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Attractors for reaction–diffusion equations in $\mathbb{R}^N$ with continuous nonlinearity

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Abstract. In this paper we prove the existence of a compact global attractor for a reaction–diffusion equation on $\mathbb{R}^N$. We do not assume that the nonlinear term is differentiable (just continuous) and, also, we do not guarantee the uniqueness of solutions of the Cauchy problem. Besides, the growth and dissipative conditions are different from the ones used in previous papers on the topic. An application is given to the Fitz–Hugh–Nagumo system, which models the transmission of signals across axons.

Keywords: reaction–diffusion equations, set-valued dynamical system, global attractor, unbounded domain

1. Introduction

A great number of processes coming from Physics, Chemistry, Biology, Economics and other sciences can be described by reaction–diffusion equations. One of the most interesting problems concerning partial differential equations is to understand the asymptotic behaviour of the solutions of the equation when time grows to infinite. The study of the asymptotic behaviour of the system is giving us relevant information about "the future" of the phenomenon described in the model. In this context, the concept of global attractor has become very important in the literature.

In this paper we study the asymptotic behaviour of the solutions of the following reaction–diffusion system:

$$u_t = a\Delta u - f(x,u), \quad x \in \mathbb{R}^N, t > 0, \tag{1}$$

$$u(0) = u_0 \in \left[L^2(\mathbb{R}^N)\right]^d, \tag{2}$$

where u is an unknown vector function, that is, $u(x,t) = (u^1,\dots,u^d), x \in \mathbb{R}^N, t > 0$, $f(x,u) = (f^1,\dots,f^d)$, and $u_t = \partial u/\partial t$. We assume the following conditions:

($H1$) The real $d \times d$ matrix a has a positive symmetric part $\frac{1}{2}(a + a^*) \geqslant AI$, where $A > 0$.

($H2$) $f = f_0 + f_1$, $f_0(x,u) = (f_0^1,\dots,f_0^d)$, $f_1(x,u) = (f_1^1,\dots,f_1^d)$, where f_i^j are Caratheodory functions, that is, they are continuous on u and measurable on x.

($H3$) There exist positive functions $C_0(x), C_1(x) \in L^1(\mathbb{R}^N)$ and constants $\alpha\beta > 0$, $p_i \geqslant 2$ verifying

$$\left(f_0(x,u),u\right) \geqslant \alpha|u|^2 - C_0(x), \tag{3}$$

$$\big(f_1(x,u),u\big) \geqslant \beta \sum_{i=1}^{d} |u^i|^{p_i} - C_1(x). \tag{4}$$

$(H4)$ There exist positive functions $C_2(x) \in L^2(\mathbb{R}^N), C_3(x) \in L^1(\mathbb{R}^N)$, and constants $\gamma, \eta > 0$ verifying

$$\big|f_0(x,u)\big| \leqslant C_2(x) + \eta|u|, \tag{5}$$

$$\sum_{i=1}^{d} \big|f_1^i(x,u)\big|^{p_i/(p_i-1)} \leqslant C_3(x) + \gamma \sum_{i=1}^{d} |u^i|^{p_i}. \tag{6}$$

As an application, we consider the Fitz–Hugh–Nagumo system, which is a well known model of transmission of signals across axons (see [44,41]).

In the sequel, we shall use the notation $H = [L^2(\mathbb{R}^N)]^d$, $V = [H^1(\mathbb{R}^N)]^d$ and $V' = [H^{-1}(\mathbb{R}^N)]^d$, together with the respective norms $\|\cdot\|$, $\|\cdot\|_V$ and $\|\cdot\|_{V'}$. By $\|\cdot\|_r$, $|\cdot|$, $(\cdot,\cdot)_H$, $(\cdot,\cdot)$ we denote the usual norm in $L^r(\mathbb{R}^N)$, the norm in $\mathbb{R}^d$ (or $\mathbb{R}^N$), the scalar product in H and the usual scalar product in $\mathbb{R}^d$ (or $\mathbb{R}^N$), respectively, so that $(u,v)_H = \sum_{i=1}^{d} \int_{\mathbb{R}^N} u_i v_i \,\mathrm{d}x = \int_{\mathbb{R}^N} (u,v)\,\mathrm{d}x$. For simplicity, for any $u, v \in V$ we shall use also the following notation:

$$|\nabla u|^2 = \sum_{i=1}^{d} |\nabla u^i|^2 = \sum_{i=1}^{d} \sum_{j=1}^{N} \left|\frac{\partial u^i}{\partial x_j}\right|^2,$$

$$(\nabla u, \nabla v) = \sum_{i=1}^{d} \big(\nabla u^i, \nabla v^i\big) = \sum_{i=1}^{d} \sum_{j=1}^{N} \frac{\partial u^i}{\partial x_j} \frac{\partial v^i}{\partial x_j}, \qquad (\nabla u, \nabla v)_H = \int_{\mathbb{R}^N} (\nabla u, \nabla v)\,\mathrm{d}x.$$

For $p = (p_1, \ldots, p_d)$ we define the space

$$L^p\big(\mathbb{R}^N\big) = L^{p_1}\big(\mathbb{R}^N\big) \times \cdots \times L^{p_d}\big(\mathbb{R}^N\big).$$

The aim of this paper is to prove the existence of a compact global attractor of (1)–(2) in the phase space H.

The existence of attractors for reaction–diffusion equations on unbounded domains has been studied by many authors before (see [1,5,17,19,20,33,37,39,46,48]). Different phase spaces have been considered in the previous papers: weighted spaces [5], L^q spaces [1,19,39,46], Sobolev spaces [1,37,48] and spaces of bounded functions [33].

Our paper continues the line of investigation started in [46] (see also [39]).

One of the main differences is the fact that we do not assume f to be differentiable on the variable u (just continuous). Also, we do not assume conditions providing the uniqueness of solutions. Nevertheless, by defining a multivalued semiflow instead of a semigroup of operators the asymptotic behaviour of solutions can be studied in the same way (see [31]). It is worth to remark that one of the difficulties which appears as a consequence of this lack of uniqueness is that, unlike the case with uniqueness, there are some restrictions in the kind of estimates we can obtain. This occurs because we do not have a regular approximation for each solution of the equation and, as a result, we cannot obtain formally the necessary estimates and justify them after that with suitable approximations. We are only able to get the estimates that the regularity properties of the solutions allow. In particular, this problem appears when we need to prove the asymptotic compactness of the solutions. For example, in order to obtain this property in the space L^2, an estimate of the solutions in the space H^1, which is compactly imbedded in L^2 for bounded domains, is usually proved. In our case we do not know how to obtain such an estimate. To avoid this problem the method of the energy equation, which has been successfully used in many papers for equations with uniqueness of solutions (see, for example, [21,28,34,35,42,46]) and also for a wave equation without uniqueness [7], is an appropriate tool. Another useful approach, which is slightly different but in a similar line, has been used in [22,23] and [24].

Another difference appears in the growth conditions. In particular, in [46] (and also in other papers as [1,37,39,48]) the constant p has an additional restriction which depends on N. We do not assume any restrictions and, instead, we suppose the dissipative condition (4). In such a way we extend to the case of unbounded domains a type of conditions which have been used frequently for the case of bounded domains (see [4,15,23,29]). We note that, in [15] and [23], the existence of the global attractor in the case of nonuniqueness and bounded domains is proved.

Finally, in order to conclude the review and the comparison of the previous results we note that in [1], unlike the other papers, the dissipative condition (3) is weaker, because α is considered as a function of the space variable x. It is an open problem whether or not this kind of condition could be applied to the case of nonuniqueness.

The asymptotic behaviour of equations without uniqueness of the Cauchy problem has been studied by several authors in the last years. In our opinion there are two important reasons which justify the interest of the researches in such type of equations. On the one hand, they contain important models coming from Mathematical Physics, as we can see in the example of the relevant three-dimensional Navier–Stokes equations (see [6]). On the other hand, they allow to weaken the conditions imposed in the nonlinear functions involved in the equations, which are in many cases very restrictive. In this way we can extend the class of equations for which the asymptotic behaviour of solutions can be studied. Several results concerning the existence of global attractors in the case of nonuniqueness have been proved for differential inclusions (see [25,31,32]), reaction–diffusion equations (see [15,17,23]), parabolic problems (see [13,45]), phase-field equations (see [22,24]), wave equations (see [7,16]), the three-dimensional Navier–Stokes equations (see [6,9,10,16,36,40]), delay ordinary differential equations [12] or degenerate parabolic equations [18]. As far as we know the theory of attractors for equations without uniqueness was studied at first in [3].

