}$ and $c_{r,\delta}$ are two nonnegative constants depending only on $r$ and $\delta$. Then

$$
\bar{L}_{r,\delta} \left[ \sum_{k=1}^{\infty} \gamma_k^{(1+\delta)} \right]^{1/r+\delta} \leq \sum_{k=1}^{\infty} \gamma_k^{\beta} \lambda_k \leq L_{r,\delta} \left[ \sum_{k=1}^{\infty} \gamma_k^{(1-r\delta)} \right]^{1/r-\delta},
$$

where $\bar{L}_{r,\delta}$ and $L_{r,\delta}$ are two nonnegative constants depending only on $r$ and $\delta$.

**Proof.** First we have

$$
\sup_k \{\gamma_k^{\beta} k\} \leq \sum_{k=1}^{\infty} \gamma_k^{\beta}, \quad \forall \beta > 0,
$$

since for any integer $K > 0$

$$
\sum_{k=1}^{\infty} \gamma_k^{\beta} \geq \sum_{k=1}^{K} \gamma_k^{\beta} \geq K \gamma_K^{\beta}.
$$
Now for any $\delta, \delta'>0$ applying the assumption on $\lambda_k$ and the above fact we have

$$\sum_{k=1}^{\infty} \gamma_k^\delta \lambda_k \leq \sup_k \left\{ \gamma_k^{ar(1+\delta)} c_{r,\delta} k'^{r+\delta'} \right\} \sum_{k=1}^{\infty} \gamma_k^{1+r}. $$

We take $\delta'$ such that

$$r + \delta' = \frac{r(1+\delta)}{1+r},$$

i.e., $\delta' = \frac{\delta r(1+r)}{1-r\delta}$, and then we have

$$\sup_k \left\{ \gamma_k^{ar(1+\delta)} k'^{r+\delta'} \right\} = \left[ \sup_k \gamma_k^{a(1-r\delta)} k^{r(1+\delta)} \right] \leq \left[ \sum_{k=1}^{\infty} \gamma_k^{1+r} \right]^{r(1+\delta) \frac{1-r\delta}{1+r}}.$$ 

It follows that

$$\sum_{k=1}^{\infty} \gamma_k^\delta \lambda_k \leq c_{r,\delta'} \left[ \sum_{k=1}^{\infty} \gamma_k^{a(1-r\delta)} k^{r'} \right]^{r(1+\delta) \frac{1-r\delta}{1+r}},$$

which gives the right-side of the desired result.

As to the left-side of the desired result, we write for any $\delta > 0$

$$\sum_{k=1}^{\infty} \gamma_k^{a(1+\delta)} k'^{r+\delta'} = \sum_{k=1}^{\infty} \gamma_k^{a(1-r\delta)} k^{r-\delta''(1+\delta)} k^{(\delta''-r)(1+\delta)} \frac{(1+\delta)\delta''}{1+r},$$

where $\delta'' > 0$ will be fixed later. Applying Hölder’s inequality yields

$$\sum_{k=1}^{\infty} \gamma_k^{r'-\delta''} \leq \left[ \sum_{k=1}^{\infty} \gamma_k^\delta k^{r'-\delta''} \right]^{\frac{1+r}{1+r}} \left[ \sum_{k=1}^{\infty} \gamma_k^{a(1-r\delta)} k^{(1+\delta')(1+\delta)} \right]^{\frac{r''-\delta}{1+r}}.$$ 

By choosing, e.g., $\delta'' = \delta/2$, we have

$$\sum_{k=1}^{\infty} k^{(1+\delta')(1+\delta)} k^{-\delta''} < \infty.$$ 

Applying the assumption on $\lambda_k$ yields

$$\sum_{k=1}^{\infty} \gamma_k^{a(1+\delta)} k^{1+r} \leq \left[ \sum_{k=1}^{\infty} \gamma_k^a \lambda_k \right]^{\frac{1+r}{1+r}} \hat{c}_{r,\delta},$$

which gives the left-side of the desired result. The proof is complete.

The article proceeds as follows: Section 2 considers the strong tractability of quasi-Monte Carlo rules that use deterministic Niederreiter sequences for the classical problem. Section 3 considers strong tractability of quasi-Monte Carlo rules that use the randomly scrambled Niederreiter digital nets for the classical problem. Section 4 extends the strong tractability results for the classical problem to the weighted integration over the general domain. Some concluding remarks are given in Section 5.
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2. Tractability in worst-case settings for the classical problem

In this section, we deal with the strong tractability problem of multivariate integration using the deterministic Niederreiter sequence for the classical problem. We first introduce the spaces of our integrands, which are defined as in [HSW04a]. We briefly recall the definition as follows.

For a given \( p \in [1, \infty] \), let \( \mathcal{H}_{p,k} \) be the space of absolutely continuous functions \( h : [0, 1] \to \mathbb{R} \) such that \( \frac{d}{dx} h(x) \in L_p([0, 1]) \). Let \( \mathcal{H}_p^s = \otimes_{k=1}^s \mathcal{H}_{p,k} \) be the space consisting of linear combinations of functions of the following tensor product form

\[
 f : [0, 1]^s \to \mathbb{R} \text{ and } f(x) = \prod_{k=1}^s h_k(x_k) \text{ with } h_k \in \mathcal{H}_{p,k}.
\]

Given an additional parameter \( q \in [1, \infty] \) and a sequence \( \gamma = \{\gamma_k\}_k \) of positive numbers, define the space \( F^{(1)}_{p,q,\gamma,s} \) to be the completion of \( \mathcal{H}_p^s \) with respect to the norm

\[
 \|f\|_{p,q,\gamma,s} := \begin{cases} 
 \left( \sum_u \gamma_u^{-q} \|f_{u,c}\|_{L_p}^q \right)^{1/q} & \text{for } q < \infty, \\
 \max_u \{\gamma_u^{-1} \|f_{u,c}\|_{L_p}\} & \text{for } q = \infty,
\end{cases}
\]

where the derivative \( f'_{u,c} \) is defined by

\[
 f'_{u,c}(x_u) := \left( \prod_{k \in u} \frac{\partial}{\partial x_k} \right) f(x_u, c_u),
\]

here \((x_u, c_u)\) denotes the \( s\)-dimensional vector whose \( k\)th component is \( x_k \) if \( k \in u \), and is \( c_k \) if \( k \not\in u \), and \( f'_{\emptyset,c} \) and \( \|f'_{\emptyset,c}\|_{L_p} \) denote \( f(c) \) and \( |f(c)| \), respectively, and \( \gamma_{\emptyset} = 1 \). For the significance of the weights \( \gamma_k \) we refer to, e.g., [SW98, NW01], and note that in some articles including [SW98, NW01, Wan03], \( p = q = 2 \) and the definition of \( \| \cdot \|_{2,2,\gamma,s} \) uses \( \gamma_u^{-1} \) instead of \( \gamma_u^{-2} \). Hence, our weights \( \gamma_k \) are the square-roots of those in [SW98, NW01, Wan03].

