Switched Observers for Switched Linear Systems With Unknown Inputs

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Abstract—Two state observers are designed for some classes of switched linear systems with unknown inputs. The design of the proposed observers assumes that all switching subsystems fulfill a property of “strong detectability” [4] that allows to implement suitable reduced-order unknown-input switched observers. The synthesis of the observers is based on the feasibility of a certain system of LMI’s. Two main schemes are presented. For the case when the set of possible unknown-input distribution matrices are linearly dependent, an observer is suggested that guarantees the asymptotic state reconstruction without making any slow-switching dwell-time constraint about the sequence of the switching times. For the general case, the existence of a minimal average dwell-time for every switching sequence is assumed. By appropriate Lyapunov analysis, the convergence of the state estimate is proven to be exponential in both cases. Simulation results confirm the predicted performance.

Index Terms—Switched linear systems, Unknown-input observers, Strong detectability.

I. INTRODUCTION

The problem of state observation for linear time-invariant systems with unknown inputs has been widely studied during the last two decades. It was shown that, under the condition of strong detectability (see, e.g., [4], [3]) and an additional particular relative-degree condition (both conditions together were referred to in [4] as “strong detectability”), a “decoupling” state transformation can be made such that the observation error dynamics in the transformed state coordinates is not contaminated by the unknown inputs, and a reduced-order Luenberger-like observer is capable of reconstructing the entire original state vector. Unknown input observers have been widely used in the framework of fault detection and isolation [11]. It was discovered that sliding mode observers allow to reconstruct accurately the unknown input together with the system state, which is an important requirement in FDI schemes [8]. In [14], [15], second-order sliding-mode observers were suggested allowing to reconstruct the unknown input in finite time.

Concerning switched linear systems, controllability and observability issues for systems without unknown inputs are addressed in [9]. In [7] a switched version of the conventional Luenberger observer was suggested and the problem of finding a common Lyapunov function for the switching dynamics of the error system was addressed via LMI technique. In [10] the author proposed to reset the observer state at the switching times in order to guarantees the boundedness of the observation error. A step by step sliding mode observer is designed in [13] for a special class of nonlinear switched systems. Recently a general separation results for observer-based control of switched linear systems has been demonstrated [2]. Necessary and sufficient LMI-based conditions for the simultaneous design of an observer and a stabilizing compensator were developed for a class of perfectly known switched linear systems.

None of the above works addressed the observation problem for switched systems subject to unknown inputs. In [16] it was suggested an observation scheme for a class of Linear parameter-varying (LPV) uncertain systems which is capable of partially attenuating the effect of an unmeasured disturbance. In this manuscript the state observation problem for a class of switched linear systems with unknown inputs is tackled. The provided state estimate is totally insensitive against the presence of the (possibly unbounded) unmeasurable disturbance input. We propose a structure that realizes a switched version of a Luenberger-like observer. In the present approach it is instrumental the above mentioned “disturbance-decoupling” state transformation. The main difficulty in the plane application of the known stability results for switched systems [5], [6] is the fact that such a state transformation depends on the time-varying switching matrices of the plant. This implies that the transformation matrix abruptly changes at the switching instants, which causes discontinuities (“jumps”) in the transformed state vector that complicate the observer design. The problem is solved by incorporating similar jumps into the observer dynamics. A constructive LMI-based design procedure is presented, and the stability issues associated to the properties of the switching times sequence are discussed.

The paper is organized as follows. The next Section outlines the problem formulation and the main assumption about the system matrices. Section III discusses the time-varying state transformation which plays an instrumental role in the present treatment. Section IV presents a simplified observer design that requires an additional structural assumption on the system matrices. Section V presents the “jump” observer and discusses the relevant properties of stability. General comments and implementation issues are discussed in section VI. Section

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VII shows some simulation examples. The paper finishes with a conclusive section that draws possible line for future improvements of the presented approach.