We note that several approaches have been used in order to develop a general theory of attractors for equations without uniqueness. In this paper we use the method of multivalued semiflows (see [2, 14,30,31]). Another approach, which is rather similar, is the method of generalized semigroups (see [6, 18]). A comparison of these two theories can be found in [11]. We note also that the theory of trajectory attractors have been also fruitfully applied to treat equations without uniqueness (see [15,16,36,40]). The main difference with the previous approaches is the fact that a new phase space is defined. In this space the whole trajectory of any solution is a point, and the global attractor is obtained for the translation semigroup.

The main result of this paper is the following:

Theorem 1. *Let* $(H1)$–$(H4)$ *hold. Then the system* (1)–(2) *defines a multivalued semiflow in the phase-space* H*, which possesses a compact invariant global attractor* $\mathcal{A}$*. Also,* $\mathcal{A}$ *is the minimal closed attracting set.*

This paper is organized as follows. In the second section we obtain some a priori estimates and prove the existence of solutions of the Cauchy problem. In the third section we define the multivalued semiflow and prove that its graph is weakly closed. In the fourth section we check first that the semiflow is asymptotically compact (using the method of the energy equation) and then that it is upper semicontinuous and has compact values. Finally we obtain the existence of the global attractor and apply this result to the Fitz–Hugh–Nagumo model of transmission of signals across axons. In the last section we give another proof of the asymptotic compactness property, but now using the second method, which we have called the monotonicity method.

2. Existence of solutions and a priori estimates

Let $L^p(0,T;L^p(\mathbb{R}^N)) = L^{p_1}(0,T;L^{p_1}(\mathbb{R}^N)) \times \cdots \times L^{p_d}(0,T;L^{p_d}(\mathbb{R}^N))$, $p = (p_1,\ldots,p_d)$, and $q = (q_1,\ldots,q_d)$, where $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Conditions (5)–(6) imply that for any $u \in L^p(0,T;L^p(\mathbb{R}^N)) \cap L^2(0,T;H)$ we have

$$\int_0^T\int_{\mathbb{R}^N} \bigl|f_0(x,u(t,x))\bigr|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant K_0\bigl(T + \|u\|^2_{L^2(0,T;H)}\bigr), \tag{7}$$

$$\int_0^T\int_{\mathbb{R}^N} \sum_{i=1}^d \bigl|f_1^i(x,u(t,x))\bigr|^{q_i}\,\mathrm{d}x\,\mathrm{d}t \leqslant K_1\left(T + \sum_{i=1}^d \|u^i\|^{p_i}_{L^{p_i}(0,T;L^{p_i}(\mathbb{R}^N))}\right). \tag{8}$$

First we shall give the definition of a weak solution.

Definition 2. The function $u(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}^N$, is said to be a weak solution of (1) on $[0,T]$ if $u \in L^2(0,T;V) \cap L^p(0,T;L^p(\mathbb{R}^N)) \cap L^\infty(0,T;H)$ and u satisfies Eq. (1) in the distribution sense, that is,

$$-\int_0^T (u,v_t)_H\,\mathrm{d}t - \int_0^T (au,\Delta v)_H\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^N} \bigl(f(x,u),v\bigr)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{9}$$

for all $v \in [C_0^\infty([0,T]\times\mathbb{R}^N)]^d$.

It follows from this definition and (7)–(8) that the time derivative u_t of any weak solution u belongs to the space $L^2(0,T;V') + L^2(0,T;H) + L^q(0,T;L^q(\mathbb{R}^N)) \subset L^q(0,T;Y) = L^{q_1}(0,T;V' + L^{q_1}(\mathbb{R}^N)) \times \cdots \times L^{q_d}(0,T;V' + L^{q_d}(\mathbb{R}^N))$, where $Y = V' + L^q(\mathbb{R}^N)$. Since $u \in L^2(0,T;V) \subset L^q(0,T;Y)$, u belongs to $C([0,T],Y)$, and then the inclusion $u \in L^\infty(0,T;H)$ implies that $t \mapsto u(t,\cdot)$ is weakly continuous with values in the space H (see [43, Lemma 1.4, p. 263] or [27]).

It is an immediate consequence that for any $v \in L^2(0,T;V) \cap L^p(0,T;L^p(\mathbb{R}^N))$ and any weak solution u one has

$$\int_0^T \langle u_t,v\rangle_Y\,\mathrm{d}t + \int_0^T (a\nabla u,\nabla v)_H\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^N} \bigl(f(x,u),v\bigr)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{10}$$

where $\langle\cdot,\cdot\rangle_Y$ denotes pairing in the space Y. Since (10) implies (9), this is an equivalent definition of weak solution.

In fact, as in the case of bounded domains (see [15]), we shall prove that $u(t,\cdot)$ is absolutely continuous on $[0,T]$ with respect to the strong topology of the space H.

Lemma 3. *Let $\rho(x):\mathbb{R}^N \to \mathbb{R}$ be a smooth function such that $\rho \in W^{1,\infty}(\mathbb{R}^N)$. If a function u belongs to $L^2(0,T;V)\cap L^p(0,T;L^p(\mathbb{R}^N))$, and its derivative $\partial u/\partial t$ belongs to $L^2(0,T;V') + L^q(0,T;L^q(\mathbb{R}^N))$, then u is almost everywhere equal to a continuous function from $[0,T]$ into H, $\|u(t)\|^2$ is absolutely continuous on $[0,T]$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = 2\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, u\right\rangle_Y, \quad \textit{for a.a. } t\in(0,T), \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho u\|^2 = 2\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \rho^2 u\right\rangle_Y, \quad \textit{for a.a. } t\in(0,T). \tag{12}$$

Sketch of the proof. Let

$$\hat{u}(t) = \begin{cases} u(t), & \text{on } [0,T],\\ 0, & \text{on } \mathbb{R}\setminus[0,T]. \end{cases}$$

By regularizing the function $\hat{u}:\mathbb{R}\to V\cap L^p(\mathbb{R}^N)$ we obtain a sequence of functions $u_m \subset C^1([0,T], V\cap L^p(\mathbb{R}^N))$ such that

$$u_m \longrightarrow u \quad \text{in } L^2_{\mathrm{loc}}(0,T;V)\cap L^p_{\mathrm{loc}}\big(0,T;L^p\big(\mathbb{R}^N\big)\big), \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}u_m \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}u \quad \text{in } L^2_{\mathrm{loc}}\big(0,T;V'\big) + L^q_{\mathrm{loc}}\big(0,T;L^q\big(\mathbb{R}^N\big)\big), \tag{14}$$

as $m\to\infty$. Since $\rho\in W^{1,\infty}$, we have that

$$\rho^2 u_m \longrightarrow \rho^2 u \quad \text{in } L^2_{\mathrm{loc}}(0,T;V)\cap L^p_{\mathrm{loc}}\big(0,T;L^p\big(\mathbb{R}^N\big)\big). \tag{15}$$

Also, it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|u_m(t)\big\|^2 = 2\left\langle \frac{\mathrm{d}}{\mathrm{d}t}u_m(t), u_m(t)\right\rangle_Y = 2\left(\frac{\mathrm{d}}{\mathrm{d}t}u_m(t), u_m(t)\right)_H, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\rho u_m(t)\big\|^2 = 2\left\langle \frac{\mathrm{d}}{\mathrm{d}t}u_m(t), \rho^2 u_m(t)\right\rangle_Y = 2\left(\frac{\mathrm{d}}{\mathrm{d}t}u_m(t), \rho^2 u_m(t)\right)_H. \tag{17}$$

Passing to the limit in (16)–(17) in the distribution sense we have the equalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = 2\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, u\right\rangle_Y, \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\alpha u\|^2 = 2\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \rho^2 u\right\rangle_Y. \tag{19}$$

Since $\langle \frac{\mathrm{d}u}{\mathrm{d}t}, u\rangle_Y$ and $\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \rho^2 u\rangle_Y$ belong to $L^1([0,T])$, (11)–(12) hold. The rest of the proof repeats the same steps of [43, p. 262, Lemma 1.3], and we shall omit it. □

We shall obtain now some a priori estimates.