For the weighted Banach space \( F^{(1)}_{p,q,\gamma,s} \) defined above the worst-case error of the quasi-Monte Carlo quadrature \( Q_n \) is defined as

\[
 e^{wo}(Q_n, F^{(1)}_{p,q,\gamma,s}) := \sup_{\|f\|_{p,q,\gamma,s} \leq 1} |I(f) - Q_n(f)|.
\]

An expression for \( e^{wo}(Q_n, F^{(1)}_{p,q,\gamma,s}) \) is given in [HSW04a], in which the following functions, \( M_u(x,t) \), are important. For each \( k \in 1 : s = \{1, \ldots, s\} \) let

\[
 M_k(x,t) := \begin{cases} 
 1, & \text{if } c_k \leq t < x, \\
 -1, & \text{if } x \leq t < c_k, \\
 0, & \text{otherwise},
\end{cases}
\]

and for each subset \( u \subseteq 1 : s \) let

\[
 M_u(x_u, t_u) := \prod_{k \in u} M_k(x_k, t_k)
\]
with the convention that $M_\emptyset \equiv 1$. It is shown in [HSW04c] that

$$f(x) = \sum_{u \subseteq 1:s} \int_{[0,1]^s} f'_{u,c}(t_u) M_u(x_u, t_u) dt_u.$$  

Define

$$h_u(t_u) := I(M_u(\cdot, t_u)) - Q_n(M_u(\cdot, t_u)).$$  

(7)

When $c = 1$, then $h_u(t_u)$ has the following expression

$$h_u(t_u) = \text{vol}([0,t_u]) - \frac{1}{n} \sum_{j=0}^{n-1} 1_{(0,t_u)}(x^j_u).$$  

(8)

For the case where $c$ is in the interior of the unit cube $[0,1]^s$, $h_u(t_u)$ has a similar expression replacing the cube $[0,t_u)$ by a certain box, which is described below. Note that the anchor $c \in (0,1)^s$ partitions the unit cube $[0,1]^s$ into $2^s$ quadrants. Given a $t = (t_1, \ldots, t_s)^T$ in one of these quadrants, let $B(t; c)$ denotes the box with one corner at $t$ and the opposite corner given by the unique vertex of $[0,1]^s$ that lies in the same quadrant as $t$. Then $h_u(t_u)$ can be expressed as

$$h_u(t_u) = \text{vol}(B_u(t_u; c_u)) - \frac{1}{n} \sum_{j=0}^{n-1} 1_{B_u(t_u; c_u)}(x^j_u),$$  

(9)

where $B_u(t_u; c_u)$ is the projection of $B(t; c)$ onto the axes in $u$.

In terms of the functions $h_u$ the worst-case error $e^{\text{wo}}(Q_n, \mathcal{F}_p^{(1)}_{p,q,\gamma,s})$ in (6) is given by [HSW04a]

$$e^{\text{wo}}(Q_n, \mathcal{F}_p^{(1)}_{p,q,\gamma,s}) = \left\{ \begin{array}{ll} \left( \sum_{u} \gamma_u^{q^*} \|h_u\|_{L_{p^*}}^{q^*} \right)^{1/q^*} & \text{for } q > 1 \\ \max_u \{\gamma_u \|h_u\|_{L_{p^*}}\} & \text{for } q = 1, \end{array} \right.$$  

(10)

where $p^*$ and $q^*$ are the conjugates of $p$ and $q$, respectively.

Note that for any nonempty subset $u$ of the axes $1:s$, the projection of Niederreiter $(T, s)$-sequence onto the axes in $u$ forms a $|u|$-dimensional Niederreiter sequence with a certain quality parameter

$$T_u = \sum_{k \in u} (\deg(g_k) - 1),$$  

(11)

where $g_1, \ldots, g_s$ denote the first $s$ monic irreducible polynomials over the finite field $F_b$. This allows us to write the Niederreiter sequence and net as $(T_u, s)$-sequence and $(T_u, m, s)$-net, respectively, where $(T_u)$ denotes an $(2^s - 1)$-dimensional vector of the quality parameters corresponding to the all nonempty subsets $u$ of $1:s$. From [Nie92, Wan03] one has the following upper bound on $\deg(g_k)$:

$$\deg(g_k) \leq \log_b k + \log_b \log_b (k + b) + 2, \quad k = 1, 2, \ldots.$$  

(12)

The result in the following lemma will be used several times in the proofs of the main theorems in this article.
Lemma 2. Let $\alpha, \beta, \theta > 0$ and $\tau \geq 1$. For Niederreiter $((T_u), m, s)$-net in base $b$, define
\[
\Phi(\alpha, \beta, \theta, \tau) := \sum_{\emptyset \neq u \subseteq [1:s]} \gamma_k^u b^{\beta T_u(\theta \ln(\tau n))|u|},
\]
where $n = b^m$. If the $\gamma_k$ satisfy
\[
\sum_{k=1}^{\infty} \gamma_k^a (k \ln k)^\beta < \infty,
\]
then for any fixed $\epsilon > 0$ there exists a constant $C$ independent of $s$ and $n$ such that
\[
\Phi(\alpha, \beta, \theta, \tau) \leq C n^{\epsilon}.
\]

Proof. This lemma can be proved by a similar argument to that used in [Wan03, Theorem 4].

Now we can proceed to prove the strong tractability result of multivariate integration using Niederreiter sequences in the worst-case setting.

Theorem 1. Assume $p, q \in [1, \infty]$. Let $\mathcal{F}_{p,q,\gamma,s}^{(1)}$ be the Banach space of functions $f$ with norm (5). Assume that quasi-Monte Carlo quadrature $Q_n$ uses Niederreiter $((T_u), s)$-sequence in a prime power base $b$. If
\[
\sum_{k=1}^{\infty} \gamma_k^a (k \ln k)^\beta < \infty
\]
for any $a \in [1, q^*]$, then the corresponding integration is strongly tractable in the worst-case setting, and
\[
e^{\text{wc}}(Q_n, \mathcal{F}_{p,q,\gamma,s}^{(1)}) \leq C n^{-1/a + \epsilon}
\]
for any fixed $\epsilon > 0$, where $C$ is some constant independent of $s$ and $n$.

