Optical solitons in a few-cycle regime: Breakdown of slow-envelope approximation

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Abstract

The propagation of few-cycles optical pulses in a collection of two-level atoms is investigated beyond the traditional slowly varying envelope approximation. It is shown that in certain conditions, a femtosecond pulse can evolve into a stable spatiotemporal optical soliton.

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1. Introduction

The recent progress in the development of solid-state mode-locked lasers has resulted in the generation of two-cycle optical pulses [1]. Subsequently, such short pulses have been widely exploited to generate a
uni-polar single-cycle electromagnetic pulse in a variety of nonlinear media [2]. They have also resulted in theoretical studies in order to determine whether it is possible to describe correctly the dynamics of such pulses within the traditional framework of the slowly varying envelope approximation (SVEA) operating with a quasi-monochromatic field [3].

The validity of the predictions of the nonlinear Schrödinger equation (NLSE) rests on an overwhelming number of experimental results which, at least up to the sub-picosecond time scale, have never been known to contradict the basic assumptions of the model. Although the recent extensions of the SVEA were intended to describe the femtosecond (fs) regime, many restrictive assumptions must be made to render the equations applicable. The most important amongst them is a finite number of weakly interacting co-propagating waves to be taken into account.

In this paper, we apply a perturbation expansion method for femtosecond pulse propagation through a medium of two-level atoms. The purpose is to go beyond the SVEA to show the advantages of the self-consistent description based upon the coupled Maxwell–Bloch system and to impose the proper relationships among nonlinearity, dispersion, dissipation (or amplification), diffraction, and backward scattering effects.

This article is organized as follows. In Section 2, we present a perturbation expansion of a pulsed electric field to solve the Maxwell–Bloch equations. General expressions for the pulse shape are derived as a function of the pulse width and the parameters of the dipole transition. Numerical results illustrating a transverse stability of optical pulses are then presented. In Section 3, we investigate the case when the consideration can be formally reduced to the sine-Gordon equation, and we analyze its properties in this case. Section 4 presents conclusions.

2. A modified Korteweg-de Vries equation

2.1. Model

Leaving completely the concept of the slow envelope, we are able to derive an approximate model of the modified Korteweg-de Vries (mKdV) type [4]. The calculations are performed using the following simple model. We consider identical atoms described by a two-level Hamiltonian

\[ H_0 = \begin{pmatrix} \omega_a & 0 \\ 0 & \omega_b \end{pmatrix}, \]  

and assume that the evolution of the electric field can be described by the scalar Maxwell equation

\[ \frac{\partial^2 E}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2}{\partial z^2} (E + 4\pi P), \]  

where \( c \) is the light velocity in vacuum. This assumes a plane wave propagating along the \( z \)-axis, and linearly polarized along the direction of the atomic dipolar electric momentum \( \vec{\mu} \). If we denote by \( \vec{e} \) the unitary vector along this direction (\( x \)-axis), we have thus \( \vec{\mu} = \mu \vec{e} \) and \( \vec{E} = E \vec{e} \). The interaction between the atoms and the electric field is then taken into account by an energy coupling term in the total Hamiltonian \( H \), as

\[ H = H_0 - \mu E. \]
The density matrix $\rho$ is related to the polarization density $\vec{P}$ through

$$P = N \alpha(\rho \mu),$$

where $N$ is the number of atoms per volume unit, and $P$ evolves in time according to

$$i \frac{\partial \rho}{\partial t} = [H, \rho] + i \left( \frac{\rho_b}{\tau_b} - \frac{\rho_t}{\tau_t} \right) \left( \frac{\rho_t}{\tau_t} - \frac{\rho_b}{\tau_b} \right).$$

(5)

The last term on the right-hand side is a phenomenological relaxation term, with $\tau_b$ and $\tau_t$ being the relaxation times for the population and for the coherence, respectively. The set of Eqs. (2)–(5) is sometimes called the Maxwell–Bloch equations (although more often this name denotes a reduced version).