Lemma 4. *For any weak solution* u *of problem* (1)–(2) *we have*:

$$\|u\|_X \leqslant K_1(\|u_0\|, T), \tag{20}$$

$$\|u_t\|_U \leqslant K_2(\|u_0\|, T), \tag{21}$$

where K_i *are increasing functions of* $\|u_0\|$ *and* T, $X = L^2(0,T;V) \cap L^p(0,T;L^p(\mathbb{R}^N)) \cap C([0,T],H)$, *and* $U = L^2(0,T;V') + L^q(0,T;L^q(\mathbb{R}^N))$.

Proof. Using Lemma 3 we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -(a\nabla u, \nabla u)_H - \int_{\mathbb{R}^N} (f(x,u), u)\,\mathrm{d}x. \tag{22}$$

Now, assumptions $(H1)$ and $(H3)$ give

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + A\|\nabla u\|^2 \leqslant -\alpha\|u\|^2 - \beta\sum_{i=1}^{d}\int_{\mathbb{R}^N} |u^i|^{p_i}\,\mathrm{d}x + \int_{\mathbb{R}^N} C_0(x)\,\mathrm{d}x + \int_{\mathbb{R}^N} C_1(x)\,\mathrm{d}x. \tag{23}$$

Hence, using that $C_i(x) \in L^1(\mathbb{R}^N)$ and integrating we obtain

$$\|u(t)\|^2 + 2A\int_0^t \|\nabla u\|^2\,\mathrm{d}s + 2\alpha\int_0^t \|u\|^2\,\mathrm{d}s + 2\beta\sum_{i=1}^{d}\int_0^t \|u^i\|_{p_i}^{p_i}\,\mathrm{d}s \leqslant \|u_0\|^2 + 2MT,$$
$$\text{for all } t \in [0,T], \tag{24}$$

which implies (20).

It follows from (7), (8) and (20) that $f_0(x, u(t,x))$ is bounded in $L^2(0,T;H)$, and then in $L^2(0,T;V')$, and also that $f_1(x,u(t,x))$ is bounded in $L^q(0,T;L^q(\mathbb{R}^N))$. It is clear that the constants depend increasingly on $\|u_0\|$ and T. On the other hand, since $-\Delta : V \to V'$ is a linear and bounded operator, it follows from (20) that

$$\|-\Delta u\|_{L^2(0,T;V')} \leqslant K\|u\|_{L^2(0,T;V)} \leqslant KK_1.$$

Finally, the equality $u_t = a\Delta u - f_0(x,u) - f_1(x,u)$ implies (21). □

In order to prove the existence of weak solutions we consider the Dirichlet problem in a bounded domain

$$u_t = a\Delta u - f(x,u) + h(x), \quad x \in \Omega_R, t > 0, \tag{25}$$

$$u|_{\partial\Omega_R} = 0, \quad t > 0, \tag{26}$$

$$u(x,0) = u_{0,R}(x), \quad x \in \Omega_R, \tag{27}$$

where $\Omega_R = B(0,R)$ is the open ball of radius $R \geqslant 1$ centered at 0, $u_{0,R}(x) = u_0(x)\psi_R(|x|)$, and ψ_R is a smooth function verifying

$$\psi_R(\xi) = \begin{cases} 1, & \text{if } 0 \leqslant \xi \leqslant R-1, \\ 0 \leqslant \psi_R(\xi) \leqslant 1, & \text{if } R-1 \leqslant \xi \leqslant R, \\ 0, & \text{if } \xi > R. \end{cases}$$

It is well known [15, Theorem 2.1] that (25)–(27) has at least one weak solution for any $u_{0,R}(x) \in [L^2(\Omega_R)]^d$ (the definition of weak solution is the same as in Definition 2 but replacing $\mathbb{R}^N$ by Ω_R). In the sequel, we shall use the notation $H_{r_j} = [L^2(\Omega_{r_j})]^d$, $V_{r_j} = [H^1(\Omega_{r_j})]^d$ and $V'_{r_j} = [H^{-1}(\Omega_{r_j})]^d$.

Theorem 5. *Let* $(H1)$–$(H4)$ *hold. Then, problem* (1)–(2) *has a weak solution for any* $u_0 \in H$ *and* $T > 0$.

Proof. Let u_{r_j}, $r_j \to \infty$, be a sequence of solutions of (25)–(27). We note that

$$\|u_0 - u_{0,r_j}\|^2 = \int_{\mathbb{R}^N} \left(1 - \psi_{r_j}(|x|)\right)^2 |u_0|^2 \,\mathrm{d}x \leqslant \int_{|x|>r_j-1} |u_0|^2 \,\mathrm{d}x \to 0, \quad \text{if } r_j \to \infty. \tag{28}$$

Repeating the same proof of Lemma 4 we have

$$\begin{aligned} &\|u_{r_j}(t)\|^2_{H_{r_j}} + 2A\int_0^t \|\nabla u_{r_j}\|^2_{H_{r_j}} \,\mathrm{d}s + 2\alpha \int_0^t \|u_{r_j}\|^2_{H_{r_j}} \,\mathrm{d}s + 2\beta \sum_{i=1}^d \int_0^t \|u^i_{r_j}\|^{p_i}_{L^{p_i}(\Omega_{r_j})} \,\mathrm{d}s \\ &\quad \leqslant \|u_{0,r_j}\|^2 + 2T\left(\int_{\mathbb{R}^N} C_0(x)\,\mathrm{d}x + \int_{\mathbb{R}^N} C_1(x)\,\mathrm{d}x\right), \quad \text{for all } t \in [0,T]. \end{aligned}$$

Hence, (28) implies

$$\|u_{r_j}\|_{X_{r_j}} \leqslant K_1\left(\|u_{0,r_j}\|, T\right) \leqslant \widetilde{K_1}\left(\|u_0\|, T\right), \tag{29}$$

where $X_{r_j} = L^2(0,T;V_{r_j}) \cap L^p(0,T;L^p(\Omega_{r_j})) \cap C(0,T;H_{r_j})$.

Following [5, Theorem 1.3] we extend these solutions to be defined on $\mathbb{R}^N$ in the following way:

$$\hat{u}_{r_j}(x) = \begin{cases} u_{r_j}(x)\psi_{r_j}(|x|), & \text{in } B(0,r_j), \\ 0, & \text{otherwise.} \end{cases}$$

Since u_{r_j} are bounded in X_{r_j} uniformly with respect to r_j, $\hat{u}_{r_j}$ is a bounded sequence in X. Hence, there exists a subsequence of $\hat{u}_{r_j}$ (denoted again by u_{r_j}) such that

$$\begin{aligned} &u_{r_j} \to u_\infty \quad \text{weakly in } L^2(0,T;V), \\ &u_{r_j} \to u_\infty \quad \text{weakly star in } L^\infty(0,T;H), \\ &u_{r_j} \to u_\infty \quad \text{weakly in } L^p\left(0,T;L^p\left(\mathbb{R}^N\right)\right). \end{aligned} \tag{30}$$

Next, we shall prove that u_∞ is a weak solution of (1)–(2). Let r_k fixed. From the convergence $r_j \to \infty$ we can assume that $r_k \leqslant r_j - 1$. We define the projections in $B(0,r_k)$ of u_{r_j} and denote them by

$$u_{kj} = L_k u_{r_j}.$$

It is clear from (29) that u_{kj} is bounded in X_{r_k}. It follows that there exists a subsequence (denoted again by u_{r_j}) such that $u_{kj} = L_k u_{r_j} \to u_{k\infty}$ weakly in $L^2(0,T;V_{r_k})$ and $L^p(0,T;L^p(\Omega_{r_k}))$ and weakly star in $L^\infty(0,T;H_{r_k})$.

Now, we shall check the equality $L_k u_\infty = u_{k\infty}$. Let $v \in [C_0^\infty([0,T]\times\Omega_{r_k})]^d$. The weak convergence in $L^2(0,T;V_{r_k})$ gives

$$\int_0^T\int_{\Omega_{r_k}} (L_k u_{r_j}, v)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_{\Omega_{r_k}} (u_{k\infty}, v)\,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, using $v(t,x) = 0$, if $x \notin \Omega_{r_k}$, and (30) we obtain

$$\int_0^T\int_{\Omega_{r_k}} (L_k u_{r_j}, v)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^N} (u_{r_j}, v)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_{\mathbb{R}^N} (u_\infty, v)\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_0^T\int_{\mathbb{R}^N} (u_\infty, v)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\Omega_{r_k}} (L_k u_\infty, v)\,\mathrm{d}x\,\mathrm{d}t,$$

so that $u_{k\infty} = L_k u_\infty$.

