Abstract—This manuscript tackles the regulation problem of linear time invariant systems with unmatched perturbations. A high order sliding mode observer is used allowing theoretically exact state and perturbation estimation. A compensation control approach based on the identified perturbation values is proposed ensuring exact regulation of the unmatched states. A simulation example shows the feasibility of this approach.

I. INTRODUCTION

Motivation. Control under heavy uncertainties is one of the main problems of modern control theory. One of the most prospering control strategies insensitive w.r.t. uncertainties is the Sliding-mode control (SMC) (see, e.g., [1]). This robust technique is well known for its ability to withstand external disturbances and model uncertainties satisfying the matching condition. This condition is presented when the perturbation or parameters variations are implicit at the input channels, for example in the case of completely actuated systems.

The SMC design methodology involves two distinct stages: the design of a switching function which provides desirable system performance in the sliding mode and the design of the control law which will ensure that the system states are driven to the sliding manifold and thus the desired performance is attained and maintained in spite of the matched uncertainties. Nevertheless, there are some disadvantages: the necessity to measure the whole state and the lack of robustness against unmatched uncertainties of the resulting controller.

A possible solution to overcome the full state requirement is to use an observer to estimate the state, while on the other hand, to address the issue of robustness against the unmatched perturbation, the main solution has been the combination of sliding mode technique with other robust strategies. In order to reduce the effects of the unmatched uncertainties, a method that combines $H_{\infty}$ and integral sliding mode control is proposed in [2]. The main idea is to choose such a projection matrix, ensuring not only that unmatched perturbations are not amplified, but even more, that its effects are minimized. In [6] the linear time-varying system with unmatched disturbances is replaced by a finite set of dynamic models such that each one describes a particular uncertain case then, applying a min-max SMC they develop an optimal robust sliding-surface design. A new control scheme, based on block control and quasi-continuous HOSM techniques, is proposed in [7] for control of nonlinear systems with unmatched perturbations, this method assures exact finite time tracking using only output information.

Aim of the paper. A new methodology to compensate the unmatched uncertainties, while simultaneously stabilizing the underactuated dynamics is suggested here. We propose an output sliding mode type approach based on the estimation of states and the identification of perturbations.

Contribution. In this paper a robust output control law is designed to reject the unmatched uncertainties and stabilize the underactuated dynamics using a high order sliding mode observer to reconstruct the states and perturbations in finite time.

The proposed control law stabilizes the underactuated dynamics compensating the perturbations. At the same time, it is guaranteed that the trajectories of the remaining states are bounded. In order to achieve this:

- A high order sliding mode observer is designed to estimate and identify the unknown inputs achieving better precision in comparison with similar schemes [3] and [5].
- A sliding manifold is designed such that the system’s motion along the manifold meets the specified performance: the regulation of the non-actuated states and the rejection of unmatched uncertainties.
- A discontinuous control law is designed such that the system’s state is driven toward the manifold and stays there for all future time, regardless of disturbances and uncertainties.

Paper Structure. In Section II the problem formulation and control challenge are presented. The high order sliding mode observer is introduced in Section III as well as the perturbations identification algorithm. In Section IV an output sliding mode controller rejecting the unmatched uncertainty is presented. A simulation example illustrates the performance of the robust exact unmatched uncertainties compensation controller in Section V.

II. PROBLEM STATEMENT

Let us consider a linear time invariant system with unknown inputs

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw(t), \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p (1 \leq p < n)$ are the state vector, the control and the output of the
system, respectively. The unknown inputs are represented by the vector \( w(t) \in \mathbb{R}^q \), and \( \text{rank} C = p \) and \( \text{rank} B = m \).

Thus, throughout the paper the following conditions are assumed to be fulfilled.

A1. The \((A,B)\) pair is assumed to be controllable.

A2. For \( u = 0 \), the system is strongly observable, or equivalently \((A,C,D)\) has no invariant zeros.

A3. \( w(t) \) has successive derivatives up to the order \( \alpha \) bounded by the same constant \( w^+ \), i.e. \( \|w^{(\alpha+1)}(t)\| \leq w^+ \) for all \( t \geq 0 \).

Here \( \|\cdot\| \) is understood as the vector Euclidean norm.

Now, let us transform the system into a suitable canonical form. In this form, the system is decomposed into two connected subsystems. Consider an invertible matrix of elementary row operations \( T \in \mathbb{R}^{m \times n} \)

\[
T = \begin{bmatrix} B_1 & \vdots & B_n \end{bmatrix}
\]

such that

\[
TB = \begin{bmatrix} 0 & I \end{bmatrix}
\]

where \( I \in \mathbb{R}^{m \times m} \). By using the coordinate transformation \( x \leftrightarrow Tx \). The states are partitioned such that \( x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \). Applying the transformation to system (1) yields

\[
\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + D_1w(t) \quad (3)
\]

\[
\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + D_2w(t) + u(t) \quad (4)
\]

where \( x_1 \in \mathbb{R}^{n-m} \), \( x_2 \in \mathbb{R}^m \), \( D_1 \in \mathbb{R}^{(n-m) \times q} \), \( D_2 \in \mathbb{R}^{m \times q} \).

Control Goal. The objective of this work is to design a controller to regulate the perturbed non-actuated subsystem (3).

Before designing the control law it is necessary to estimate the state and identify the perturbations, next a brief observation procedure description is given.

### III. OBSERVER DESIGN

Now, to realize the observation of the state, let us introduce a high order sliding mode (HOSM) observer. The HOSM observer provides the exact value of the state vector and the unknown inputs identification in a finite time. Basically, the observer works in two stages: first, a linear observer is used to maintain the estimation error between a linear observer and the original state bounded; then, by means of a differentiation scheme, the state vector and unknown inputs identification vector are found. Below a general description of the observer is given, for more details see [4].

