Impact of Phase Noise on OFDM and SC-CP

J. L. Zamorano\(^2\) J. Nsenga\(^{1,2}\) W. Van Thillo\(^{1,2}\) A. Bourdoux\(^2\) F. Horlin\(^3\)

\(^1\)ESAT-KUL
Kasteelpark Arenberg 10
B-3001 Heverlee, Belgium

\(^2\)IMEC
Kapeldreef 75
B-3001 Heverlee, Belgium

\(^3\)ULB
Avenue F. D. Roosevelt 50
B-1050 Bruxelles, Belgium

Abstract—Single-Carrier with Cyclic Prefix (SC-CP) is seen as an interesting air interface to replace Orthogonal Frequency-Division Multiplexing (OFDM) because it features a lower Peak-to-Average Power Ratio (PAPR) while still allowing low complexity Frequency Domain Equalization (FDE). Both air interfaces are highly sensitive to Phase Noise (PN) that degrades their system performances. In this paper, we study and compare analytically the PN impact on both air interfaces. Simulations are also carried out to validate the analytical results. PN causes the same Common Phase Error (CPE) on both air interfaces, as well as it leads to Inter-Carrier Interference (ICI) in OFDM and Inter-Symbol Interference (ISI) in SC-CP. However OFDM is found to be slightly less affected than SC-CP in both flat channels and frequency selective channels. It is shown also that CPE is the dominant impact if the PN cut-off frequency is smaller than the subcarrier spacing.

I. INTRODUCTION

We are witnessing today an explosive demand for wireless communication systems. Because the available spectrum is limited, the spectral efficiency of wireless systems should be as high as possible. On the one hand, Orthogonal Frequency-Division Multiplexing (OFDM) has usually been used for indoor and outdoor wireless communications because it can mitigate the multipath propagation at a low complexity with a Frequency Domain Equalization (FDE). On the other hand, Single-Carrier with Cyclic Prefix (SC-CP) is seen as a possible alternative since it offers the same low complexity with a FDE. Moreover it is characterized by a low Peak-to-Average Power Ratio (PAPR), enabling an efficient non-linear power amplification [1].

Front-End (FE) non-idealities are known to degrade the system performance. One of the most important non-idealities is the oscillator Phase Noise (PN). The main PN effects on OFDM are well known: a Common Phase Error (CPE) and Inter-Carrier Interference (ICI) [2]. Comparison of the sensitivity of OFDM and SC with Time Domain Equalization (TDE) to PN have been carried out for different scenarios. Bit Error Rate (BER) sensitivity is higher for OFDM than for SC in Additive White Gaussian noise (AWGN) channel [3]. However the authors approximate the contribution of the PN by an additional noise component, which is accurate enough only if the degradation is mainly determined by the ICI [4]. On the other hand, signal-to-noise ratio (SNR) sensitivity is the same for both air interfaces before attempting to correct the CPE [5]. These statements will be extended to SC with FDE (SC-CP) in this paper and generalized to multipath channels.

We evaluate the PN impact on both air interfaces, OFDM and SC-CP. We consider a wireless system working around 60 GHz, offering promising perspectives for high rate Wireless Personal Area Network (WPAN) communications. Because the system bandwidth is higher compared to 5 GHz systems, the integrated PN power is also about 20 dB higher [6]. The letter is organized as follows. The system model is provided in section II. In section III, the PN impact is studied analytically. To check the theoretical results, simulations are presented in section IV. Our conclusions are drawn in Section V.

II. SYSTEM MODEL

A block diagram of the system is depicted in Fig. 1. The information symbols, \(s[n]\), which are assumed independent and of variance \(\sigma_s^2\), are first serial-to-parallel converted into blocks of \(Q\) symbols, leading to the symbol block sequence, \(\hat{s}[n] := [s[nQ], \ldots, s[(n+1)Q-1]]^T\). The blocks \(\hat{s}[n]\) are linearly precoded with the square matrix \(\theta\) of size-\(Q\):

\[
\hat{s}[n] := \theta \cdot s[n].
\]

(1)

The precoding matrix can be defined such that OFDM and SC-CP modulations are instantiated:

- OFDM is obtained by setting \(\theta = I\);
- SC-CP is obtained by setting \(\theta = E\).