Finally, we have to prove that $L_k u_\infty$ is a weak solution in $[0,T]\times\Omega_{r_k}$, and then, since r_k is arbitrary and for any $v \in [C_0^\infty([0,T]\times\mathbb{R}^N)]^d$ we can find r_k such that $v \in [C_0^\infty([0,T]\times\Omega_{r_k})]^d$, we can conclude what u_∞ is a weak solution of (1)–(2).

Let $v \in [C_0^\infty([0,T]\times\Omega_{r_k})]^d$. Since $B(0,r_k) \subset B(0,r_j)$, it follows that v belongs to $[C_0^\infty([0,T]\times\Omega_{r_j})]^d$, and using that u_{r_j} is a weak solution in Ω_{r_j} we have

$$\begin{aligned}
&\int_0^T\int_{\Omega_{r_k}} \bigl(-(L_k u_{r_j}, v_t) - (aL_k u_{r_j}, \Delta v) + \bigl(f(x, L_k u_{r_j}), v\bigr)\bigr)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \int_0^T\int_{\Omega_{r_j}} \bigl(-(u_{r_j}, v_t) - (au_{r_j}, \Delta v) + \bigl(f(x, u_{r_{j_k}}), v\bigr)\bigr)\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned} \tag{31}$$

Let us prove that $f_0(x, L_k u_{r_j}) \to f_0(x, L_k u_\infty)$ weakly in $L^2(0,T;H_{r_j})$ and also that $f_1(x, L_k u_{r_j}) \to f_1(x, L_k u_\infty)$ weakly in $L^q(0,T;L^q(\Omega_{r_k}))$. We already know that $L_k u_{r_j} \to u_{k\infty}$ weakly in the spaces $L^2(0,T;V_{r_k})$ and $L^p(0,T;L^p(\Omega_{r_k}))$. Arguing as in Lemma 4 we obtain (up to a subsequence) that

$$f_0(x, L_k u_{r_j}) \to \chi_0 \quad \text{weakly in } L^2(0,T;H_{r_k}), \tag{32}$$

$$f_1(x, L_k u_{r_j}) \to \chi_1 \quad \text{weakly in } L^q\bigl(0,T;L^q(\Omega_{r_k})\bigr), \tag{33}$$

$$L_k \frac{\partial u_{r_j}}{\partial t} = \frac{\partial L_k u_{r_j}}{\partial t} \to \frac{\partial u_{k\infty}}{\partial t} \quad \text{weakly in } L^2\bigl(0,T;V'_{r_k}\bigr) + L^q\bigl(0,T;L^q(\Omega_{r_k})\bigr). \tag{34}$$

We have to prove that $\chi_i = f_i(x, u_{k\infty})$. Put $s_i = \max\{1, N(\frac{1}{q_i} - \frac{1}{2})\}$. Then the Sobolev imbedding theorems imply that $L^{q_i}(\Omega_{r_k}) \subset H^{-s_i}(\Omega_{r_k})$ with continuous injection. Hence, $\frac{\partial L_k u_{r_j}}{\partial t} \to \frac{\partial u_{k\infty}}{\partial t}$

weakly in $L^q(0,T;H^{-s}(\Omega_{r_k}))$, where $s=(s_1,\dots,s_d)$, $H^{-s}(\Omega_{r_k})=H^{-s_1}(\Omega_{r_k})\times\cdots\times H^{-s_d}(\Omega_{r_k})$, $L^q(0,T;H^{-s}(\Omega_{r_k}))=L^{q_1}(0,T;H^{-s_1}(\Omega_{r_k}))\times\cdots\times L^{q_d}(0,T;H^{-s_d}(\Omega_{r_k}))$. Since the injection $H_0^1(\Omega_{r_k})\subset L^2(\Omega_{r_k})$ is compact and the injection $L^2(\Omega_{r_k})\subset H^{-s_i}(\Omega_{r_k})$ continuous, we can apply the compacity theorem (see [38] or [26]) to get the existence of a subsequence strongly convergent in $L^2(0,T;H_{r_k})$ to $u_{k\infty}$, that is,

$$L_k u_{r_j}\to u_{k\infty}\quad\text{strongly in } L^2(0,T;H_{r_k}),$$

and then

$$L_k u_{r_j}\to u_{k\infty}\quad\text{a.e. in } [0,T]\times\Omega_{r_k}.$$

Since f_i are continuous functions, we have

$$f_i(x,L_k u_{r_j})\to f_i(x,u_{k\infty})\quad\text{a.e. in } [0,T]\times\Omega_{r_k}.$$

Lemma 8.3 in [38] (see also [26]) implies $f_0(x,L_k u_{r_j})\to f_0(x,u_{k\infty})$, $f_1(x,L_k u_{r_j})\to f_1(x,u_{k\infty})$ weakly in $L^2(0,T;H_{r_k})$ and $L^q(0,T;L^q(\Omega_{r_k}))$, respectively.

Finally, passing to the limit in (31) we obtain that

$$\int_0^T\int_{\Omega_{r_k}}\bigl(-(L_k u_\infty,v_t)-(aL_k u_\infty,\Delta v)+\bigl(f(x,L_k u_\infty),v\bigr)\bigr)\,\mathrm{d}x\,\mathrm{d}t=0,$$

so that u_∞ is a weak solution. □

We note that, although the theorem of existence of solutions is proved on a finite interval $[0,T]$, since the concatenation of solutions is a solution (see the proof of Lemma 7 below), each solution can be extended to a global one defined for $t\in[0,+\infty)$. Let us denote the set of all global solutions corresponding to the initial condition u_0 by $\mathcal{D}(u_0)$. It is clear that any $u\in\mathcal{D}(u_0)$ belongs to the space $L^2_{\mathrm{loc}}(0,+\infty;V)\cap L^p_{\mathrm{loc}}(0,+\infty;L^p(\mathbb{R}^N))\cap C([0,+\infty),H)$. To conclude this section we obtain an exponential bound of the solutions, proving that in fact $u\in L^\infty(0,+\infty;H)$ for all $u\in\mathcal{D}(u_0)$.

Lemma 6. *If u is a weak solution of the problem* (1)–(2)*, then*

$$\|u(t)\|^2+2A\int_0^t \mathrm{e}^{-\alpha(t-s)}\|\nabla u\|^2\,\mathrm{d}s\leqslant\|u(0)\|^2\,\mathrm{e}^{-2\alpha t}+D,\quad \textit{for all } t\geqslant 0,\tag{35}$$

where $D=(\|C_0\|_{L^1(\mathbb{R}^N)}+\|C_1\|_{L^1(\mathbb{R}^N)})/\alpha$.

Proof. The proof follows directly from (23) and Gronwall lemma. □

3. Definition of the multivalued semiflow

In this section we shall define a multivalued semiflow G associated with the solutions of (1)–(2) and prove that its graph is weakly closed.

Let $u_0 \in H$ and denote by $P(H)$ the set of all nonempty subsets of H. We define the (in general multivalued) map $G : \mathbb{R}^+ \times H \to P(H)$ by

$$G(t, u_0) = \{ z \in H \colon \exists u \in \mathcal{D}(u_0) \text{ such that } u(0) = u_0 \text{ and } u(t) = z \}.$$

Lemma 7. *$G(t, G(s, x)) = G(t+s, x)$, for all $x \in H$, $s, t \in \mathbb{R}^+$, i.e., G is a strict multivalued semiflow.*

Proof. First, we shall prove that $G(t+s, x) \subset G(t, G(s, x))$. Let $y \in G(t+s, x)$. Then there exists $u(\cdot) \in \mathcal{D}(x)$ verifying $u(0) = x$ and $u(t+s) = y$. It is clear that $u(s) \in G(s, x)$ and the result follows if we prove $y \in G(t, u(s))$. Let $\bar{u}(\cdot) = u(\cdot + s)$. It is straightforward to prove by a change of variable that $\bar{u}$ is a weak solution and $\bar{u}(t) = u(t+s) = y$, $\bar{u}(0) = u(s)$. Then $y \in G(t, u(s)) \subset G(t, G(s, x))$.