Proof. Noting from the expression in (10) for the worst-case error and the fact $\|h_u\|_{L^p} \leq \|h_u\|_{L^\infty}$ for any $p^* \in [1, \infty]$, it is sufficient to consider the case with $p^* = \infty$. For the case of $c = 1$
\[
\|h_u\|_{L^\infty} = \sup_{t_u \in [0,1]^u} \text{vol}([0, t_u)) - \frac{1}{n} \sum_{j=0}^{n-1} 1_{[a,t_u)}(x^j_u) = D_u^{*,\infty}(P)
\]
due to the expression (8), where $D_u^{*,\infty}(P)$ is the local star discrepancy of $P$ corresponding to the subset $u$. Note that if $\{x^j\}_{j \geq 0}$ is a Niederreiter $((T_u), s)$-sequence in base $b$, then its projection onto the axes in $u$ is a Niederreiter $(T_u, |u|)$-sequence in base $b$, where $T_u$ is as given by (11). It follows from [Wan03, Lemma 1] that
\[
D_u^{*,\infty}(P) \leq n^{-1} b^{T_u(\theta \ln(bn))|u|},
\]
where $\theta = b / \ln b$. Noting that $\|h_u\|_{L_\infty} = D_{u,\infty}^*(P) \leq 1$, we then have for $q \in [1, \infty]$ and any $\tilde{a} \in [1, q^*]$

$$\left( \sum_u \gamma_u^q \|h_u\|_{L_\infty}^q \right)^{1/q} \leq \left( \sum_u \gamma_u^q \|h_u\|_{L_\infty}^\tilde{a} \right)^{\tilde{a}/q} \leq \left( \sum_u \gamma_u^q \|h_u\|_{L_\infty}^\tilde{a} \right)^{\tilde{a}/q} \leq n^{-\tilde{a}/q} \left( \sum_u \gamma_u^q \|h_u\|_{L_\infty}^{\tilde{a}} \right)^{\tilde{a}/q}$$

$$= n^{-\tilde{a}/q} \Phi(q^*/\tilde{a}, 1, \theta, b)^{\tilde{a}/q}.$$ 

Set $a = q^*/\tilde{a}$, and then $a \in [1, q^*]$ since $1 \leq \tilde{a} \leq q^*$. Applying Lemma 2 to $\Phi(a, 1, \theta, b)$ gives the upper bound (16) for $e^{\omega}(Q_n, \mathcal{F}(1)_{p,q,\gamma,s})$ if the condition (15) holds.

For the case with $c$ in the interior of the unit cube $[0, 1]^s$, from the expression (9) we have

$$\|h_u\|_{L_\infty} = \sup_{t_u \in [0,1]^u} \left| \text{vol}(B_u(t_u; c_u)) - \frac{1}{n} \sum_{j=0}^{n-1} 1_{B_u(t_u; c_u)}(x_j^u) \right| \leq \sup_{J_u} \left| \text{vol}(J_u) - \frac{1}{n} \sum_{j=0}^{n-1} 1_{J_u}(x_j^u) \right| = D_{u,\infty}(P),$$

where the $J_u$ denote subintervals of $[0, 1]^s$ of the form $\prod_{k \in u} [\alpha_k, \beta_k]$, and $D_{u,\infty}(P)$ is the local extreme discrepancy (or unanchored discrepancy) of $P$ corresponding to the subset $u$. From Proposition 2.4 in [Nie92] we have

$$D_{u,\infty}(P) \leq 2|u| D_{u,\infty}^*(P).$$

Then the desired result follows from the previous argument for $c = 1$. $

\textbf{Remark 1.}$ From Theorem 1 the following facts are observed:

i. If condition (15) holds for $a = q^*$, then the worst-case error is $O(n^{-1+\epsilon})$, and in the case with $p = q = 2$, this result is the same as that obtained in [Wan03] for reproducing kernel Hilbert spaces. Therefore, our result in Theorem 1 is an extension of that in [Wan03].

ii. Although the convergence rate for $a < q^*$ is smaller than that for $a = q^*$, but, by introducing the number $a \in [1, q^*)$, the sequence of weights $\{\gamma_k\}_k$ is transformed at the same time and the strong tractability condition is weaker. Therefore, when considering the question of strong tractability, this theorem allows one to accept a lower convergence rate in turn for less restrictive conditions on the weights. This was done in [HSW04b].

\section{Tractability in randomized settings for the classical problem}

In this section we consider the strong tractability problem of integration using randomly scrambled Niederreiter digital net rules in randomized settings for the classical problem. The
function spaces $\mathcal{F}^{(2)}_{p,q;\gamma,s}$ defined below are different from $\mathcal{F}^{(1)}_{p,q;\gamma,s}$ considered in previous section in norms. The reason for this is that the arguments for the randomized setting are different from that used in previous section. In analyzing the randomized error, the mean square error of the quadrature rule will be expressed in terms of Fourier coefficients of the integrand under the orthonormal system of multivariate Haar wavelets. By making use of integration by parts each of the Fourier coefficients will be expressed by an integral of the product of two functions: one is unanchored mixed first derivative of the integral, and another is defined via the Haar wavelet.

For $p \in [1, \infty]$ let $\mathcal{H}^s_p$ be defined as previous section. Given an additional parameter $q \in [1, \infty]$ and a sequence $\gamma = \{\gamma_k\}_k$ of positive numbers, $\mathcal{F}^{(2)}_{p,q;\gamma,s}$ is defined to be the completion of $\mathcal{H}^s_p$ with respect to the norm

$$\|f\|_{p,q;\gamma,s} := \begin{cases} \left( \sum_u \gamma_u^{-q} \|f'_u\|^q_{L^p} \right)^{1/q} & \text{for } q < \infty, \\ \max_u \{\gamma_u^{-1} \|f'_u\|_{L^p}\} & \text{for } q = \infty, \end{cases} \quad (17)$$

where the derivative $f'_u(x)$ is defined by

$$f'_u(x) := \left( \prod_{k \in u} \frac{\partial}{\partial x_k} \right) f(x),$$

$f'_0(x)$ denotes $f(x)$, and $\gamma_0 = 1$. For $p = q = 2$ the space becomes the Sobolev space considered in Section 3.1 of [SW04].

For the weighted Banach spaces defined above, we define the randomized error of the quadrature rule $Q_n$ as

$$e^{ra}(Q_n, \mathcal{F}^{(2)}_{p,q;\gamma,s}) := \sup_{\|f\|_{p,q;\gamma,s} \leq 1} \sqrt{E|I(f) - Q_n(f)|^2}, \quad (18)$$

where the expectation $E$ is taken with respect to the random samples.

In what follows, we first give some backgrounds about the randomly scrambled digital sequences and Haar wavelets. Then we deal with the randomized error of the quadrature rules.

### 3.1 Randomly scrambled digital sequences

The sequence $\{x^j\}_{j \geq 0}$ of points in the unit cube $[0,1]^s$ is generated in following way. Let $b \geq 2$ be a prime power base. The $k$th component of the $j$th point $x^j = (x^j_1, \ldots, x^j_s)^T$ is determined by the $b$-ary expression

$$x^j_k = x^j_{k1} \frac{1}{b} + x^j_{k2} \frac{1}{b^2} + \cdots.$$ 

Here, the digits $x^j_{kl}$ are generated by

$$\begin{pmatrix} x^j_{k1} \\ x^j_{k2} \\ \vdots \end{pmatrix} = L_k C_k \begin{pmatrix} j_1 \\ j_2 \\ \vdots \end{pmatrix} + e_k \mod b, \quad k = 1, \ldots, s,$$
where \((j_1, j_2, \cdots)^T\) is the vector of \(b\)-ary digits of \(j \in \mathbb{Z}_+\), i.e., \(j = j_1 + j_2 b + j_3 b^2 + \cdots\), the \(C_k\) are the prescribed \(\infty \times \infty\) generator matrices, the \(L_k\) are lower triangular \(\infty \times \infty\) scrambling matrices, and the \(e_k\) are \(\infty \times 1\) digital shifts. The scrambling matrices and shifts are chosen randomly.