Matrix \(E\) is the size-\(Q\) identity matrix and matrix \(F\) is the size-\(Q\) Fast Fourier Transform (FFT) matrix defined as:

\[
F := \frac{1}{\sqrt{Q}} \left[ e^{-2\pi j mn/Q} \right]_{m,n=0}^{Q-1}.
\]

(2)

The next operation involves the transformation of the frequency-domain block sequence, \(\bar{x}[n]\), into the time-domain block sequence:

\[
x[n] = F^H \cdot \bar{x}[n].
\]

(3)

Finally, the \(K \times Q\) \((K \geq Q)\) transmit matrix, \(T_{ex}\), adds the CP of length \(L\) to the time-domain blocks:

\[
u[n] = T_{ex} \cdot x[n]
\]

(4)

with \(K = Q + L\) and \(T_{ex} := [\emptyset_{\nu \times (Q-L)}; L; L; \ldots; L; L]\), in which \(\emptyset_{\nu \times (Q-L)}\) is a matrix of zeros of size \(L \times (Q-L)\). The resulting transmitted block sequence, \(u[n]\), is parallel-to-serial converted into the corresponding scalar sequence, \([u[nK], \ldots, u[(n+1)K-1]]^T := u[n]\), and transmitted over the air at a rate \(1/T\), where \(T\) is the sample period.
The transmitted sequence, \( u[n] \), is convolved with the digital equivalent channel impulse response, \( h[n] \), that models the frequency-selective multipath propagation, including the effects of the transmit/receive filters. After AWGN addition, each sample at the receiver is rotated by a random phase, \( \phi[n] \), of zero mean and characterized by a power spectral density flat at the low frequencies and exhibiting a -20 dB/dec decrease at high frequencies [7]. The received sequence, \( v[n] \), is given by:

\[
v[n] = e^{j\phi[n]} (u[n] \otimes h[n] + w[n]),
\]

(5)

in which \( w[n] \) is the AWGN sequence of variance \( \sigma_w^2 \).

Assuming perfect time and frequency synchronization, the received sequence \( v[n] \) is serial-to-parallel converted into the corresponding block sequence, \( \Psi[n] := [v[nK], \ldots, v[(n+1)K-1]]^T \). From the scalar input/output relationship in (5), we can derive the corresponding block input/output relationship:

\[
\Psi[n] = \Phi \cdot (H[0] \cdot \Psi[n] + H[1] \cdot u[n-1] + w[n]),
\]

(6)

where \( H[0] \) is a \( K \times K \) lower triangular Toeplitz matrix with entries \( H[0]_{p,q} = h[p-q] \), and \( H[1] \) is a \( K \times K \) upper triangular Toeplitz matrix with entries \( H[1]_{p,q} = h[K+p-q] \). The delay-dispersive nature of multipath propagation gives rise to the so-called inter-block interference (IBI) between successive blocks, which is modeled by the second additive term in (6). The additive noise block sequence is denoted by \( w[n] := [w[nK], \ldots, w[(n+1)K-1]]^T \). The received vector is multiplied by a diagonal matrix \( \Phi \) of size \( K \), with entries \( e^{j\phi[n]} \) on its diagonal.

The \( Q \times K \) receive matrix, \( R_{Q \times kp} \), removes the redundancy from the blocks, that is, \( \Psi[n] := R_{Q \times kp} \cdot \Psi[n] \). With \( R_{Q \times kp} := \begin{bmatrix} L^{-1} \\ L^{-1} \end{bmatrix} \otimes I_Q \), it discards the \( L \) samples corresponding to the cyclic prefix. By defining a new PN matrix \( \Psi \), equal to the matrix \( \Psi \cdot \Phi \), in which the first \( L \) rows and columns have been removed, we obtain \( R_{Q \times kp} \cdot \Psi = \Psi \cdot R_{Q \times kp} \).