Now we shall prove that $G(t, G(s, x)) \subset G(t+s, x)$. Let $y \in G(t, G(s, x))$, then there exist z_1, $u_1(\cdot) \in \mathcal{D}(x)$, and $u_2(\cdot) \in \mathcal{D}(z_1)$, verifying

$$\begin{aligned} &u_1(0) = x, \qquad u_1(s) = z_1, \\ &u_2(0) = z_1, \qquad u_2(t) = y. \end{aligned}$$

Further, we shall check that there is $u(\cdot) \in \mathcal{D}(u_0)$ verifying $u(0) = x$, $u(t+s) = y$. Define u as

$$u(r) = \begin{cases} u_1(r), & \text{if } 0 \leqslant r \leqslant s, \\ u_2(r-s), & \text{if } s \leqslant r. \end{cases}$$

If we prove that u is a weak solution, then it is evident that $y \in G(t+s, x)$. For any $v \in [C_0^\infty([0,T] \times \mathbb{R}^N)]^d$, using the change of variable $\tau = r - s$, and the definition of u_1 and u_2, we have

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial}{\partial r} u, v \right\rangle_Y \mathrm{d}r + \int_0^T \left[(a\nabla u, \nabla v)_H + \int_{\mathbb{R}^N} (f(x,u), v)\, \mathrm{d}x \right] \mathrm{d}r \\ &\quad = \int_0^s \left\langle \frac{\partial}{\partial r} u_1, v \right\rangle_Y \mathrm{d}r + \int_0^s \left[(a\nabla u_1, \nabla v)_H + \int_{\mathbb{R}^N} (f(x,u), v)\, \mathrm{d}x \right] \mathrm{d}r \\ &\qquad + \int_s^T \left\langle \frac{\partial}{\partial r} u_2(r-s), v \right\rangle_Y \mathrm{d}r + \int_s^T \left[\big(a\nabla u_2(r-s), \nabla v\big)_H + \int_{\mathbb{R}^N} \big(f\big(x, u_2(r-s)\big), v\big)\, \mathrm{d}x \right] \mathrm{d}r \\ &\quad = \int_0^s \left\langle \frac{\partial}{\partial r} u_1, v \right\rangle_Y \mathrm{d}r + \int_0^s \left[(a\nabla u_1, \nabla v)_H + \int_{\mathbb{R}^N} (f(x,u_1), v)\, \mathrm{d}x \right] \mathrm{d}r \\ &\qquad + \int_0^{T-s} \left\langle \frac{\partial}{\partial r} u_2, v \right\rangle_Y \mathrm{d}r + \int_0^{T-s} \left[(a\nabla u_2, \nabla v)_H + \int_{\mathbb{R}^N} (f(x,u_2), v) \right] \mathrm{d}r. \end{aligned}$$

Since u_1, u_2 are weak solutions, (10) implies that the two last integrals are equal to zero. Hence, u is a weak solution. □

We are now in a position to establish that the graph of the multivalued map $G(t,\cdot)$ is weakly closed for each $t \geqslant 0$. This result will be necessary for the proof of the asymptotic compactness in the next section.

Lemma 8. *The graph of $G(t,\cdot)$ is weakly closed, i.e., if $\xi_n \to \xi_\infty$, $\beta_n \to \beta_\infty$ weakly in H, where $\xi_n \in G(t,\beta_n)$, then $\xi_\infty \in G(t,\beta_\infty)$.*

Proof. We have to prove the existence of a weak solution $u(\cdot)$ verifying

$$u(0) = \beta_\infty, \qquad u(t) = \xi_\infty.$$

From $\xi_n \in G(t,\beta_n)$ we have that there exists a sequence u_n of weak solutions verifying $u_n(0) = \beta_n$ and $u_n(t) = \xi_n$. Let $T \geqslant t$. Since $u_n(0)$ is bounded, Lemma 4 implies that

$$u_n \text{ is bounded in } L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^p(0,T;L^p),$$

$$\frac{\partial u_n}{\partial t} \text{ is bounded in } L^2(0,T;V') + L^q(0,T;L^q),$$

so that there exists a subsequence of u_n verifying

$$\begin{aligned}
&u_n \to u_\infty \quad \text{weakly in } L^2(0,T;V),\\
&u_n \to u_\infty \quad \text{weakly in } L^p(0,T;L^p),\\
&\frac{\partial u_n}{\partial t} \to \frac{\partial u_\infty}{\partial t} \quad \text{weakly in } L^2(0,T;V') + L^q(0,T;L^q),\\
&u_n \to u_\infty \quad \text{weakly star in } L^\infty(0,T;H).
\end{aligned} \tag{36}$$

We shall show that u_∞ is a weak solution, i.e., we shall check the equality (9). First, we note that u_n verifies (9), and then the restriction of u_n to the ball $\Omega_k = B(0,k)$, denoted as before by $L_k u_n$, verifies for any $v \in [C_0^\infty([0,T] \times \Omega_k)]^d$ the same equality on $(0,T) \times \Omega_k$, i.e.,

$$-\int_0^T\int_{\Omega_k} (L_k u_n, v_t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Omega_k} (aL_k u_n, \Delta v)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_k} \big(f(x, L_k u_n), v\big)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{37}$$

It is obvious that $L_k u_n$ converges to some $u_{k\infty}$ in the same sense as in (36) (but changing $\mathbb{R}^N$ by Ω_k) in the respective spaces. On the other hand, since $L^q(0,T;L^q(\Omega_k)) + L^2(0,T;V'_{r_k}) \subset L^q(0,T;H^{-s}(\Omega_k))$ for $s = (s_1,\ldots,s_d)$, $s_i = \max\{1, N(\frac{1}{q_i} - \frac{1}{2})\}$, we have $L_k \frac{\partial u_n}{\partial t} = \frac{\partial L_k u_n}{\partial t} \to \frac{\partial u_{k\infty}}{\partial t}$ weakly in $L^q(0,T;H^{-s}(\Omega_k))$. As in the proof of Theorem 5 we can check that $u_{k\infty} = L_k u_\infty$ and also that $f_0(x, L_k u_n) \to f_0(x, L_k u_\infty)$ weakly in $L^2(0,T;H_{r_k})$, $f_1(x, L_k u_n) \to f_1(x, L_k u_\infty)$ weakly in $L^q(0,T;L^q(\Omega_k))$. Finally, passing to the limit in (37) we obtain that

$$\int_0^T\int_{\Omega_k} \big[(-L_k u_\infty, v_t) - (aL_k u_\infty, \Delta v) + \big(f(x, L_k u_\infty), v\big)\big]\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Since k is arbitrary, u_∞ is a weak solution.

Finally, we shall show that

$$u_\infty(0) = \beta_\infty, \tag{38}$$

$$u_\infty(t) = \xi_\infty. \tag{39}$$

For this aim we shall deduce first that $L_k u_n(r) \to L_k u_\infty(r)$, for all $r \in [0,T]$, weakly in H_{r_k}. First, we note that $\frac{\partial L_k u_n}{\partial t}$ is a bounded sequence of the space $L^q(0,T;H^{-s}(\Omega_k))$, so that $L_k u_n(t) : [0,T] \to H^{-s}(\Omega_k)$ is an equicontinuous family of functions. For each fixed $r \in [0,T]$ from (24) we obtain that the sequence $L_k u_n(r)$ is bounded in H_{r_k}, and then it is precompact in $H^{-s}(\Omega_k)$. Applying the Ascoli–Arzelà theorem we deduce that $\{L_k u_n(t)\}$ is a precompact sequence in $C([0,T],H^{-s}(\Omega_k))$. Hence, since $L_k u_n \to L_k u_\infty$ weakly in $L^2(0,T;H^{-s}(\Omega_k))$, passing to a subsequence we have $L_k u_n \to L_k u_\infty$ in $C([0,T],H^{-s}(\Omega_k))$. The boundedness of $L_k u_n(r)$ in H_{r_k} implies by a standard argument that $L_k u_n(r) \to L_k u_\infty(r)$ weakly in H_{r_k} for all r.

In particular, we have $L_k u_n(0) \to L_k u_\infty(0)$, so that $L_k u_\infty(0) = L_k \beta_\infty$. Since k is arbitrary, we get (38). A similar argument gives (39). □

4. Asymptotic compactness and the global attractor: the method of the energy equation

As before for a given $k > 0$ we denote by $\Omega_k = \{x \in \mathbb{R}^N\colon |x| < k\}$ a ball of radius k centered at 0.