II. PROBLEM FORMULATION

Let us consider the following class of switched linear systems

$$
\begin{align*}
\dot{x}(t) &= A_{\lambda(t)}x(t) + B_{\lambda(t)}u(t) + E_{\lambda(t)}w(t) \\
y(t) &= C_{\lambda(t)}x(t)
\end{align*}
$$

where $x(t) \in \mathbb{R}^n$ represents the continuous state vector, $\lambda(t)$ is the piecewise-constant discrete state function that takes values on the discrete set $\{1, \ldots, N\}$, with $N$ being the number of “modes” that compose the overall switched dynamics, $y(t) \in \mathbb{R}^p$ is the system output, $u(t) \in \mathbb{R}^h$ is a known input to the system, and $w(t) \in \mathbb{R}^m$ is an unknown input term. We say that $q \in \{1, \ldots, N\}$ is the active mode at time $t$ if $\lambda(t) = q$, to which it corresponds a specific instance of the matrix quadruple $(A_q, B_q, C_q, E_q)$. Let $\{t_1, t_2, \ldots, t_k\}$ ($k$ might be infinite) denotes the sequence of the “switching times”, the time instants at which the system mode is changing (i.e. $\lambda(t_i^+) \neq \lambda(t_i^-)$). Let $t_0 = 0$ and let the system discrete state be given as follows

$$
\lambda(t) = \lambda_i, \quad \lambda_i \in \{1, 2, \ldots, N\}, \quad t_i \leq t < t_{i+1},
$$

with $i = 0, 1, 2, \ldots$. Let the system output $y(t)$, the discrete state $\lambda(t)$ and the input signal $u(t)$ be all available for measurement, while $w(t)$ is completely uncertain. Let also the matrix quadruples $(A_q, B_q, C_q, E_q)$, $(q = 1, 2, \ldots, N)$, be known.

The task is that of providing the asymptotic reconstruction of the continuous system state $x(t)$. Henceforth, it will be assumed that $\forall q = 1, 2, \ldots, N$ the following properties hold:

A1. The invariant zeros of the matrix triplets $(A_q, C_q, E_q)$ have negative real part.

A2. $\text{rank}(C_qE_q) = \text{rank} E_q = m$.

Condition A1 is sometimes referred to as the “strong detectability condition [3, 4]. Assumption A2 allows expressing in closed form the state transformation that will be introduced below. Both conditions are equivalent to the strong detectability* concept introduced in [4]. As it is discussed later (see remark 1), the strong detectability of the triplets $(A_q, C_q, E_q)$ can be verified in a simplified way by means of a standard detectability criterion.

III. SYSTEM DYNAMICS IN NEW COORDINATES

A preliminary step of the observer design is that of introducing a nonsingular, time-varying, state and output change of coordinates that decouples the unknown input $w$ from a subset of the transformed state coordinates. Let the nonsingular matrices $T_q$ and $U_q$ ($q = 1, 2, \ldots, N$) be defined as follows:

$$
T_q = \begin{bmatrix} E_q^\perp \\ (C_qE_q)^+ C_q \end{bmatrix}, \quad U_q = \begin{bmatrix} (C_qE_q)^+ \\ (C_qE_q)^+ \end{bmatrix}
$$

where $\Gamma^\perp$ is a matrix such that $\Gamma^\perp \Gamma = 0$ and $\Gamma^\perp$ is linearly independent of $\Gamma^T \Gamma$. $\Gamma^\perp = [\Gamma^T \Gamma]^{-1} \Gamma^T$ is the left pseudo-inverse of $\Gamma$. The inverse matrix $T_q^{-1}$ takes the following form:

$$
T_q^{-1} = \begin{bmatrix} I - E_q (C_qE_q)^+ C_q (E_q^+) & (E_q^+) \\ -T_q & E_q \end{bmatrix} = \begin{bmatrix} \bar{T}_q & E_q \end{bmatrix}
$$