Choosing the first \(n = b^m\), \(m \in \mathbb{Z}_+\) points of a digital sequence in base \(b\) gives a digital net in base \(b\). The quality of the digital net is defined below.

For and vector \(\ell = (\ell_1, \ell_2, \cdots)^T \in \mathbb{Z}_+^\infty\) with \(\|\ell\|_1 := \sum_{\alpha} \ell_\alpha < \infty\), let \(C(\ell)\) be the \(\|\ell\|_1 \times \infty\) matrix formed by the first \(\ell_1\) rows of \(C_1\) followed by the first \(\ell_2\) rows of \(C_2\), etc. For any integer \(m \in \mathbb{Z}_+\) let \(C(\ell, m)\) denote the \(\|\ell\|_1 \times m\) matrix formed by the first \(m\) columns of \(C(\ell)\). Define

\[
A(\ell, m) = m - \text{rank}(C(\ell, m)),
\]

which is the dimension of the null space of \(C(\ell, m)\). For a fixed \(u \subseteq 1 : s\) define the parameter \(T_u\) to be the smallest number \(T\) for which any \(\ell\) with \(\|\ell\|_1 = m - T\) and \(U(\ell) \subseteq u\) yields \(A(\ell, m) = T\), where \(U(\ell)\) denotes the set of all \(\alpha\) for which \(\ell_\alpha > 0\). That is,

\[
T_u := \min\{T : A(\ell, m) = T \hspace{1em} \forall m, \forall \ell \text{ with } U(\ell) \subseteq u, \|\ell\|_1 = m - T\}.
\]

Smaller values of \(T_u\) corresponds to better nets.

### 3.2 Haar wavelets

To deals with the randomized error of the randomly scrambled digital net rules, we require Haar wavelets. We first introduce some notation. For any \(\nu \in \mathbb{Z}_+\) define the function

\[
\lg(\nu) := \begin{cases} 
\lfloor \log_b(\nu) \rfloor + 1, & \text{if } \nu > 0, \\
0, & \text{if } \nu = 0,
\end{cases}
\]

which means that the base \(b\) representation of \(\nu\) has \(\lg(\nu)\) digits if one ignores leading zeros. Let \(\tilde{\nu}\) denote the leading digit of \(\nu\) when written in base \(b\), and define

\[
z_\nu := \nu b^{1-\lg(\nu)} - \tilde{\nu}.
\]

For any \(\nu = (\nu_1, \cdots, \nu_s)^T \in \mathbb{Z}_+^s\) define

\[
\lg(\nu) := (\lg(\nu_1), \cdots, \lg(\nu_s))^T, \quad z_\nu := (z_{\nu_1}, \cdots, z_{\nu_s})^T.
\]

Also, let \(U(\nu)\) denote the set of all \(k\) for which \(\nu_k > 0\) and let \(|U(\nu)|\) denote the cardinality of \(U(\nu)\). Moreover, for \(u \subseteq 1 : s\) let \(1_u\) denote the \(|u|\)-dimensional vector whose \(k\)th component is 1 for \(k \in u\) and 0 otherwise. For any \(x = (x_1, \cdots, x_s)^T, y = (y_1, \cdots, y_s)^T \in [0, 1]^s\) and \(\ell = (\ell_1, \cdots, \ell_s)^T \in \mathbb{Z}_+^s\) let \(\delta(x, y, \ell) = 1\) if the first \(\ell_k\) digits of \(x_k\) and \(y_k\) are the same for all \(k = 1, \cdots, s\), and let \(\delta(x, y, \ell) = 0\) otherwise.

Multivariate Haar wavelets \(\psi_\nu(x)\) for \(\nu \in \mathbb{Z}_+^s\) are piecewise constant functions, which are defined as

\[
\psi_\nu(x) := b^{\|\lg(\nu)\|_1 - |U(\nu)|} / 2 \exp \left( \frac{2\pi i}{b} \sum_{k=1}^s \tilde{\nu}_k x_k \kappa_{\lg(\nu_k)} \right) \delta(x, z_\nu, \lg(\nu) - 1_{U(\nu)}), \quad (19)
\]
where \( x_{k\ell} \) denotes the \( \ell \)th \( b \)-ary digit of the \( k \)th component of \( x \), and \( i := \sqrt{-1} \). Note that the support of \( \psi_\nu \) is a box, \( S(\nu) \), of volume \( b^{U(\nu) - \|lg(\nu)\|_1} \). In fact,

\[
S(\nu) = \prod_{k \in U(\nu)} \left[ z_{\nu_k}, z_{\nu_k} + b^{1 - \|lg(\nu_k)\|} \right] \times [0,1)^{1 : s \setminus U(\nu)}.
\]

(20)

It is known that \( \{\psi_\nu(x)\}_\nu \) is a sequence of complex-valued, integrable, orthonormal basis functions. For any \( f \in L_2([0,1]^s) \) it can be represented as infinite series

\[
f(x) = \sum_{\nu} F(\nu) \psi_\nu(x), \quad x \in [0,1]^s,
\]

(21)

where the \( F(\nu) \) are Fourier coefficients given by

\[
F(\nu) := \int_{[0,1]^s} f(x) \overline{\psi_\nu(x)} \, dx,
\]

(22)

where \( \overline{\psi_\nu(x)} \) denotes the conjugate of \( \psi_\nu(x) \). The following facts will play an important role in our randomized error analysis. For each \( \nu \in Z_+ \) and any \( x \in [0,1] \) let

\[
\xi_\nu(x) := \int_0^x \overline{\psi_\nu(t)} \, dt.
\]

For each \( \nu \in Z_+^s \) and any \( x \in [0,1]^s \) let

\[
\xi_\nu(x) := \prod_{k \in U(\nu)} \xi_{\nu_k}(x_k).
\]

(23)

Note that the support of \( \xi_\nu \) is the same as that of \( \psi_\nu \).

The following upper bounds on the different norms of \( \xi_\nu \) can be verified by immediate calculation which are omitted here.

**Lemma 3.** For each \( \nu \in Z_+^s \)

\[
\|
\xi_\nu\|_{L_1} \leq 2^{\|U(\nu)\| - 3(\|lg(\nu)\|_1 - \|U(\nu)\|)}/2,
\]

\[
\|
\xi_\nu\|_{L_2} \leq 3^{\|U(\nu)\|/2} b^{\|lg(\nu)\|_1 + \|U(\nu)\|},
\]

\[
\|
\xi_\nu\|_{L_{\infty}} \leq b^{\|lg(\nu)\|_1 - \|U(\nu)\|}/2.
\]

### 3.3 Upper bounds for the randomized error

This subsection will give upper bounds for the randomized error defined in (18) for different values of \( p \in [1, \infty] \), and find sufficient conditions under which multivariate integration using the randomly scrambled Niederreiter digital nets is strongly tractable in the randomized setting.