Lemma 9. *For any weak solution $u \in \mathcal{D}(u_0)$, where $u_0 \in B$, a bounded subset of H, and any $\varepsilon > 0$, there exist $T(\varepsilon,B)$, $K(\varepsilon,B)$ verifying*

$$\int_{|x| \geqslant \sqrt{2}k} |u(t,x)|^2 \, dx \leqslant \varepsilon, \quad \forall t \geqslant T,\ k \geqslant K.$$

Proof. Let $s \in \mathbb{R}^+$. We define a smooth function verifying

$$\theta(s) = \begin{cases} 0, & 0 \leqslant s \leqslant 1, \\ 0 \leqslant \theta(s) \leqslant 1, & 1 \leqslant s \leqslant 2, \\ 1, & s \geqslant 2, \end{cases}$$

which obviously satisfies $|\theta'(s)| \leqslant C$, for all $s \in \mathbb{R}^+$. Moreover, we assume that $\sqrt{\theta}$ is also smooth.

We can apply Lemma 3 with $\rho(x) = \sqrt{\theta(\frac{|x|^2}{k^2})}$. It follows from the definition of weak solution that the equality (1) is satisfied for a.a. $t \in (0,T)$ in the sense of the space Y. Hence,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)|u|^2 \, dx = \langle u_t, \rho^2 u\rangle_Y = \langle a\Delta u, \rho^2 u\rangle_Y - \int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)\bigl(f(x,u),u\bigr) \, dx, \tag{40}$$

for a.a. t.

Now, we shall obtain an estimate of the first term in the last expression:

$$\langle a\Delta u, \rho^2 u\rangle_Y = -\sum_{i=1}^{d}\left[\frac{2}{k^2}\int_{\mathbb{R}^N}\left(\bigl(\nabla(au)^i, x\bigr)\theta'\left(\frac{|x|^2}{k^2}\right)u^i\right) dx\right] - \bigl(\rho^2 a\nabla u, \nabla u\bigr)_H. \tag{41}$$

Using that $|\theta'(s)| \leqslant C$, $\theta'(\frac{|x|^2}{k^2}) = 0$, for $|x| < k$ and $|x| > \sqrt{2}k$, Lemma 6 and $(H1)$ we obtain

$$\begin{aligned}\langle a\Delta u, \rho^2 u\rangle_Y &\leqslant \frac{\widetilde{C}}{k}\int_{K\leqslant|x|\leqslant\sqrt{2}K}|\nabla u||u|\,\mathrm{d}x - A\int_{\mathbb{R}^N}\theta|\nabla u|^2\,\mathrm{d}x\\ &\leqslant \frac{\widehat{C}}{k}\big(1+\|\nabla u\|^2\big) \leqslant \varepsilon'\big(1+\|\nabla u\|^2\big),\end{aligned}$$

for any $k \geqslant K_1(\varepsilon')$, where ε' is arbitrary small.

For the second term in (40) condition $(H3)$ implies

$$\begin{aligned}-\int_{\mathbb{R}^N}\theta\biggl(\frac{|x|^2}{k^2}\biggr)\big(f(x,u),u\big)\,\mathrm{d}x &\leqslant -\alpha\int_{\mathbb{R}^N}\theta\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\theta\biggl(\frac{|x|^2}{k^2}\biggr)C_0(x)\,\mathrm{d}x\\ &\quad - \beta\sum_{i=1}^d\int_{\mathbb{R}^N}\theta\biggl(\frac{|x|^2}{k^2}\biggr)|u^i|^{p_i}\,\mathrm{d}x + \int_{\mathbb{R}^N}\theta\biggl(\frac{|x|^2}{k^2}\biggr)C_1(x)\,\mathrm{d}x\\ &\leqslant -\alpha\int_{\mathbb{R}^N}\theta\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,\mathrm{d}x + 2\varepsilon',\end{aligned}\tag{42}$$

if $k \geqslant K_2(\varepsilon')$. Denoting $Y(t) = \int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|u|^2\,\mathrm{d}x$ and using (40)–(42) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Y(t) + \alpha Y(t) \leqslant 3\varepsilon' + \varepsilon'\|\nabla u\|^2,$$

if $k \geqslant K = \max\{K_1, K_2\}$. Applying Gronwall's lemma and using Lemma 6 we obtain

$$\begin{aligned}Y(t) &\leqslant Y(0)\,\mathrm{e}^{-2\alpha t} + \frac{3}{\alpha}\varepsilon' + \varepsilon'\int_0^t \mathrm{e}^{-2\alpha(t-s)}\|\nabla u\|^2\,\mathrm{d}s\\ &\leqslant Y(0)\,\mathrm{e}^{-2\alpha t} + \frac{3}{\alpha}\varepsilon' + \frac{\varepsilon'}{2A}\big(\|u_0\|^2 + D\big).\end{aligned}$$

Choosing ε', $T(\varepsilon, B)$ such that $\frac{3}{\alpha}\varepsilon' + \frac{\varepsilon'}{2A}(\|u_0\|^2 + D) \leqslant \varepsilon/2$, $Y(0)\,\mathrm{e}^{-2\alpha t} \leqslant \varepsilon/2$, for all $u_0 \in B$, $t \geqslant T$, we obtain $Y(t) \leqslant \varepsilon$, and then

$$\int_{|x|\geqslant\sqrt{2}k}|u|^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N}\theta\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,\mathrm{d}x \leqslant \varepsilon. \qquad \square$$

Remark 10. In [46] it is used an estimate of the norm $\|u(t)\|_V$. We are not able to get such an estimate in our case but, instead, it is sufficient to have a bound of $\int_0^t \mathrm{e}^{-2\alpha(t-s)}\|\nabla u\|^2\,\mathrm{d}s$.

For a bounded set $B \subset H$ denote $\gamma_T^+(B) = \bigcup_{t\geqslant T} G(t,B)$. We recall that the multivalued semiflow G is asymptotically compact if any sequence $\xi_n \in G(t_n, u_n)$, where $t_n \to +\infty$, $u_n \in B$, which is a bounded subset satisfying that $\gamma_T^+(B)$ is bounded for some $T(B) \geqslant 0$, is precompact in H. Now we are ready to prove the following:

Proposition 11. *The multivalued semiflow G is asymptotically compact.*

Proof. Let $\xi_n \in G(t_n, v_n)$, $v_n \in B$, a bounded set in H. Since $\gamma^+_{T(B)}(B)$ is bounded and $\xi_n \in G(t_n, v_n) \subset \gamma^+_{T(B)}(B)$, for $n \geqslant n_0$, there exists a subsequence (again denoted by ξ_n) weakly convergent in H to some ξ.

Let $T_0 > 0$ be an arbitrary number. Using Lemma 7 we have that $\xi_n \in G(t_n, v_n) = G(T_0, G(t_n - T_0, v_n))$, and then there must be $\beta_n \in G(t_n - T_0, v_n)$ satisfying $\xi_n \in G(T_0, \beta_n)$. We can choose $N(B, T_0)$ such that $t_n - T_0 \geqslant T(B)$ for all $n \geqslant N(B, T_0)$, so that $G(t_n - T_0, v_n) \subset \gamma^+_{T(B)}(B)$ is bounded and $\beta_n \to \xi_{T_0}$ weakly in H. From Lemma 8 the graph of $G(T_0, \cdot)$ is weakly closed, so that $\xi \in G(T_0, \xi_{T_0})$ and

$$\liminf_{n\to\infty} \|\xi_n\| \geqslant \|\xi\|. \tag{43}$$

If we prove that $\limsup_{n\to\infty} \|\xi_n\| \leqslant \|\xi\|$, then $\xi_n \to \xi$ strongly in H, as we need.

From (22) and $(H1)$ any weak solution satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{1}{2}\|u\|^2 + \|\nabla cu\|^2 = -\int_{\mathbb{R}^N} \big(f(x,u), u\big)\,\mathrm{d}x + \frac{1}{2}\|u\|^2, \quad \text{a.e. on } [0,T], \tag{44}$$

where c is a real matrix such that $\frac{a+a^t}{2} = c^t c$.