2.2. Derivation of a nonlinear evolution equation

We consider the situation where the wave duration $t_w$ is long with regard to the period $t_r = \frac{2\pi}{\Omega}$, with $\Omega = \omega_b - \omega_a$, corresponding to the resonance frequency of the two-level atoms. We assume that $t_w$ is about one optical period, say about one femtosecond. Thus, we assume that the resonance frequency $\Omega$ is large with regard to optical frequencies. In order to obtain soliton-type propagation, the nonlinearity must balance dispersion, thus the two effects must arise simultaneously in the propagation. This involves a small amplitude approximation. Further, we can speak of a soliton only if the pulse shape is maintained over a large propagation distance. Therefore, we use the reductive perturbation method as defined in [5].

We expand the electric field $E$, the polarization density $P$ and the density matrix $\rho$ as power series of a small parameter $\epsilon$ as

$$E = \sum_{n \geq 1} \epsilon^n E_n, \quad P = \sum_{n \geq 1} \epsilon^n P_n, \quad \rho = \sum_{n \geq 0} \epsilon^n \rho_n,$$

(6)

and introduce the slow variables

$$\tau = \epsilon \left( t - \frac{z}{V} \right), \quad \zeta = \epsilon^3 \zeta.$$

(7)

Further, since the physical values of the relaxation times $\tau_b$ and $\tau_t$ are in the picosecond range, or even slower, i.e. very large with regard to optical frequencies, we write $\tau_j = \hat{\tau}_j / \epsilon^2$ for $j = b$ and $t$. We assume that, in the unperturbed state, all atoms are in their fundamental state $a$.

The order by order resolution of the perturbative scheme leads to the following nonlinear evolution equation:

$$\frac{\partial E_1}{\partial \zeta} = \frac{4\pi N |\mu|^2}{nc \Omega} E_1 + \frac{8\pi N |\mu|^4}{nc^2 \Omega^2} \frac{\partial (E_1^*)}{\partial \tau},$$

(8)

where $n = c/V = \sqrt{1 + \frac{8\pi N |\mu|^2}{c^3}}$. The mKdV equation (8) is completely integrable by means of the inverse scattering transform [8]. The $N$-soliton solution has been given by Hirota [9]. It has appeared in various branches of physics such as anharmonic lattices and Alfvén waves in collisionless plasma. Moreover, it is needed to eliminate secularly growing terms in a formal asymptotic expansion over a small parameter [15].

The general solutions of Eq. (8) are governed by the relative sign between the nonlinear and dispersion terms, the asymptotic values of the field (the boundary conditions), and the pulse width. For an adequate...
Fig. 1. Second-order soliton solution of the mKdV equation, using dimensionless parameters.

choice of parameters, the two-soliton solution can have the behavior shown in Fig. 1. The corresponding spectrum and pulse profile are given in Fig. 2. They are comparable to the experimental pulses given by [10]. It can thus be thought that the two-cycle pulses produced experimentally could propagate as solitons in certain media, according to the mKdV model.

In the case of the non-zero boundary conditions, $E_1(\zeta = \pm \infty) = E_\infty$, the general single soliton solution takes the form [11]:

$$E_1(\zeta, \tau) = E_\infty \left[ 1 - \frac{4e^{-\beta}(1 - E_d/E_0)}{(1 - E_d/E_0 - 2\delta e^{-\beta}\beta^2 + 2\delta e^{-2\beta})} \right].$$

Fig. 2. (a) Pulse profile and (b) spectrum, of the second-order soliton solution of the mKdV equation of Fig. 1. Dimensionless parameters.
Fig. 3. Normalized intensity vs. normalized time for single bright soliton with non-zero boundary conditions (9): (a) $\delta = 0.75$, (b) $\delta = 1.0$, and (c) $\delta = 2.0$.