With the transformed state and output variables $\bar{x} = T_{\lambda(t)}x$, and $\bar{y} = U_{\lambda(t)}y$, let us partition the transformed state and output as follows

$$
\bar{x}_1 = \begin{bmatrix} \bar{x}_1^T \\ \bar{x}_2^T \end{bmatrix}, \quad \bar{x}_1 \in \mathbb{R}^{n-m}, \quad \bar{x}_2 \in \mathbb{R}^m
$$

$$
\bar{y}_1 = \begin{bmatrix} \bar{y}_1^T \\ \bar{y}_2^T \end{bmatrix}, \quad \bar{y}_1 \in \mathbb{R}^{p-m}, \quad \bar{y}_2 \in \mathbb{R}^m
$$

The system dynamics in the new coordinates takes the form:

$$
\begin{align*}
\dot{\bar{x}}_1(t) &= A_{\lambda(t)}\bar{x}_1(t) + A_{\lambda(t)}\bar{x}_2(t) + B_{\lambda(t)}u(t) \\
\dot{\bar{x}}_2(t) &= A_{\lambda(t)}\bar{x}_1(t) + A_{\lambda(t)}\bar{x}_2(t) + B_{\lambda(t)}u(t) + w(t)
\end{align*}
$$

with implicit definition of the matrices $A_{\lambda,1}, \ldots, A_{\lambda,4}, B_{\lambda,1}, B_{\lambda,2}$ which depend on the time-varying discrete state function $\lambda = \lambda(t)$. The matrix $C_{\lambda(t)}$ results from the following equation

$$
U_{\lambda(t)}C_{\lambda(t)}T_{\lambda(t)}^{-1} = \begin{bmatrix} \bar{C}_{\lambda(t)} & 0 \\ 0 & I \end{bmatrix}
$$

Remark 1: It is easy to show (see [11]) that the strong detectability of the matrix triplets $(A_q, C_q, E_q)$ is equivalent to the simple detectability of the matrix pairs $(A_{q,1}, C_q)$.

Because of the time-varying form of the transformation (4)-(6), which depends on the piece-wise constant function $\lambda(t)$, the state $\bar{x}$ of the transformed system features jumps at the switching instants $t = t_i$ which are governed by the following reset equation:

$$
\bar{x}(t_i^+) = T_{\lambda(t_i)}x(t_i^-) = T_{\lambda(t_i)}T_{\lambda(t_i)}^{-1}\bar{x}(t_i^-), \quad i = 1, 2, \ldots
$$

The observation problem for the switched system (8)-(11), subject to the state reset equations (12), is now addressed. It is a peculiarity of the transformed system (8)-(11) that the state component $\bar{x}_2$ is directly available through the measurement of $\bar{y}_2$. Therefore, only $\bar{x}_1$ needs to be reconstructed. Indeed, once $\bar{x}_1$ is available, one can reconstruct the original state vector $x$ through the inverse mapping

$$
x = T_{\lambda(t)}^{-1}\bar{x} = T_{\lambda(t)}^{-1}\begin{bmatrix} \bar{x}_1 \\ \bar{y}_2 \end{bmatrix} = T_{\lambda(t)}^{-1}(C_{\lambda(t)}E_{\lambda(t)})^+ y
$$

Furthermore, the dynamics (8) of $\bar{x}_1$ is not contaminated by the unknown input $w(t)$.

1 A Matlab instruction for computing $M_b = M^+$ for a generic matrix $M$ is $M_b = \text{null}(M')$. 

Two main results are demonstrated in the two next sections

A. An observer scheme that works for any arbitrary switching sequence $t_i$ is developed in section IV for the special case when the matrices $E_1, E_2, ..., E_N$ are pairwise linearly dependent.

B. An observer scheme that works under an average dwell-time constraint on the switching times $t_i$ is developed in section V for the general case.