**Lemma 4.** Let \( \psi_\nu(x) \), \( \nu \in Z_+^s \), be the Haar wavelets defined by (19), and let \( Q_n \) be the quasi-Monte Carlo quadrature that uses randomly scrambled digital nets with \( n = b^m \),
where $m \in \mathbb{Z}_+$. Then for any $f \in L_2([0,1]^s)$
\[
\text{MSE}(Q_n,f) := E|I(f) - Q_n(f)|^2 = \sum_{\nu \neq 0} |F(\nu)|^2 E[Q_n(\psi_\nu)Q_n(\overline{\psi}_\nu)] \leq \sum_{\nu : \|\text{lg}(\nu)\|_1 + T_{U(\nu)} > m} |F(\nu)|^2 3^{U(\nu)/b_{T_{U(\nu)}} - m},
\]
where the $F(\nu)$ are the coefficients of $f$ under the Haar wavelets.

Proof. Making use of the expansion in (21) and noting that $F(\mathbf{0}) = I(f)$ yields
\[
I(f) - Q_n(f) = -\frac{1}{n} \sum_{j=0}^{n-1} \sum_{\nu \neq 0} F(\nu)\psi_\nu(x^j) = -\sum_{\nu \neq 0} F(\nu)Q_n(\psi_\nu),
\]
and then
\[
|I(f) - Q_n(f)|^2 = \sum_{\nu \neq 0} \sum_{\omega \neq 0} F(\nu)\overline{F(\omega)}Q_n(\psi_\nu)Q_n(\overline{\psi}_\omega).
\]
It is proved in [HD04, Lemma 10] that for scrambled digital nets
\[
E[Q_n(\psi_\nu)Q_n(\overline{\psi}_\omega)] = 0 \quad \text{for } \nu \neq \omega,
\]
\[
E[Q_n(\psi_\nu)Q_n(\overline{\psi}_\nu)] = 0 \quad \text{for } \|\text{lg}(\nu)\|_1 + T_{U(\nu)} \leq m,
\]
\[
E[Q_n(\psi_\nu)Q_n(\overline{\psi}_\nu)] \leq 3^{U(\nu)/b_{T_{U(\nu)}} - m} \quad \text{for } \|\text{lg}(\nu)\|_1 + T_{U(\nu)} > m.
\]
The results in (24) then follows immediately from the facts mentioned above. 

For $f \in \mathcal{F}^{(2)}_{p,q,\gamma,s}$, making use of integration by parts we can express the Fourier coefficients $F(\nu)$ in terms of the functions $\xi_\nu$ defined in (23) as follows
\[
F(\nu) = (-1)^{U(\nu)} \int_{S(\nu)} \xi_\nu(x)f_{U(\nu)}(x)dx,
\]
where $S(\nu)$ is the support of $\xi_\nu$ given in (20). By Hölder’s inequality applied to (25),
\[
|F(\nu)| \leq \|f_{U(\nu)}\|_{L_p,S(\nu)}\|\xi_\nu\|_{L_{p^*}},
\]
where
\[
\|f_{U(\nu)}\|_{L_p,S(\nu)} := \left( \int_{S(\nu)} |f_{U(\nu)}(x)|^p dx \right)^{1/p}, \quad \|\xi_\nu\|_{L_{p^*}} := \left( \int_{[0,1]^s} |\xi_\nu(x)|^{p^*} dx \right)^{1/p^*}.
\]
It follows from (24) that
\[
\text{MSE}(Q_n,f) \leq \sum_{\nu : \|\text{lg}(\nu)\|_1 + T_{U(\nu)} > m} \|f_{U(\nu)}\|_{L_p,S(\nu)}^2 3^{U(\nu)/b_{T_{U(\nu)}} - m} = \sum_{\nu : \|\text{lg}(\nu)\|_1 + T_{U(\nu)} > m} \|f_{U(\nu)}\|_{L_p,S(\nu)}^2 3^{\nu/b_{T_{U(\nu)}} - m}. \]

\[
(26)
\]
For simplicity in notation, we define $N_{u,m}$ as the set of $\nu$ such that $U(\nu) = u$ and $\|\lg(\nu)\|_1 + T_u > m$, i.e.,

$$N_{u,m} := \{\nu : U(\nu) = u, \|\lg(\nu)\|_1 + T_u > m\},$$

and let

$$\Psi_{u,m,p} := \sum_{\nu \in N_{u,m}} \|f'_u\|_{L_p(S(\nu))}^2 \|\xi_\nu\|^2_{L_p^*}.$$ (27)

Then (26) becomes

$$\text{MSE}(Q_n, f) \leq \sum_{u \neq \emptyset} \Psi_{u,m,p}^3 |u|^b T_u - m.$$ (28)

Before stating the main results for the randomized strong tractability, we give the following lemma that will be used in the proof of Theorem 2. This lemma can be proved by the binomial theory.

**Lemma 5.** Let $m, t, r$ and $b$ be integers with $m \geq t \geq 0, r \geq 1$ and $b \geq 2$. Then for $\eta \in (0, \infty)$

$$\Omega(\eta, m, t, r) := \sum_{l=m-t+1}^{\infty} \frac{(l-1)}{(r-1)} b^{-nl} < \left(\frac{by_m}{b^r - 1}\right)^r b^{-\eta(m-t)}.$$ (29)

**Theorem 2.** Let $p, q \in [1, \infty]$. Let $\mathcal{F}_{p,q,\gamma,s}$ be the Banach space of functions with norm (17). Let $Q_n$ be the quasi-Monte Carlo quadrature that uses randomly scrambled Niederreiter digital $((T_u), m, s)$-nets in prime power base $b$.

(i) For $p \in [1, \infty]$, if

$$\sum_{k=1}^{\infty} (\gamma_k k \ln k)^{\min(p,2)} < \infty,$$ (30)

then the corresponding integration is strongly tractable in the randomized setting, and

$$e^{\text{ra}}(Q_n, \mathcal{F}_{p,q,\gamma,s}) \leq C n^{-1+\epsilon}$$ (31)

for any fixed $\epsilon > 0$, where $C$ is some constant independent of $s$ and $n$.

(ii) In particular, for $p \in [2, \infty]$, if

$$\sum_{k=1}^{\infty} \gamma_k^2 (k \ln k)^{3} < \infty,$$ (32)

then the corresponding integration is strongly tractable in the randomized setting, and

$$e^{\text{ra}}(Q_n, \mathcal{F}_{p,q,\gamma,s}) \leq C n^{-\frac{3}{2}+\epsilon}$$ (33)

for any fixed $\epsilon > 0$, where $C$ is some constant independent of $s$ and $n$.