Here

$$\beta = \frac{(\tau - \zeta/v)}{t_w} + \beta_0, \quad E_1^2 = 2E_\infty^2 + E_0^2, \quad \delta = \frac{E_\infty}{E_0},$$

where $E_0$ labels the maximum amplitude of the bright soliton whose displacement from $E_\infty$ is given by $E_d$ which, in turn, is to be found for a given set of boundary conditions. It turns out that the solutions $E_1(\zeta, \tau)$ lie in the range

$$E_\infty \leq E_1 \leq \sqrt{2}E_\infty - E_\infty.$$

They describe either a hump-like pulse superimposed on a continuous wave background, i.e., an unbound soliton, or a hyperbolic secant solitary pulse. The behavior depends on the magnitude of $E_\infty$, with the transition between bound and unbound solitons occurring when $E_\infty > 0$. In Fig. 3, we plot the intensity of the single bright soliton with non-zero boundary conditions for different ratios $\delta$. A typical spreading out of hump amplitudes at $\delta \geq 1$ is readily seen. Notice also the appearance of asymmetry in the pulse profile for $\delta > 0.5$. Physically, this is due to the line broadening by the dc field $E_\infty$ that shifts the dispersion contour of the two-level atom.

Let us now return back to Eq. (8). The relative contributions to the polarization from nonlinearity and dispersion effects are of the same sign. This dictates our choice of the non-zero asymptotic soliton in the form of Eq. (9) and rules out the dark soliton solution:

$$E_1(\zeta, \tau) = E_\infty \left[ 1 - \frac{4e^{-\delta}}{1 + 2\delta^2e^{-\delta^2} - 2\delta^2e^{-\delta}} \right].$$

Although such topology for the femtosecond field should be regarded as an illustrative one, it is depicted in Fig. 4 and describes the bifurcation of the dark-grey soliton state into the coupled state of two dark-black solitons of equal width, with $\delta \to \sqrt{2}/2$ as the point of bifurcation.
2.3. A generalization

Both direct computation, and the general theory [6] have shown that the coefficient of the dispersive term $\partial^3 E_1 / \partial \tau^3$ in Eq. (8) is $\frac{1}{6} \frac{\partial k}{\partial \omega^3}$. To get a general expression of the nonlinear term, we consider the long wave limit of the NLSE which describes the evolution of a short pulse envelope in the same medium (given in [7], 6.5.32). In this way it is found that the nonlinear coefficient is $\frac{6\pi}{nc} \chi^{(3)}$. The relevant component of the third order nonlinear susceptibility tensor $\chi^{(3)}$ can be calculated both from the above model in [7] and from quasi-adiabatic following approximation [11]. Comparison of the coefficient computed this way with its expression given in Eq. (8) shows that the general expression is valid in the particular case considered. Thus, we can write Eq. (8) as

$$\frac{\partial E_1}{\partial \zeta} = \frac{1}{6} \frac{\partial k}{\partial \omega^3} \frac{\partial E_1}{\partial \tau} - \frac{6\pi}{nc} \chi^{(3)}(\omega, \omega, -\omega) \frac{\partial (E_1^3)}{\partial \tau}. \tag{11}$$

It can be reasonably conjectured that Eq. (11) will still hold in the more general case of an arbitrary number of atomic levels, when the inverse of the characteristic pulse duration is much smaller than any of the transition frequencies of the atoms.

2.4. Multidimensional localization of optical solitons in a few-cycle regime

In what follows we construct numerically solutions of the generalized Kadomtsev–Petviashvili equation (gKPE), which is a natural extension of the mKdV equation to $(2+1)$ dimensions. In the notations of ref. [12]:

$$\frac{\partial E_1}{\partial \xi} + \frac{1}{c_g} \frac{\partial E_1}{\partial \tau} + c_1 E_1^2 \frac{\partial E_1}{\partial \tau} + \frac{\partial^2 E_1}{\partial \tau^2} = c \int_{-\infty}^{\tau} \frac{\partial^2 E_1}{\partial \tau^2} d\tau. \tag{12}$$

We integrated Eq. (12) numerically, and the results are shown in Fig. 5, presenting two examples of pulses at the two different input peak fields. These are plots of the output intensity as a function of the...
Fig. 5. Transverse structure of the pulse at $z = z_{sf}$ for different input intensities: $I_{in} = I_{th}$ (a) and $I_{in} = 2I_{th}$ (b); the intensity units are arbitrary, and the scales are defined by the input pulse parameters: initial duration of 40 fs and transverse width of 50 μm.