By Gronwall's lemma we have

$$\begin{aligned}\|u(T_0)\|^2 = {} & \mathrm{e}^{-T_0}\|u(0)\|^2 - 2\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|\nabla cu\|^2\,\mathrm{d}s \\ & - 2\int_0^{T_0}\!\!\int_{\mathbb{R}^N} \mathrm{e}^{-(T_0-s)}\big(f(x,u),u\big)\,\mathrm{d}x\,\mathrm{d}s + \int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|u\|^2\,\mathrm{d}s.\end{aligned} \tag{45}$$

Let $u_n(\cdot)$ be a sequence of weak solutions verifying $u_n(T_0) = \xi_n$ and $u_n(0) = \beta_n$. Obviously, u_n satisfy (45), so that

$$\begin{aligned}\|\xi_n\|^2 = {} & \mathrm{e}^{-T_0}\|\beta_n\|^2 - 2\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|\nabla cu_n\|^2\,\mathrm{d}s \\ & - 2\int_0^{T_0}\!\!\int_{\mathbb{R}^N} \mathrm{e}^{-(T_0-s)}\big(f(x,u_n),u_n\big)\,\mathrm{d}x\,\mathrm{d}s + \int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|u_n\|^2\,\mathrm{d}s.\end{aligned} \tag{46}$$

We know from Lemma 8 that u_n converges to some weak solution u in the sense of (36) and $u(0) = \xi_{T_0}$, $u(T_0) = \xi$.

We need to handle each term in (46) separately for the sequence u_n. First, since β_n is bounded, it is clear that

$$\mathrm{e}^{-T_0}\|\beta_n\|^2 \leqslant \mathrm{e}^{-T_0}M, \quad \text{for all } n. \tag{47}$$

Further we have

$$\begin{aligned}&\limsup_{n\to\infty}\left(-2\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|\nabla cu_n\|^2\,\mathrm{d}s\right) = -\liminf_{n\to\infty} 2\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|\nabla cu_n\|^2\,\mathrm{d}s \\ &\quad \leqslant -2\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|\nabla cu\|^2\,\mathrm{d}s.\end{aligned} \tag{48}$$

On the other hand, we consider the splitting

$$\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|u_n\|^2\,\mathrm{d}s = \int_0^{T_0}\int_{\Omega_k} \mathrm{e}^{-(T_0-s)}|u_n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^{T_0} \mathrm{e}^{-(T_0-s)}\int_{|x|\geqslant k}|u_n|^2\,\mathrm{d}x\,\mathrm{d}s.$$

We note that Lemma 7 implies $u_n(s) \in G(s, G(t_n - T_0, v_n)) = G(s + t_n - T_0, v_n)$. By Lemma 9 for any $\varepsilon > 0$ there exist $T(\varepsilon, B), K_1(\varepsilon, B) > 0$ such that $\int_{|x|\geqslant k}|u_n(s)|^2\,\mathrm{d}x \leqslant \varepsilon$, if $k \geqslant K_1$, $t_n - T_0 \geqslant T$. As in the proof of Theorem 5 we can check that (up to a subsequence) $L_k u_n \to L_k u$ strongly in $L^2(0, T; H_k)$, so that

$$\limsup_{n\to\infty}\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|u_n\|^2\,\mathrm{d}s$$
$$\leqslant \int_0^{T_0}\int_{\Omega_k} \mathrm{e}^{-(T_0-s)}|u|^2\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\,\mathrm{d}s \leqslant \int_0^{T_0} \mathrm{e}^{-(T_0-s)}\|u\|^2\,\mathrm{d}x\,\mathrm{d}s + \varepsilon. \tag{49}$$

Finally, we have to handle the nonlinear term. We note first that $(H3)$ implies

$$-2\int_0^{T_0}\int_{|x|\geqslant k} \mathrm{e}^{-(T_0-s)}\big(f(x,u_n),u_n\big)\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant 2\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\int_{|x|\geqslant k} C_1(x)\,\mathrm{d}x\,\mathrm{d}s - 2\beta\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\int_{|x|\geqslant k}|u|^p\,\mathrm{d}x\,\mathrm{d}s$$
$$+ 2\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\int_{|x|\geqslant k} C_0(x)\,\mathrm{d}x\,\mathrm{d}s - 2\alpha\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\int_{|x|\geqslant k}|u|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant 4\varepsilon\int_0^{T_0} \mathrm{e}^{-(T_0-s)}\,\mathrm{d}s \leqslant 4\varepsilon,$$

if $k \geqslant K_2(\varepsilon)$. We have seen that $u_n \to u$ strongly in $L^2(0, T; H_k)$, so that $u_n(t, x) \to u(t, x)$ for a.a. $(t, x) \in (0, T_0) \times \Omega_k$. Since $f(x, u)$ is continuous on u, we have that $f(x, u_n(t, x)) \to f(x, u(t, x))$ a.e. Then Lebesgue–Fatou's lemma (see [47]) and the inequality $(f(u_n(t, x)), u_n(t, x)) \geqslant -(C_0(x)+C_1(x))$, for a.a. (t, x), imply

$$\limsup_{n\to\infty}\left(-2\int_0^{T_0}\int_{\Omega_k} \mathrm{e}^{-(T_0-s)}\big(f(x,u_n),u_n\big)\,\mathrm{d}x\,\mathrm{d}s\right)$$
$$\leqslant -2\int_0^{T_0}\int_{\Omega_k} \mathrm{e}^{-(T_0-s)}\liminf_{n\to\infty}\big(f(x,u_n),u_n\big)\,\mathrm{d}x\,\mathrm{d}s = -2\int_0^{T_0}\int_{\Omega_k} \mathrm{e}^{-(T_0-s)}\big(f(x,u),u\big)\,\mathrm{d}x\,\mathrm{d}s.$$

Therefore, the splitting of the integral gives

$$\limsup_{n\to\infty}\left(-2\int_0^{T_0}\int_{\mathbb{R}^N} \mathrm{e}^{-(T_0-s)}\big(f(x,u_n),u_n\big)\,\mathrm{d}x\,\mathrm{d}s\right)$$
$$\leqslant -2\int_0^{T_0}\int_{\Omega_k} \mathrm{e}^{-(T_0-s)}\big(f(x,u),u\big)\,\mathrm{d}x\,\mathrm{d}s + 4\varepsilon. \tag{50}$$

We use now the equality (46) for u_n. Passing to the limit as $k \to \infty$ in (50) and using (45) and (47)–(50) we find that

$$\limsup_{n\to\infty} \|\xi_n\|^2 \leqslant \mathrm{e}^{-T_0} M - 2\int_0^{T_0}\int_{\mathbb{R}^N} \mathrm{e}^{-(T_0-s)} |\nabla c u|^2 \,\mathrm{d}x\,\mathrm{d}s$$

$$+ \int_0^{T_0}\int_{\mathbb{R}^N} \mathrm{e}^{-(T_0-s)} |u|^2 \,\mathrm{d}x\,\mathrm{d}s - 2\int_0^{T_0}\int_{\mathbb{R}^N} \mathrm{e}^{-(T_0-s)} \big(f(x,u),u\big)\,\mathrm{d}x\,\mathrm{d}s + 5\varepsilon$$

$$= \|\xi\|^2 + \mathrm{e}^{-T_0} M - \mathrm{e}^{-T_0}\|\xi_{T_0}\|^2 + 5\varepsilon. \tag{51}$$

Taking the limit as $T_0 \to +\infty$, and then letting $\varepsilon \to 0$ we get the inequality

$$\limsup_{n\to\infty} \|\xi_n\|^2 \leqslant \|\xi\|^2,$$

and, thus, the proof is complete. □

Finally, we shall prove a continuity property of the semiflow G. We recall that the multivalued map $G(t,\cdot) : H \to P(H)$, where t is fixed, is upper semicontinuous if for any $x_0 \in H$ and any neighborhood O of $G(t,x_0)$ there exists $\delta > 0$ such that $G(t,x) \subset O(G(t,x_0))$, as soon as $\|x - x_0\| < \delta$. We say that the multivalued map $G(t,\cdot)$ has compact values if the set $G(t,x_0)$ is compact for all $t \geqslant 0$.