IV. OBSERVATION WITH ARBITRARY SWITCHING

Let us make the following additional assumption:

A3. The input distribution matrices $E_1, E_2, ..., E_N$ are pairwise linearly dependent, i.e.,

$$\text{rank } [E_i \ E_j] = \text{rank } E_i = m \quad \forall i, j$$

Assumption A3 means that there exist $E \in \mathbb{R}^{n \times m}$ and $F_q \in \mathbb{R}^{n \times m}$ such that $E_q = EF_q$, $q = 1, 2, ..., N$, where $E$ is a constant matrix and $F_1, F_2, ..., F_N$ are nonsingular. Furthermore, $E_q^\perp = E^\perp$ for all $q$, since $E^\perp E_q = E^\perp EF_q = 0$. Therefore, from the equalities

$$\hat{x}_1 = E_q^\perp x = E^\perp x,$$

it can be concluded that there are no jumps in the vector $\hat{x}_1$ at the switching times $t_i$, i.e. the assumption A3 guarantees that $\hat{x}_1(t_i^+) = \hat{x}_1(t_i^-)$.

We consider the subsystem relevant to $\hat{x}_1$ only by combining (8), and (10)

$$\dot{\hat{x}}_1(t) = A_{\lambda(t),1}\hat{x}_1(t) + A_{\lambda(t),2}\bar{y}_2(t) + B_{\lambda(t),1}u(t)$$

$$\bar{y}_1(t) = C_{\lambda(t)}\hat{x}_1(t)$$

Under assumptions A1 and A2, all pairs $(A_{1,1}, C_1), (A_{2,1}, C_2), ..., (A_{N,1}, C_N)$ are detectable. The proposed observer has the form

$$\dot{\hat{x}}(t) = T_{\lambda(t)}\begin{bmatrix} \hat{x}_1(t) \\ \bar{y}_2(t) \end{bmatrix}$$

where the estimate $\hat{x}$ is defined as follows:

$$\dot{\hat{x}}_1(t) = A_{\lambda(t),1}\hat{x}_1(t) + A_{\lambda(t),2}\bar{y}_2(t) + B_{\lambda(t),1}u(t)$$

$$+ L_{\lambda(t)}(C_{\lambda(t)}\hat{x}_1(t) - \bar{y}_1(t))$$

The matrices $L_1, L_2, ..., L_N$ are the observer gains to be designed in order to achieve an asymptotically stable error dynamics. Define the observation error $\bar{e}_1 := \bar{x}_1 - \hat{x}_1$. From (16), (17) and (19), the error dynamics is:

$$\dot{\bar{e}}_1(t) = (A_{\lambda,1} - L_{\lambda}C_\lambda)\bar{e}_1(t) = \hat{A}_{\lambda,1}\bar{e}_1(t)$$

The continuity of $\bar{x}_1$ implies that there are no jumps in the trajectories of $\bar{e}_1(t)$.

Lemma 1: Let us assume that observer gain matrices $L_1, L_2, ..., L_N$ and positive definite matrices $Q_1, Q_2, ..., Q_N > 0$ can be found so that a common solution $P$ to the following Lyapunov equations ($q = 1, 2, ..., N$) exists

$$(A_{q,1} - L_qC_q)^T P + P (A_{q,1} - L_qC_q) = -Q_q,$$ (21)  

Then the error dynamics (20) is exponentially stable and $\bar{x}(t)$ tends exponentially to $x(t)$.

Proof: If a matrix $P$ satisfies all the Lyapunov equations (21), then $V = \bar{e}_1^T P \bar{e}_1$ represents a common Lyapunov function for each mode of the switching system (20). The time derivative of the common Lyapunov function along the trajectories of the switched system (20) fulfills the following chain of inequalities

$$\dot{V} = -\bar{e}_1^T Q_{\lambda} \bar{e}_1 \leq -\min_\lambda (Q_{\lambda}) \bar{e}_1^T \bar{e}_1 \leq -\frac{\min_\lambda (Q_{\lambda})}{\max(P)} V \leq \frac{\inf_{q \in Q} \min_\lambda (Q_{\lambda})}{\max(P)} V \leq \frac{\inf_{q \in Q} \alpha_1}{\max(P)} V$$