Although our calculations contain the full structure of the pulse, we display only its landscape in the ($\xi, \tau$) space for the sample length $z = z_{sf}$ where $z_{sf} = 10\tau_p/c_1 E_0^2$ is the self-focusing length. We see that at the threshold of self-focusing the Gaussian pulse transforms into a soliton-like structure (a), that is, into a hyperbolic secant spatiotemporal profile. It is an outcome of the interplay between the group velocity and absorption on the left-hand side of Eq. (12) and transient diffraction on the right-hand side. The polarization wave that is induced does not cause any observable distortion at the pulse front or in the pulse itself.

Furthermore, in Fig. 5(b) we illustrate the case in which the peak intensity is twice as large as the threshold for self-focusing, $I_{in} = c/c_1 \alpha^2 \xi_0^2$, where $\xi_0$ is the spatial characteristic scale of the input field along the $\xi$ axis. The Gaussian pulse smoothly propagates up to the self-focusing point where, instead of collapsing, it then forms a coupled pair of $\pi$-shifted sech-like solitons.

The following explanation can be given to this phenomenon. Initially, as the process of self-focusing starts up, the nonlinearity pulls the energy in from the wings and generates higher spectral components.
On the other hand, the photons which propagate outward, that is, which diffracts more, have a smaller velocity along the axis of the propagation, and, consequently, experience a stronger time delay. This give rise to memory effects in the spatial structure of the pulse field. In addition, as the dispersion begins to take its toll, the pulse breaking occurs. The subsequent evolution can be understood as follows: the leading pulse acts as a potential field source which creates a field minimum where the trailing one is trapped by means of combined action of the nonlinearity, diffraction, and dispersion.

3. A sine-Gordon equation

3.1. Coherent absorber

We consider again the two-level model of Section 2.1, but assuming now that the resonance frequency $\Omega$ of the atoms is below the optical frequencies [4]. Thus, the characteristic pulse duration $t_\omega$ is very small with regard to $t_r = 2\pi/\Omega$, and we use a short wave approximation. We now introduce a small perturbative parameter $\varepsilon$, such that the resonance period $t_r = \varepsilon t_\omega$. Consequently, the Hamiltonian $H_0$ of the atom is replaced in the Schrödinger Eq. (5) by $\varepsilon \hat{H}_0$. We introduce a retarded time $\tau = t - z/V$ and a slow propagation variable $\zeta = \varepsilon z$. The electric field $E$ is expanded as $E = \sum_{n=0}^{\infty} \varepsilon^n E_n$, and so on. The pulse duration $t_\omega$ is still assumed to be about one femtosecond, corresponding to an optical pulse of a few cycles, and the relaxation times $\tau_b$ and $\tau_t$ are very long with regard to $t_\omega$. Since the above scaling uses $t_\omega$ as zero-order reference time, it can be expressed by setting $\tau_j = \hat{\tau}_j/\varepsilon$ for $j = b$ and $t$. Although formally different, this assumption is physically the same as that presented in Section 2.1. This scaling is equivalent to the standard short wave approximation formalism developed, e.g. in [13,14].

We now introduce the population inversion $w = \rho_{bb} - \rho_{ba}$. We show by solving the perturbative scheme that $w$ and the leading term $E_0$ in the expansion of the electric field satisfy the following set of equations:

$$\frac{\partial E_0}{\partial \zeta} = \frac{4\pi i \Omega N}{c} p, \quad \frac{\partial p}{\partial \tau} = -i |\mu|^2 E_0w, \quad \frac{\partial w}{\partial \tau} = -4i E_0 p,$$

(13)

where we have set

$$p = -i |\mu|^2 \int^\tau E_0w.$$

(14)