Lemma 12. *The map $G(t,\cdot)$ is upper semicontinuous and has compact values for any $t \geqslant 0$.*

Proof. Let $\xi_n \in G(t,x_n)$ and $x_n \to x_0$. We claim that the sequence ξ_n is precompact in H. In view of Lemma 6 the sequence ξ_n is bounded, so that up to a subsequence it is weakly convergent to some ξ. Arguing in a similar way as in the proof of Proposition 11 there exist weak solutions $u_n(\cdot)$, $u(\cdot)$ such that $u_n(t) = \xi_n$, $u_n(0) = x_n$, $u(t) = \xi$, $u(0) = x_0$ and u_n converges to u in the sense of (36). Moreover, they satisfy the equality

$$\big\|u(t)\big\|^2 = \big\|u(0)\big\|^2 - 2\int_0^t \|\nabla c u\|^2 \,\mathrm{d}s - 2\int_0^t\int_{\mathbb{R}^N} \big(f(x,u),u\big)\,\mathrm{d}x\,\mathrm{d}s. \tag{52}$$

Repeating the same arguments of Proposition 11 we obtain

$$\limsup_{n\to\infty} \|\xi_n\|^2 \leqslant \lim_{n\to\infty} \|x_n\|^2 - 2\int_0^t\int_{\mathbb{R}^N} |\nabla c u|^2 \,\mathrm{d}x\,\mathrm{d}s - 2\int_0^t\int_{\mathbb{R}^N} \big(f(x,u),u\big)\,\mathrm{d}x\,\mathrm{d}s + 4\varepsilon$$

$$= \|\xi\|^2 + 4\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \|\xi\|^2.$$

Hence, $\xi_n \to \xi$ strongly in H. An immediate consequence of this property is that $G(t,x_0)$ is precompact for any x_0. Lemma 8 implies that $G(t,x_0)$ is weakly closed, hence closed, so that $G(t,x_0)$ is a compact set.

Now, if $G(t,\cdot)$ is not upper semicontinuous, then there exists a point x_0, a neighborhood O of $G(t,x_0)$ and a sequence $\xi_n \in G(t,x_n)$ with $\|x_n - x_0\| \to 0$ such that $\xi_n \notin O$. Passing to a subsequence we have $\xi_{n_k} \to \xi$, $x_{n_k} \to x_0$, strongly in H. Lemma 8 implies that $\xi \in G(t,x_0)$, which is a contradiction. □

We conclude this section by proving Theorem 1, the main result of the paper. Summarizing the results obtained so far we have:

1. Lemma 6 implies that $\gamma_0^+(B)$ is bounded for any bounded set B and that a bounded absorbing set B_0 exists, i.e., a set satisfying that for any bounded set B there exists $T(B)$ such that

$$G(t,B) \subset B_0, \quad \text{for any } t \geqslant T.$$

2. Lemma 11 implies that G is asymptotically compact.
3. Lemma 12 implies that $G(t,\cdot)$ is upper-semicontinuous and has compact values for all $t \geqslant 0$.

A global attractor $\mathcal{A}$ for the multivalued semiflow G is a set satisfying the following two properties:

1. It is attracting, i.e.,

$$\operatorname{dist}\bigl(G(t,B),\mathcal{A}\bigr) \to 0, \quad \text{as } t \to +\infty,$$

for any bounded set B, where $\operatorname{dist}(C,A) = \sup_{c\in C} \inf_{a\in A} \|c-a\|$.
2. It is negatively semi-invariant, i.e.,

$$A \subset G(t,A), \quad \text{for all } t \geqslant 0.$$

The global attractor is said to be invariant if $A = G(t,A)$, for all $t \geqslant 0$. The given properties and the equality $G(t,G(s,x)) = G(t+s,x)$ (see Lemma 7) imply the existence of a global compact invariant attractor (see [31, Theorem 3 and Remark 8]), which is the minimal closed attracting set (that is, for any closed attracting set C we have $A \subset C$). Hence, Theorem 1 is proved.

Consider now the Fitz–Hugh–Nagumo system (see [44,41]):

$$\begin{cases} u_t - d_1 u_{xx} + g(x,u) + v = h_1(x), \\ v_t - d_2 v_{xx} - \delta u + \xi v = h_2(x), \end{cases}$$

where $d_i, \xi, \delta > 0$, $h_i \in L^2(\mathbb{R}^N)$ and $g(x,u)$ is a Caratheodory function. Usually, g is a cubic polynomial with respect to u. For example, we can take $g(x,u) = u^3 + \omega(x)u^2 + \sigma u$, where $\omega \in L^4(\mathbb{R}^N)$, $\sigma > 0$. We assume also that $\sigma\xi > (1-\delta)^2/2$.

Denote $z = (u,v)$, $a = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ and

$$f_0(x,z) = \begin{pmatrix} \sigma u + v + h_1(x) \\ -\delta u + \mu\xi v + h_2(x) \end{pmatrix}, \quad \frac{(1-\delta)^2}{2\sigma\xi} < \mu < 1,$$
$$f_1(x,z) = \begin{pmatrix} u^3 + \omega(x)u^2 \\ (1-\mu)\xi v \end{pmatrix}.$$

It is easy to check that conditions $(H1)$–$(H4)$ are satisfied with $p = (4,2)$. Hence, the statement of Theorem 1 holds.

5. Asymptotic compactness: the method of monotonicity

As we have pointed out in the introduction there exists another approach for proving that the semiflow is asymptotically compact, which has been used already in [23] for reaction–diffusion equations in bounded domains and in [22,24] for phase-field equations. Now we shall show that this method also works in our case.

The first part of the proof is the same as in Proposition 11. The difference appears in the proof of the inequality $\limsup_{n\to\infty} \|\xi_n\| \leqslant \|\xi\|$. From (44) and condition $(H3)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2A\|\nabla u\|^2 + 2\alpha\|u\|^2 + 2\beta\sum_{i=1}^{d}\|u^i\|_{p_i}^{p_i} \leqslant 2\|C_0\|_{L^1(\mathbb{R}^N)} + 2\|C_1\|_{L^1(\mathbb{R}^N)}$$
$$\leqslant C \quad \text{a.e. on } [0,T],$$

for any weak solution, so that the functions $J_n(t) = \|u_n(t)\|^2 - Ct$, $J(t) = \|u(t)\|^2 - Ct$, where u_n, u are the same solutions defined in the proof of Proposition 11, are non-increasing. We shall show that $\limsup_{n\to\infty} J_n(T_0) \leqslant J(T_0)$, from which the result follows. We know from Proposition 11 that (up to a subsequence) $L_k u_n \to L_k u$ strongly in $L^2(0,T;H_k)$, and also that for any $\varepsilon > 0$ there exist $T(\varepsilon,B), K_1(\varepsilon,B) > 0$ such that $\int_{|x|\geqslant k} |u_n|^2\,\mathrm{d}x \leqslant \varepsilon$, for any $s \in [0,T_0]$, if $k \geqslant K_1$, $t_n - T_0 \geqslant T$. It follows also that

$$\int_{\Omega_k} \left|u_n(s,x)\right|^2 \mathrm{d}x \to \int_{\Omega_k} \left|u(s,x)\right|^2 \mathrm{d}x, \quad \text{for a.a. } s \in (0,T_0).$$

Let t_m be a sequence such that $t_m < T_0$, $t_m \to T_0$, as $m \to \infty$, and $\int_{\Omega_k} |u_n(t_m,x)|^2\,\mathrm{d}x \to \int_{\Omega_k} |u(t_m,x)|^2\,\mathrm{d}x$, as $n \to \infty$, for any fixed m. Hence, using the continuity of J and the monotonicity of J_n, J we have that for any $\varepsilon > 0$ there exist m, $N(m)$, T and K_1 such that

$$\begin{aligned} J_n(T_0) - J(T_0) &= J_n(T_0) - J_n(t_m) + J_n(t_m) - J(t_m) + J(t_m) - J(T_0) \\ &\leqslant \left|J_n(t_m) - J(t_m)\right| + \left|J(t_m) - J(T_0)\right| \\ &\leqslant \left|\int_{\Omega_k} \left|u_n(t_m,x)\right|^2 \mathrm{d}x - \int_{\Omega_k} \left|u(t_m,x)\right|^2 \mathrm{d}x\right| \\ &\quad + \int_{|x|\geqslant k} \left|u_n(t_m,x)\right|^2 \mathrm{d}x + \int_{|x|\geqslant k} \left|u(t_m,x)\right|^2 \mathrm{d}x + \varepsilon \leqslant 5\varepsilon. \end{aligned}$$

Thus, the proof is complete.

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