**Proof.** For the item (i), we first note that $\mathcal{F}_{p,q,\gamma,s} \subseteq \mathcal{F}_{2,q,\gamma,s}$ for any $p > 2$, and then

$$e^{\text{ra}}(Q_n, \mathcal{F}_{p,q,\gamma,s}) \leq e^{\text{ra}}(Q_n, \mathcal{F}_{2,q,\gamma,s}), \quad \forall p > 2.$$ (34)

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Hence it is sufficient to consider the case \( p \in [1, 2] \). From (28) it requires to deals with the quantity \( \Psi_{u, m, p} \) defined in (27). For \( p \in [1, 2] \), we have \( p^* \in (1, \infty) \) and

\[
\| \xi \|_{L^p}^2 \leq \| \xi \|_{L^\infty}^2 \leq b^{-\| \log(\nu) \|_1 + |U(\nu)|}
\]

by Lemma 1. Then by the definition (27) of \( \Psi_{u, m, p} \)

\[
\Psi_{u, m, p} \leq \sum_{\nu \in \mathcal{N}_{u, m}} \| f'_u \|_{L^p, S(\nu)}^2 b^{-\| \log(\nu) \|_1 + |u|}
\]

\[
= b^{[u]} \sum_{\nu \in \mathcal{N}_{u, m}} \left( b^{-\frac{|u|}{2} \| \log(\nu) \|_1} \int_{S(\nu)} |f'_u(x)|^p dx \right)^{2/p}
\]

\[
\leq b^{[u]} \left( \sum_{\nu \in \mathcal{N}_{u, m}} b^{-\frac{|u|}{2} \| \log(\nu) \|_1} \int_{S(\nu)} |f'_u(x)|^p dx \right)^{2/p}
\]

where the last inequality holds due to \( \frac{2}{p} \geq 1 \). By the definition of the support \( S(\nu) \) in (20), for a fixed \(|s|\)-dimensional vector \( e = (e_1, \ldots, e_s)^T \) with \( e_k > 0 \) for \( k \in u \) and \( e_k = 0 \) for \( k \in \bar{u} \) we have

\[
\sum_{\nu : U(\nu) = u, |\log(\nu) = e|} \int_{S(\nu)} |f'_u(x)|^p dx = (b - 1)^{|u|} \int_{[0,1]^n} |f'_u(x)|^p dx = (b - 1)^{|u|} \| f'_u \|_{L^p}^p.
\]  

(35)

Define the set

\[
\mathcal{L}_{u, m} := \{ \ell : \| \ell \|_1 = m - T_u, \text{ and } \ell_k > 0 \ \forall k \in u \}.
\]

(36)

Note that for a given positive integer \( l \) with \( l > m - T_u \) there are total number \( \binom{l-1}{|u|-1} \) of the vectors \( \ell \) in \( \mathcal{L}_{u, m} \) such that \( \| \ell \|_1 = l \), and we then have

\[
\Psi_{u, m, p} \leq [b(b - 1)^{2/p}]^{[u]} \left( \sum_{\ell \in \mathcal{L}_{u, m}} b^{-\frac{|u|}{2} \| \ell \|_1} \| f'_u \|_{L^p}^2 \right)^{2/p}
\]

\[
= \| f'_u \|_{L^p}^2 \left[ b(b - 1)^{2/p} \right]^{[u]} \sum_{l = m - T_u + 1}^{\infty} \left( \frac{l - 1}{|u| - 1} \right)^{\frac{1}{2}} b^{-\frac{|u|}{2} l} \right)^{2/p}
\]

\[
= \| f'_u \|_{L^p}^2 \left[ b(b - 1)^{2/p} \right]^{[u]} \Omega(p/2, m, T_u, |u|)^{2/p},
\]

where \( \Omega \) is as defined in Lemma 5. Making use of the inequality in Lemma 5 and the fact \( m = \log_b n = \ln n / \ln b \) yields

\[
\Psi_{u, m, p} \leq \| f'_u \|_{L^p}^2 \left[ \frac{b^p(b - 1)}{(b^p/2 - 1) \ln b} \ln n \right]^{\frac{2}{p}} b^{-m + T_u}.
\]

Applying this inequality to (28) we have

\[
\text{MSE}(Q_n, f) \leq n^{-2} \sum_{u \neq \emptyset} \| f'_u \|_{L^p}^2 b^{2T_u} (\log n)^{\frac{2}{p}} |u|,
\]

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where $\theta_1 = 3^{p/2}b^{(b-1)/([b^{p/2}]-1)} \ln b$. For $q \in [1, \infty]$, by Hölder inequality and the definition of norm in (17) we have

$$\text{MSE}(Q_n, f) \leq n^{-2} \left[ \sum_{u \neq \emptyset} \left( \sum_{u \neq \emptyset} \left( \gamma_u^{-2} \|f_u^p\|_{L_p}^2 \right)^{q} \right)^{1/q} \right]^{1/q^*} \leq n^{-2} \left\| f \right\|_{p, q, \gamma, s}^{2} \left\{ \sum_{u \neq \emptyset} \left( \gamma_u^{-2} b_2^{2T_u} (\theta_1 \ln n)^{\frac{2}{p}|u|} \right)^{q} \right\}^{1/q^*} \leq n^{-2} \left\| f \right\|_{p, q, \gamma, s}^{2} \left[ \Phi(p, p, \theta_1, 1) \right]^{2/p} \Omega(2, m, T_u, |u|).$$

where the last inequality follows from $2q^*/p \geq 1$ for $p \in [1, 2]$, and $\Phi$ is defined by (13).

Applying Lemma 2 to $\Phi(p, p, \theta_1, 1)$ gives the desired result for the case $p \in [1, 2]$ and then for $p \in (2, \infty)$.

As to the item (ii), it is also sufficient to consider the case $p = 2$ due to the fact in (34). For $p = 2$ ($p^* = 2$), from (27), (35) and Lemma 3 we have

$$\Psi_{u,m,2} \leq \left( \frac{b^2}{3} \right)^{|u|} \sum_{\nu \in \mathcal{N}_{u,m}} \|f_u^p\|_{L_2, S(\nu)} b^{-2\|\nu\|_1}$$

$$= \left( \frac{b^2}{3} \right)^{|u|} \sum_{l \in \mathcal{L}_{u,m}} b^{-2|\ell|_1} (b-1)^{|u|} \|f_u^p\|_{L_2}$$

$$= \|f_u^p\|_{L_2}^2 \left( \frac{b^2(b-1)}{3} \right)^{|u|} \sum_{l=m-T_u+1}^{\infty} \left( |u| - 1 \right) b^{-2l}$$

$$= \|f_u^p\|_{L_2}^2 \left( \frac{b^2(b-1)}{3} \right)^{|u|} \Omega(2, m, T_u, |u|).$$

From Lemma 5 we then have

$$\Psi_{u,m,2} \leq \|f_u^p\|_{L_2}^2 \left( \frac{b^4}{3(b+1) \ln b} \ln n \right)^{|u|} b^{-2m+2T_u}.$$ 