(22)

where $Q \equiv \{1, 2, ..., N\}$. The continuity of $\bar{e}_1(t)$ implies the continuity of $V(\bar{e}_1(t))$, which, with (22), allows $V$, and thus $\bar{e}_1$, to tend exponentially to zero. By (13), (18) and (3), we obtain

$$\|x(t) - \hat{x}(t)\| = \left\| T_{\lambda(t)} \begin{bmatrix} \bar{e}_1(t) \\ 0 \end{bmatrix} \right\| \leq \|\hat{T}_{\lambda(t)}\| \|\bar{e}_1(t)\|$$

Since $Q \equiv \{1, 2, ..., N\}$ is a finite set, the maximum of $\|\hat{T}_{\lambda}\|$ over the set $q \in Q$ can be evaluated. Therefore, the following inequality is obtained,

$$\|x(t) - \hat{x}(t)\| \leq \sup_{q \in Q} \|\hat{T}_{\lambda}\| \|\bar{e}_1(t)\|$$

(23)

In view that $\|\bar{e}_1(t)\|$ is exponentially vanishing, there are positive constants $\alpha$ and $\gamma$ such that $\|\bar{e}_1(t)\| \leq \alpha e^{-\gamma t} \|\bar{e}_1(0)\|$. Therefore, defining $\hat{\alpha} = \alpha \sup_{q \in Q} \|\hat{T}_{\lambda}\|$, (23) yields the inequality $\|x(t) - \hat{x}(t)\| \leq \hat{\alpha} e^{-\gamma t} \|\bar{e}_1(0)\|$. Since $\bar{e}_1(t) = E^\perp (x(t) - \hat{x}(t))$, we finally get the following inequality that proves the lemma: $\|x(t) - \hat{x}(t)\| \leq \hat{\alpha} \|E^\perp\| e^{-\gamma t} \|x(0) - \hat{x}(0)\|$. Lemma 1 is proven. ■

The observer design according to Lemma 1 entails finding a common solution to the Lyapunov equations (21) [5]. The effectiveness of this stability criterion heavily relies on the fact that vector $\bar{e}_1(t)$ does not jump at the switching instants.

The Lyapunov equations (21) require the simultaneous finding of the matrices $P$ and $Q_q$, then a non convex LMI problem would naturally arise since the product between two unknowns is not allowed in the conventional LMI setup. By making algebraic manipulations, the problem can be reduced to a standard LMI problem according to the following Lemma 2 (see [7]).

Lemma 2 ([7]): The Lyapunov equations (21) admit a common solution $P > 0$ if, and only if, the following LMI system with the new unknowns $P$ and $Y_i, i = 1, ..., N$ is feasible

$$A_{i,1}^T P - C_{i}^T Y_i^T + PA_{i,1} - Y_i C_{1} < 0$$

(24)

If the above LMI system is feasible then the observer gains are recovered as $L_i = P^{-1} Y_i, i = 1, ..., N$.

On the basis of the previous Lemmas and considerations, the following Theorem is proven.

Theorem 1: Consider system (1), fulfilling assumptions A1, A2 and A3, and the state and output transformations (4)-(7). Consider the observer (18)-(19) with the gain matrices $L_1, ..., L_N$ computed by solving the LMI system as described...
in Lemma 2. Then, the exponentially-converging observation of the state vector \( x \) is guaranteed for any arbitrary switching sequence.

V. OBSERVER DESIGN FOR THE GENERAL CASE

We now dispense with the assumption A3. Such an assumption was instrumental in the previous section for guaranteeing the absence of jumps in the dynamics of the transformed state \( \bar{x}_1 \). Actually, considering (5), jumps of the transformed state variable \( \bar{x}_1 \) occur at the switching instants, which are governed by

\[
\bar{x}_1(t^+) = E_{\lambda(t)}^{-1}x(t^-) \quad i = 1, 2, \ldots
\]  

(25)

As compared with the previous Section, the observer reconstruction equation (18) remains the same; it actually changes the formulation of the switched Luennberger observer, that now contains state jumps at the switching instants according to

\[
\dot{\hat{x}}_1(t) = A_{\lambda(t)}\hat{x}_1(t) + B_{\lambda(t)}u(t) + L_{\lambda(t)}(\hat{C}_{\lambda(t)}\hat{x}_1(t) - \bar{y}_1(t))
\]