Equations (13) coincide with the equations for self-induced transparency, although the physical situation is quite different: the characteristic frequency $1/t_\omega$ of the pulse is far above the resonance frequency $\Omega$, while the self-induced transparency occurs when the optical field oscillates at the frequency $\Omega$. The quantities $E$ and $w$ here describe the electric field and the population inversion themselves, and not amplitudes modulating a carrier with frequency $\Omega$. Notice that $E$ and $w$ here also are real quantities, and not complex ones as in the case of the self induced transparency. Further, $p$ is not the polarization density, but is proportional to its $\tau$-derivative. Another difference is the absence of a factor $1/2$ in the right-hand side of Eq. (13).
System (13) can be reduced to
\[
\frac{\partial^2 u}{\partial Z \partial T} = \sin u.
\]
(15)
in which
\[
Z = \frac{z}{L}, \quad T = \frac{1}{t_w} \left( t - \frac{z}{c} \right), \quad E = \frac{E_r}{2} \int \sin u, \quad w = w_i \cos u.
\]
(16)
The electric field and propagation length scaling parameters are
\[
E_r = \sqrt{\frac{\mu}{\Omega t_w}}, \quad \hat{L} = \frac{c}{D_0 4 \pi N \mu^2 w_i},
\]
(17)
in which the initial population inversion $w_i$ and typical pulse duration $t_w$ are given.

The sine-Gordon Eq. (15) is completely integrable [15]. A $N$-soliton solution can be found using either the IST or the Hirota method. The two-soliton solution, obtained for adequate values of the parameters, becomes a second-order soliton or breather, analogous to the solution of the mKdV equation shown in Fig. 2. It is thus able to describe soliton-type propagation of a pulse in the two-cycle regime. The pulse profile, with the corresponding population inversion and spectrum, are shown in Fig. 6. They are comparable with the experimental observation of [10].

![Fig. 6. (a) Pulse profile, (b) population inversion and (c) spectrum, of the second-order soliton solution of the sine-Gordon equation. Dimensionless parameters.](image)
3.2. Coherent amplification and Raman shift suppression

Eq. (15) can also be exploited in the coherent amplification by means of corresponding change of the right-hand side. Its general solution can be expressed in terms of Painlevé transcendental functions [16,17]. In order to avoid re-writing redundant equations, let us further restrict the consideration by noticing that the forward-propagating pulse leads to the following self-consistent solution in physical units:

\[ E(z,t) \approx \frac{2\pi}{c} \mu N \Omega z \text{sech} \left[ \frac{4\pi}{c} \mu^2 N \Omega z \left( t - \frac{z}{c} \right) \right]. \] (18)

This solution can be interpreted as a pulse self-compression, which is due to coherent amplification, and a blue shift of the carrier frequency \( \omega_c \) of the pulse,

\[ \omega_c(z) \approx \frac{4\pi}{c} \mu^2 N \Omega z, \] (19)

which is due to the strong pulsed field pushing away the coupled levels of the two-level atom. This dynamical blue shift is the main point of this subsection. It should be stressed that the energy of the fs pulse grows up, owing to the increasing energy of photons taken from the inverted medium. What is more important is that the number of photons remains constant, contrary to the implication of the SVEA, where the growth of the pulse energy is caused by increasing the number of photons of the same frequency. This effect can be exploited to balance a red-shift of the carrier frequency that is due to the Raman scattering in a femtosecond optical fiber-amplifier system [18].

4. Conclusions

In conclusion, a systematic perturbation expansion is used to solve the complete set of Maxwell–Bloch equations in the lowest order approximation. This enables us to demonstrate the possibility of a femtosecond soliton transmission without resorting to the SVEA. One advantage of the soliton solution found is that it does not imply the existence of a carrier frequency. Another hidden advantage is that a meaningful comparison with already known solutions to the non-reduced Maxwell–Bloch equations, and with pertinent numerical results, is possible. Because almost all experimental data have been interpreted by using different modifications of the NLSE and selecting the ones giving the best results, it would be desirable to check those parameters anew by using the integrable equations delivered here. Regarding applications, the sensitivity to boundary conditions on the pulse width and pulse shape can make the predicted solitons an effective tool for optical switching, signal processing and information carrying.

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