Therefore, from (28)

$$\text{MSE}(Q_n, f) \leq n^{-3} \sum_{u \neq \emptyset} \|f_u^p\|_{L_2}^2 b^{3T_u} (\theta_3 \ln n)^{|u|},$$
where \( \theta_3 = b^4/[(b + 1) \ln b] \). For \( q \in [1, \infty] \), by Hölder inequality

\[
\text{MSE}(Q_n, f) \leq n^{-3} \| f \|_{2,q,\gamma,s}^2 \left[ \sum_{u \neq \emptyset} \left( \gamma_2 b^{3T_u} (\theta_3 \ln n)^{|u|} \right)^q \right]^{1/q^*} \leq n^{-3} \| f \|_{2,q,\gamma,s}^2 \sum_{u \neq \emptyset} \gamma_2 b^{3T_u} (\theta_3 \ln n)^{|u|} = n^{-3} \| f \|_{2,q,\gamma,s}^2 \Phi(2, 3, \theta_3, 1).
\]

Applying Lemma 2 to \( \Phi(2, 3, \theta_3, 1) \) gives the desired result for the case \( p = 2 \) and then for \( p \in (2, \infty] \). This completes the proof for the theorem. ■

**Remark 2.** For different values of \( p \) in \([2, \infty]\), we have two kinds of sufficient condition for strong tractability in the randomized setting. The convergence rates are different under these two conditions. The condition under which the rate is higher is a little bit stronger.

**Remark 3.** The results in Theorem 2 are just for randomly scrambled Niederreiter digital nets but not for sequences. The difficulty for randomly scrambled sequences is the calculation for the expectation, \( E[Q_n(\psi_\nu)Q_n(\overline{\psi_\nu})] \).

### 4. Weighted integration over a general domain

In this section, we extend the results presented in previous two sections to the weighted integration problem over a general domain \( D = (a_1, b_1) \times \cdots (a_s, b_s) \) and the general weight function \( \rho \) defined in (3).

#### 4.1 Worst-case error analysis

The general problem for the worst case of integration can be reduced to the classical problem with domain \([0, 1]^s\) and uniform weight \( \rho \equiv 1 \). Specifically, we define transformations:

\[
W_k(x) := \int_{a_k}^x \rho_k(z) dz, \quad k = 1, \ldots, s.
\]

Then \( W_k : (a_k, b_k) \to [0, 1] \) is onto and increasing. Define

\[
W(x) = (W_1(x_1), \ldots, W_s(x_s))^T
\]

and \( d = W(c) \) for a given anchor \( c \in D \). By \( W^{-1} \) denote the inverse transformation of \( W \).

By these transformations the integral \( I_\rho(f) = \int_D f(x) \rho(x) dx \) may be written as

\[
\int_{[0,1]^s} g(y) dy \quad \text{where} \quad g(y) = f(W^{-1}(y)).
\]

Let \( \mathcal{F}_{p,q,\gamma,s}^{(1)} \) be the weighted Banach space of functions defined on \( D \) with the corresponding norm of the form (5). The worst-case error

\[
e_{wo}(Q_n, \mathcal{F}_{p,q,\gamma,s}^{(1)}) = \sup_{\| f \|_{p,q,\gamma,s} \leq 1} |I_\rho(f) - Q_n(f)|
\]
is still of the form (10) [HSW04a], where

\[ h_u(t_u) = I_p(M_u(\cdot; t_u)) - Q_u(M_u(\cdot; t_u)), \quad t_u \in D_u := \prod_{k \in u} (a_k, b_k). \]

Denote the \(|u|\)-dimensional vectors \(d_u = (W_k(c_k))_{k \in u}, \ y_u = (W_k(t_k))_{k \in u}, \ z_u = (W_k(x_k))_{k \in u}\). By \(u_-\) denote the subset of \(u\) containing those indices \(k\) for which \(t_k < c_k\). Then \(h_u(t_u)\) can be expressed as

\[ h_u(t_u) = (-1)^{u_-} \left[ \text{vol}(B_u(y_u, d_u)) - \frac{1}{n} \sum_{j=0}^{n-1} 1_{B_u(y_u, d_u)}(z_u^j) \right], \]

where \(B_u\) is defined as in Section 2. It follows that for \(p \in [1, \infty]\) with conjugate \(p^*\)

\[ ||h_u||_{L_{p^*}} = \left( \int_{D_u} |h_u(t_u)|^{p^*} \rho_u(t_u) dt_u \right)^{1/p^*} \]

\[ = \left( \int_{[0,1]^u} \left| \text{vol}(B_u(y_u, d_u)) - \frac{1}{n} \sum_{j=0}^{n-1} 1_{B_u(y_u, d_u)}(z_u^j) \right|^{p^*} dy_u \right)^{1/p^*} \]

\[ = D_{u,p^*}(z^0, \ldots, z^{n-1}), \]

which is the local \(L_{p^*}\) anchored discrepancy of the point set \(\{z^0, \ldots, z^{n-1}\}\). It is known that [Nie92]

\[ D_{u,p^*}(z^0, \ldots, z^{n-1}) \leq \kappa D_{u,\infty}(z^0, \ldots, z^{n-1}), \]

where \(\kappa = 1\) if \(d = W(c) = 1\), and \(\kappa = 2^{|u|}\) if \(d = W(c)\) is in the interior of the unit cube \([0, 1]^u\). Therefore, we have the following strong tractability result immediately followed from Theorem 1.

**Theorem 3.** Let \(p, q \in [1, \infty]\) and \(\mathcal{F}(1)_{p,q,2,\gamma,s}\) be defined as in Section 2 for functions on the domain \(D = [a, b]\). Assume that \(\{z^j\}_{j \geq 0}\) is a Niederreiter \((T_u), s)\)-sequence in base \(b\), and \(\{x^j\}_{j \geq 0}\) is the transformed sequence according to the transformations in (37). If

\[ \sum_{k=1}^{\infty} \gamma_k^a k \ln k < \infty \]

for any \(a \in [1, q^*]\), then the corresponding integration is strongly tractable in the worst-case setting, and

\[ e^{w_0}(Q_n, \mathcal{F}(1)_{p,q,2,\gamma,s}) \leq C n^{-1/\alpha + \epsilon} \]

for any fixed \(\epsilon > 0\), where \(C\) is some constant independent of \(s\) and \(n\).

### 4.2 Randomized error analysis

Here we note that the assumption, \(\|f\|_{p,q,2,\gamma,s} < \infty\), in the space \(\mathcal{F}(2)_{p,q,2,\gamma,s}\) defined in Section 3 might be too restrictive when \(D\) is unbounded. To alleviate this problem, we consider a modification following the approach from [HSW04b].