(26)

\[
\hat{x}_1(t^-) = E_{\lambda(t)}^\perp \hat{x}_1(t^-) = E_{\lambda(t)}^\perp T_{\lambda(t)} \hat{x}_1(t^-), \quad i = 1, 2, \ldots
\]

(27)

where the matrices \( L_q \) \((q = 1, 2, \ldots, N)\) are the observer gains to be designed. The state of the observer \( \hat{x}_1 \) features intentionally introduced jumps at the switching instants that need to compensate the destabilizing effect of the corresponding jumps of \( \bar{x}_1 \). From (8), (25), (26), and (27), the dynamics of the observation error variable \( \hat{e}_1 = \bar{x}_1 - \hat{x}_1 \) is

\[
\dot{\hat{e}}_1(t) = (A_{\lambda(t)} - L_{\lambda(t)}C_{\lambda(t)}) \hat{e}_1(t) + \bar{A}_{\lambda(t)}\hat{e}_3(t)
\]

(28)

\[
\hat{e}_1(t^+) = E_{\lambda(t)}^\perp \hat{e}(t^-) = E_{\lambda(t)}^\perp T_{\lambda(t)} \hat{e}_1(t^-)
\]

(29)

Since matrix \( E_{\lambda(t)}^\perp \) features discontinuities at the switching instants, the trajectories of \( \hat{e}_1(t) \) are also discontinuous at each switching instant. The stability analysis of (28)-(29), that is going to be presented, assume the feasibility of the same LMI system of the previous case, and, as the main difference, it leads to a stability condition involving an average dwell-time constraint on the switching sequence \( t_i, i = 1, 2, \ldots \).

Firstly, let us recall the concept of average dwell time (see, e.g. [5]). Let \( N_\lambda(t, \xi) \) be the number of discontinuities of the switching signal \( \lambda(t) \) on the interval \((\xi, t)\). We say that \( \lambda(t) \) has an average dwell time \( \tau_a \) if there exist two positive numbers \( N_0 \) and \( \tau_a \) such that

\[
N_\lambda(t, \xi) \leq N_0 + \frac{t - \xi}{\tau_a} \quad \text{for all} \ t \geq \xi \geq 0
\]

(30)

By definition \( N_\lambda(t) = N_\lambda(t, 0) \), that is \( N_\lambda(t) \) is the number of discontinuities of the switching function \( \lambda(t) \) from the initial time instant \( t = 0 \) until the current time \( t \).

Theorem 2: Consider system (1), fulfilling the assumptions A1 and A2, and the state and output transformations (4)-(7). Consider the observer (18), (26) with the gain matrices \( L_1, \ldots, L_N \) computed by solving the LMI system as described in the Lemma 2. Then, the asymptotic reconstruction of the continuous state vector \( x \) is obtained provided that the switching sequence fulfills the average dwell-time constraint in (30) with \( N_0 \) being an arbitrary positive number and \( \tau_a \) sufficiently large according to

\[
\tau_a > \frac{\ln(\kappa_1)}{\kappa_2}, \quad i = 0, 1, 2, \ldots
\]

(31)

where \( \kappa_1 \) and \( \kappa_2 \) are two constants defined as follows

\[
\kappa_1 = \sup_{1 \leq i, j \leq N} \left\| P^{1/2}E_{\lambda(t)}^\perp V(t, t^-) \right\|^2, \quad \kappa_2 = \inf_{1 \leq i, j \leq N} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P)}
\]

(32)

\[
Q_i = -(A_{q,i} - L_qC_q)^T P - P (A_{q,i} - L_qC_q),
\]

(33)