The purpose of the transmit/receive pair is twofold. First, it allows to remove the IBI, that is, \( R_{Q \times kp} \cdot H[1] \cdot T_{kp} = 0_{Q \times kp} \), provided the CP length to be at least the maximum channel order \( L \). Second, it enables low-complexity frequency-domain processing by making the linear channel convolution appear circulant to the received block. This results in a simplified block input/output relationship in the time-domain:

\[
y[n] = \Psi \cdot \hat{H} \cdot \hat{x}[n] + \hat{z}[n],
\]

(7)

where \( \hat{H} := R_{Q \times kp} \cdot H[0] \cdot T_{kp} \) is a circulant channel matrix, and \( \bar{z}[n] := \bar{R}_{Q \times kp} \cdot \Phi \cdot w[n] \) is the corresponding noise block sequence. Note that circulant matrices can be diagonalized by FFT operations, that is, \( \bar{R}_{Q \times kp} \cdot \Phi \cdot w[n] \) is a diagonal matrix composed of the frequency-domain channel response.

At the receiver, the operations performed at the transmitter are successively inverted. An FFT of size \( Q \) is applied on the received block to transform the signal to the frequency-domain, where the channel is inverted (zero-forcing equalizer by \( \bar{\Delta}_H \)). Next, the pre-coding is compensated by multiplying the result by the matrix \( \bar{\theta}_H \). In case of OFDM, \( \bar{\theta}_H = L_Q \) and no operation is performed. In case of SC-CP, \( \bar{\theta}_H = E^H \) corresponding to an Inverse FFT (IFFT) is applied to transform the signal back to the time-domain. By combining the equations (1), (3), (7) and the diagonalization of \( \bar{H} \) by FFT operations, we get:

\[
\hat{y}[n] = \bar{\theta}_H \cdot \bar{\Delta}_H^{-1} \cdot E \cdot \Psi \cdot \bar{H} \cdot \bar{\Delta}_H \cdot \hat{\theta} \cdot \hat{s}[n] + \bar{z}[n].
\]

(8)

### III. Phase Noise Impact

It is well known that PN causes a CPE and ICI in the case of OFDM [2] [8]. In this section, we will generalize these results to both OFDM and SC-CP transmissions.

In order to highlight the CPE for both air interfaces, the PN diagonal matrix \( \Psi \) should be seen as the addition of the average of the PN elements \( \hat{\Psi} := \frac{1}{Q} \text{trace}(\Psi) \) plus a deviation diagonal matrix \( \tilde{\Delta} \):

\[
\Psi = \hat{\Psi} + \tilde{\Delta}.
\]

(9)

By substituting (9) in (8) we get:

\[
\hat{y}[n] = \bar{\theta}_H \cdot \bar{\Delta}_H^{-1} \cdot E \cdot \hat{\Psi} \cdot \bar{H} \cdot \bar{\Delta}_H \cdot \hat{\theta} \cdot \hat{s}[n] + \bar{z}[n].
\]

(10)

The first term, due to the average of the phase errors, is approximately equal to a common phase rotation independent from the air interface (OFDM or SC-CP). For small values of the phase errors, it is indeed approximately equal to:

\[
\bar{\Psi} := \frac{1}{Q} \sum_{n=0}^{Q-1} e^{j\phi[n]} \approx 1 + \frac{j}{Q} \sum_{n=0}^{Q-1} \phi[n] \approx e^{j\frac{1}{2} \sum_{n=0}^{Q-1} \phi[n]}.
\]

(11)
in which we have used the linear approximation \( e^{j\phi[n]} \approx 1 + j\phi[n] \) for small values of the phase error \( |\phi[n]| << 1 \).

The second term corresponds to the interference between the symbols of the block (ICI in the case of OFDM, Inter-Symbol Interference (ISI) in the case of SC-CP). It is useful to develop the inner matrix product as:

\[
F' \cdot \Delta \cdot P \cdot \Delta' \cdot P^H \cdot F
\]

in which \( P \) is a permutation matrix equal to \( P := F \cdot F^H \) and the matrix \( \Delta' := P \cdot \Delta \cdot P^H \) is a diagonal matrix composed of the diagonal elements of \( \Delta \) in the reverse order.