Consider a transformation of variables

\[ y = W(x) = (W_1(x_1), \ldots, W_s(x_s))^T, \]

where each \( W_k \) is a cumulative distribution function on interval \((a_k, b_k)\) with density \( w_k(x_k) = W_k'(x_k)\). Then integral \( I_\rho(f) = \int_D f(x)\rho(x)dx \) may be written as

\[ \int_{[0,1]^s} g(y)dy \quad \text{where} \quad g(y) = f(W^{-1}(y))\phi(W^{-1}(y)), \] (39)

\[ \phi(x) = \frac{\rho(x)}{w(x)} \quad \text{and} \quad w(x) = \prod_{k=1}^s w_k(x_k). \]

See [HSW04b] for a discussion about the necessary of introducing \( w(x) \).

We now suppose that \( \{z^0, \ldots, z^{n-1}\} \) is a randomly scrambled digital net, and the integral of \( g \) in (39) is approximated by \( n^{-1} \sum_{j=0}^{n-1} g(z^j) \). This is equivalent to the rule

\[ Q_n(f) = \frac{1}{n} \sum_{j=0}^{n-1} f(W^{-1}(z^j))\phi(W^{-1}(z^j)) \] (40)

for \( I_\rho(f) \). By Lemma 4 the error of this approximation has the following upper bound

\[ \sum_{\nu: \|\nu\|_1 + T_{\nu} > m} |G(\nu)|^23^{L(\nu)}\bar{\gamma}^{T_{\nu} - m}, \]

where the \( G(\nu) \) are the Fourier coefficients of \( g(y) = f(W^{-1}(y))\phi(W^{-1}(y)) \) under the Haar wavelets \( \psi_\nu \). In terms of \( \xi_\nu \) defined in (23) \( G(\nu) \) can be expressed as

\[ G(\nu) = (-1)^{L(\nu)} \int_{S(\nu)} \xi_\nu(y)g_{\nu}(y)dy, \]

where \( S(\nu) \) is the support of the wavelet \( \psi_\nu \), which is given in (20). Now for each nonempty subset \( u \subseteq 1:s \)

\[ g_u'(y) = \frac{1}{w_u(W^{-1}(y))} \frac{\partial^{\nu_u}(f\phi)(x)}{\partial x_u} \bigg|_{x = W^{-1}(y)}, \]

where \( f\phi \) just denotes the multiplication of the two functions, and \( w_u(x) = \prod_{k \in u} w_k(x_k) \). Applying Hölder inequality yields that

\[ |G(\nu)| \leq \left( \int_{S(\nu)} \left| \frac{\partial^{\nu_u}(f\phi)(x)}{\partial x_u} \bigg|_{x = W^{-1}(y)} \right| \frac{1}{w_u(W^{-1}(y))} dy \right)^{1/p} \left( \int_{[0,1]^s} |\xi_\nu(y)|^p dy \right)^{1/p^*}. \]

Define the norm

\[ \|f\|_{p,q;\gamma,s,\rho,w} := \begin{cases} \left( \sum_{u \subseteq 1:s} \gamma_u^{-q}\|(f\rho/w)_u^{-1}w_u^{-1}\|_{L_p}^q \right)^{1/q} & \text{for } q < \infty, \\ \max_u \left\{ \gamma_u^{-1}\|(f\rho/w)_u^{-1}w_u^{-1}\|_{L_p} \right\} & \text{for } q = \infty. \] (41)
Then we modify the space $\mathcal{F}^{(2)}_{p,q,\gamma,s}$ to $\mathcal{F}^{(2)}_{p,q,\gamma,s,p,w}$, i.e., we let $\mathcal{F}^{(2)}_{p,q,\gamma,s,p,w}$ be the weighted Banach space of all absolutely continuous functions $f$ defined on $D$ with $\|f\|_{p,q,\gamma,s,p,w} < \infty$. The randomized error is defined as

$$e^{\text{ra}}(Q_n, \mathcal{F}^{(2)}_{p,q,\gamma,s,p,w}) := \sup_{\|f\|_{p,q,\gamma,s,p,w} \leq 1} \sqrt{E[I_p(f) - Q_n(f)]^2}.$$

From Theorem 2 we have the strong tractability results of integration over the general domain in the randomized setting.

**Theorem 4.** Consider the integration problem of approximating integral (1) by the rule (40) with randomly scrambled Niederreiter digital net in base $b$. Let $\mathcal{F}^{(2)}_{p,q,\gamma,s,p,w}$ be defined above.

(i) For $p \in [1, \infty]$, if

$$\sum_{k=1}^{\infty} (\gamma_k k \ln k)^{\min(p,2)} < \infty,$$

then the corresponding integration is strongly tractable in the randomized setting, and

$$e^{\text{ra}}(Q_n, \mathcal{F}^{(2)}_{p,q,\gamma,s,p,w}) \leq Cn^{-1+\epsilon}$$

for any fixed $\epsilon > 0$, where $C$ is some constant independent of $s$ and $n$.

(ii) In particular, for $p \in [2, \infty]$, if

$$\sum_{k=1}^{\infty} \gamma_k^2 (k \ln k)^3 < \infty,$$

then the corresponding integration is strongly tractable in the randomized setting, and

$$e^{\text{ra}}(Q_n, \mathcal{F}^{(2)}_{p,q,\gamma,s,p,w}) \leq Cn^{-3/2+\epsilon}$$

for any fixed $\epsilon > 0$, where $C$ is some constant independent of $s$ and $n$.

5. Concluding remarks

We have considered the strong tractability problems for multivariate integration using deterministic and randomly scrambled Niederreiter sequences. The spaces of integrands are weighted Banach spaces with parameters $p, q, \gamma, s$. The definitions of these spaces are slightly different in the worst-case and randomized settings. The main results of this article are summarized below.

Each of the conditions we found for strong tractability in worst-case and randomized settings is of the form $\sum_{k=1}^{\infty} \gamma_k^\alpha (k \ln k)^\beta < \infty$ for some positive numbers $\alpha$ and $\beta$. The values of $\alpha$ and $\beta$ are determined by $p$ or $q$. The larger $\alpha$ and smaller $\beta$ implies the weaker conditions of strong tractability. In the worst-case setting, the parameter $p$ has little influence on the convergence rate of the worst-case error, however, the parameter $q$ plays a significant role in determining strong tractability. In the randomized setting, only the parameter $p$
plays a significant role in determining both strong tractability and convergence rate of the randomized error.

The factors \((k \ln k)^\beta\) in each term of the summation in the strong tractability conditions come from the quality parameter vector \((T_u)\) of Niederreiter sequence due to its telescopic property. These factors make the strong tractability conditions in this article are slightly stronger than the conditions on integration using lattice rules. It is shown in [HSW04b, Theorem 3] that integration using lattice rules for the weighted Banach spaces is strongly tractable under the condition that \(\sum_{k=1}^{\infty} \gamma_k^a < \infty\) for \(a \in [1, q^*]\), and the convergence rate of the worst-case error is \(O(n^{-1/(a+\epsilon)})\). We do not know yet whether integration using Niederreiter sequences is strongly tractable under the same condition as using lattice rules.

References