Proof: By assumption, for suitably chosen observer matrix gains \( L_1, \ldots, L_N \) there exists a common solution \( P > 0 \) to the system (21) of Lyapunov equations. Then, the Lyapunov function \( V(t) = \hat{e}_1^T(t)P\hat{e}_1(t) \) fulfills the next inequality for \( t \in [t_i, t_{i+1}] \)

\[
V(t) \leq \exp(-\kappa_2(t_{i+1} - t_i)) V(t_i^+)
\]

(34)

\[
\leq \exp(-\kappa_2(t_{i+1} - t_i)) V(t_i^+)
\]

(35)

The Lyapunov function \( V(t) \) features jumps at the switching instants due to the corresponding jumps of \( \hat{e}_1 \). Thus in general one has that \( V(t_i^+) \neq V(t_i^-) \). Indeed, by well-known properties of positive definite matrices, there is \( P^{1/2} > 0 \) such that \( P = P^{1/2}P^{1/2} \). Hence, from (29), we have

\[
V(t_i^+) \leq \left\| P^{1/2}E_{\lambda(t)}^\perp \hat{e}_1(t_i^-) \right\|^2 \leq \left\| P^{1/2}E_{\lambda(t)}^\perp \hat{e}_1(t_i^-) \right\|^2
\]

(36)

\[
V(t_i^-) \leq \left\| P^{1/2}E_{\lambda(t)}^\perp \hat{e}_1(t_i^-) \right\|^2 V(t_i^-) \leq \kappa_1 V(t_i^-)
\]

(37)

By iterating (37) from \( i = 0 \) to \( i = N_\lambda(t) - 1 \), and taking into account (34), we obtain the following inequality

\[
V(t_i^-) \leq \left\| P^{1/2}E_{\lambda(t)}^\perp \hat{e}_1(t_i^-) \right\|^2 V(t_i^-) \leq \kappa_1 N_{\lambda(t)} \exp(-\kappa_2 t) V(0)
\]

(38)

Since (30) holds by construction, relation (38) yields the next estimation

\[
V(t_i^-) \leq \kappa_1 N_{\lambda(t)} \exp\left(t \frac{\ln(\kappa_1)}{\tau_a} - \kappa_2 t\right) V(0)
\]

(39)

Therefrom, according to (31), inequality (39) implies that \( V(t_i^-) \) converges exponentially to zero as \( t \) tends to infinity, which in turns implies that \( \hat{e}_1 \) converges exponentially to zero.
VI. GENERAL COMMENTS AND IMPLEMENTATION ISSUES

Even if they appear to be quite stringent, assumptions A1 and A2 are necessary and sufficient, also in the case of non-switching dynamics ($N = 1$), to solve the problem of designing an observer whose estimation error tends to zero regardless the presence of unmeasurable, and possibly unbounded, disturbances. Further, the existence of a quadratic Common Lyapunov Function (CLF) as implied by Lemma 1, is, in general, a very stringent requirement in the context of switched systems stability analysis. This limitation is particularly felt when the characteristic matrices of the switching dynamics are fixed and pre-assigned. The problem under investigation in the present paper has, however, the useful peculiarity that the dynamics of the switching systems are not pre-assigned, but can be affected through the choice of the observer gain matrices $L_1, L_2, ..., L_N$. This gives the designer the useful additional degree of freedom of selecting, however not arbitrarily, the characteristic matrices $A_{i,1} - L_i C_i$ ($i = 1, 2, ..., N$) of the switching dynamics (20) or (28). In some cases it might be not possible to find a quadratic CLF even disposing of the observer matrix gains $L_1, L_2, ..., L_N$ as free design parameters. In this case, one might try to use one of the many alternative stability results available for switched linear dynamics, nicely surveyed in [5]. The multiple Lyapunov functions approach, for instance, is known to be able of demonstrating the asymptotic stability of switched systems in some cases where quadratic CLF cannot be found.