In the case of an AWGN channel \( (\Delta_H = I_{q^2}) \), the equality (10) reduces to:

\[
\hat{s}[n] = \tilde{\Psi} \cdot s[n] + F^H \cdot \Delta' \cdot F \cdot \hat{s}[n] + \tilde{z}[n]
\]

for OFDM, and to:

\[
\hat{s}[n] = \tilde{\Psi} \cdot s[n] + \Delta \cdot \tilde{z}[n] + \tilde{z}[n]
\]

for SC-CP. There is ICI in the case of OFDM because the matrix \( F^H \cdot \Delta' \cdot F \) is a circulant matrix with zeros on its diagonal (the average is already taken in the first term of (13)). On the contrary, there is no ISI in the case of SC-CP because the matrix \( \Delta \) is diagonal.

In the case of multipath channels, there are both ICI for OFDM and ISI for SC-CP. It is interesting to note that the average powers of the ICI and ISI are equal (this is easily shown by computing the trace of the auto-correlation of the second term for each air interface).

**IV. simulation results**

In this section, the impact of the PN on both OFDM and SC-CP is illustrated. Firstly we observe the received constellations after FDE to confirm our previous analytical results. Secondly we measure the Error Vector Magnitude (EVM) and finally evaluate the BER degradation. For both air interfaces (OFDM and SC-CP), we use a QPSK constellation. We assume a carrier frequency at 60 GHz. The channel is modeled using the Saleh-Valenzuela model [9]. We use a bandwidth of 1 GHz. The block size and the CP length are respectively 256 and 64 symbols. The integrated PN power is \(-16 \text{ dBc} \) and the cut-off frequency of the PN is 1 MHz [10].

**A. Analysis of the constellations after FDE**

**Fig. 2** shows the received constellations after the FDE for both flat and frequency selective channels. We observe first a common rotation of the received symbols within one block (CPE). The CPE can be corrected by rotating the constellation in the opposite direction. When the channels are flat, the PN results in circular clouds around the desired constellation points in the case of OFDM. In the case of SC-CP, the PN results in arcs around the constellation points (there is only a rotation of the transmitted symbols). When the channels are frequency selective, a dispersion of the received constellation is observed in the OFDM and SC-CP cases. It is important to note that PN looks more destructive for SC-CP than for OFDM because the points of the clouds are more distant from the initial constellation points. Consequently, SC-CP will be slightly more sensitive to PN as it will be confirmed by BER simulations.

**B. Impact of PN cut-off frequency on the EVM**

The EVM measures the distance between the transmitted and the received signals. It is mathematically defined as \( \text{EVM}(s[n]) := E\left[|\hat{s}[n] - s[n]|^2\right] \), where \( E \) denotes the expectation over the symbols and the noise. Fig. 3 shows the EVM before and after the CPE correction as a function of the PN cut-off frequency. The EVM is the same for OFDM and SC-CP since the CPE and the average powers of ICI and ISI are equal. If the cut-off frequency is smaller than the subcarrier spacing (1GHz/256 = 4 MHz), the predominant
effect is the CPE [2] that can be compensated. If the PN cut-off frequency is larger than the subcarrier spacing the ICI/ISI dominates the CPE. The impact of the PN is more pronounced in case of frequency selective channels than in the case of frequency flat channels. Note that we have assumed an ideal correction of the CPE while the practical methods are pilot-based and their performance increases with the number of subcarriers.

C. Impact of PN on BER performance

The PN cut-off frequency is fixed to 1 MHz so the system suffers from both CPE and ICI/ISI. Results are shown in Fig. 4. In the case of a flat channel, the performance of OFDM is slightly better than the one of SC-CP as foreseen. The CPE correction brings a slight BER improvement. In the case of a frequency selective channel, the same conclusion can be drawn (compare the points where OFDM and SC-CP perform equally well). The CPE correction brings a negligible BER improvement in the case of SC-CP.

V. CONCLUSION

We have studied the impact of the PN on OFDM and SC-CP. Simulations have been carried out to validate the analytical results. The PN has two effects in both schemes: CPE and ICI/ISI. Considering a 60 GHz communication system, it has been shown that OFDM outperforms slightly SC-CP even when the CPE has been corrected. This is explained by the shape of the received constellation points after equalization: they are spread as circles in the case of OFDM and as arcs in the case of SC-CP, so that they can be more distant from the ideal constellation points in the case of SC-CP.

REFERENCES