Interestingly, it was pointed out in [12] that the condition A1 can be significantly relaxed if one correspondingly relaxes the problem statement in the sense that, instead of aiming at reconstructing the entire state vector $x$, one only wants to reconstruct a certain number of linear combinations between the state variables. More precisely, the approach in [12] allows for the reconstruction of the quantity $K_C x$, with $K_C$ being a rectangular matrix of appropriate dimension. In most observer-based output-feedback control problems, it is enough for control purposes to reconstruct the “controller output” $K_C x$ directly, rather than reconstructing the whole state vector $x$, and then multiplying it by matrix $K_C$.

VII. EXAMPLES

Let us consider an instance of (1) with $n = 5$ states, $p = 3$ outputs, $h = 0$ known inputs (i.e., $B = 0$), $m = 2$ unknown inputs, and $N = 3$ possible modes of operation. The unknown input vector is defined as $w(t) = [w_1(t), w_2(t)]^T$ with $w_1(t) = 2\sin(5t) + 3$ and $w_2(t) = 4\cos(6t) + \sin(0.4t) - 5$. The corresponding matrix triplets $(A_i, E_i, C_i)$ have been selected to satisfy the assumptions A1 and A2.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ & -1 & 0 & 1 & 0 \\ & 1 & -1 & -2 & -1 & -1 \\ & 1 & 1 & 0 & -1 & 1 \\ & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad C_1^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In the first TEST 1, let us consider a system (Model 1) that does satisfy the assumption A3 as well, and whose $E_i$ matrices are:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The LMI system in Lemma 2 has been solved by means of the Matlab LMI toolbox, resulting in a feasible LMI problem. From the corresponding solutions, the following observer gain matrices $L_1, L_2, L_3$ are derived:

$$L_1 = \begin{bmatrix} -7.62 \\ 4.63 \\ 11.62 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 19.14 \\ -7.01 \\ -28.37 \end{bmatrix}, \quad L_3 = \begin{bmatrix} -8.53 \\ 3.07 \\ 14.23 \end{bmatrix}$$

According to Theorem 1, the asymptotic recovery of the system state $x$ is guaranteed regardless of the actual switching sequence, i.e., with no dwell time constraint.

Figure 1 shows the results of TEST 1, namely the time evolutions of the five estimation error $\hat{x}(t) - x(t)$ together with the discrete state $\lambda(t)$.

In the second TEST 2, it is considered a different system (model 2) with the three distribution matrices $E_1, E_2, E_3$ that do not fulfill the assumption A3. They are given by

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving the new LMI system one derives the observer matrices:

$$L_1 = \begin{bmatrix} -0.04 \\ 3.12 \\ 5.97 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2.55 \\ -2.15 \\ -5.61 \end{bmatrix}, \quad L_3 = \begin{bmatrix} -1.26 \\ 1.00 \\ 3.47 \end{bmatrix}$$

The minimal average dwell time $\tau_a$ is now computed. By evaluating the $\kappa_1$ and $\kappa_2$ constants the following values were obtained $\kappa_1 = 1.011, \kappa_2 = 1.296$, yielding the inequality $\tau_a \geq 1.282$. The jump observer described in the Theorem 2 has been implemented by considering a switching sequence that fulfills the previously derived average dwell time constraint, that is reported in the Figure 2. The upper plot of the same figure shows the convergence to zero of the state estimation errors.

In the TEST 3 a different switching sequence which does not fulfill the average dwell time constraint is considered.

Figure 3 shows the resulting unstable behavior of the state
estimation errors. In the TEST 4 it is considered the same switching sequence as in the TEST 2, but the reset equations (27) are not implemented in the observer. The state estimation error does not tend to zero as shown in the Figure 4, which confirms the critical importance of (27) in guaranteeing the observer convergence.

VIII. CONCLUSIONS

The observer design for some classes of switched linear systems with unknown inputs has been addressed under the main structural assumption of strong detectability for the subsystems of the switching dynamics. A couple of reduced-order unknown-input switched observer are proposed, one of which containing jumps in the observer system, that are designed by an LMI based procedure. The convergence of the state estimate has been proven to be exponential in both cases. Designing an estimator for the unknown inputs is a possible future work. Further, studying a combined observer/controller design problem by using the proposed methodology is another task for future work.

REFERENCES