Before introducing HOSMO, let us define the following notation. For any matrix \( X \in \mathbb{R}^{r \times c} \) having rank \( X = h \), \( X^\perp \in \mathbb{R}^{r-h \times c} \) represents one of the matrices fulfilling \( X^\perp X = 0 \) and rank \( X^\perp = r - h \). Let \( f(t) \) be a vector function, \( f^{(i)} \) represents the \( i \)-th anti-derivative of \( f(t) \), i.e. \( f^{(i)}(t) = \int_0^{\tau_1} f^{(i-1)}(\tau_2) d\tau_2 \cdots d\tau_1 \).

Thus, throughout the paper the following conditions are assumed to be fulfilled.

Stage I: Let us design a linear observer in order to bound the observation error.

\[
\tilde{x}(t) = A\tilde{x}(t) + Bu(t) + L(y(t) - \tilde{y}(t))
\]

where \( \tilde{y}(t) = C\tilde{x}(t) \) and \( L \) must be designed such that the matrix \( \tilde{A} := (A - LC) \) is Hurwitz. Let \( e(t) := x(t) - \tilde{x}(t) \). Thus, \( e(t) \) converges to a ball of known radius in a finite time \( T_e \), such that

\[
\|e(t)\| \leq e^+ \quad \text{for all} \quad t > T_e \quad (5)
\]

Stage II: This part of the state reconstruction is based on an algorithm that allows decoupling the unknown inputs from the successive derivatives of the output of the linear estimation error system \( y_e = y - \tilde{y} \).

1. Define \( M_1 := C \).

2. Derive a linear combination of the output \( y_e := y - \tilde{y} \), ensuring that the derivative of this combination is unaffected by the uncertainties, i.e., \( \frac{d}{dt} (M_1D)^{\perp} y_e(t) = (M_1D)^{\perp} C\tilde{A}e(t) \), and construct the extended vector

\[
\begin{bmatrix}
\frac{d}{dt} (M_1D)^{\perp} y_e(t) \\
M_2
\end{bmatrix} = \begin{bmatrix}
(M_1D)^{\perp} C\tilde{A}e(t) \\
M_2
\end{bmatrix}
\]

rearranging the terms, with \( I_p \in \mathbb{R}^p \) and \( J_1 := (M_1D)^{\perp} \), the following equation is obtained

\[
M_2 e(t) = \frac{d}{dt} \begin{bmatrix}
J_1 & 0 \\
0 & I_p
\end{bmatrix} \begin{bmatrix}
y_e(t) \\
y_e(t)
\end{bmatrix}
\]

j. Step \( j \) can be summarized as follows. Derive a linear combination of the entries of the vector \( M_{j-1}e(t) \) that are unaffected by the uncertainties, that is, \( \frac{d}{dt} (M_{j-1}D)^{\perp} M_{j-1}e(t) = (M_{j-1}D)^{\perp} M_{j-1}\tilde{A}e(t) \). Then, form the extended vector

\[
\begin{bmatrix}
\frac{d}{dt} (M_{j-1}D)^{\perp} M_{j-1}e(t) \\
M_2
\end{bmatrix} = \begin{bmatrix}
(M_{j-1}D)^{\perp} M_{j-1}\tilde{A}e(t) \\
M_2
\end{bmatrix}
\]

moving out the differentiation operator, the identity

\[
M_j e(t) = \frac{d^{j-1}}{dt^{j-1}} \begin{bmatrix}
J & 0 \\
0 & I_p
\end{bmatrix} Y^{(j-1)}
\]

holds, where \( J := (M_{j-1}D)^{\perp} \begin{bmatrix}
J_{j-2} & 0 \\
0 & I_p
\end{bmatrix} \).

Under A3 there exists a matrix \( M_k \) \( (k \leq n) \), generated recursively by (6), that satisfies the condition \( \text{rank} M_k = n \) (see, e.g., [18]). This means that the algebraic equation

\[
M_k e(t) = \frac{d^{k-1}}{dt^{k-1}} \begin{bmatrix}
J & 0 \\
0 & I_p
\end{bmatrix} Y^{(k-1)}
\]
has a unique solution for \( e(t) \). Such solution could be found by means of the pre-multiplication of both sides of the previous equation by \( M_k^+ := (M_k^T M_k)^{-1} M_k^T \). That is

\[
e(t) = \frac{d^{k-1}}{dt^{k-1}} M_k^+ \left[ \begin{array}{c} J_{k-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \right] Y^{[k-1]}(t)
\]

(7)

From the above expression, the reconstruction of \( x(t) \) is equivalent to the reconstruction of \( e(t) \), which can be carried out by a linear combination of the output \( y_e \) and its \((k - 1) - th\) derivatives. Hence, a real high order sliding mode differentiator will be used in order to provide the theoretically exact observation and unknown inputs identification.

The statement A3 allows realizing a \((\alpha + k - 1) - th\) order sliding mode differentiator, which is the highest order we can construct for this case. The HOSM differentiator is given by

\[
\begin{align*}
\dot{z}_0 &= \lambda_0 \Gamma \frac{d}{dt} |z_0 - H(t)| \frac{d}{dt} (z_0 - H(t)) + z_1 \\
\dot{z}_1 &= \lambda_1 \Gamma \frac{d}{dt} |z_1 - z_0| \frac{d}{dt} (z_1 - z_0) + z_2 \\
&\quad \vdots \\
\dot{z}_{i-1} &= \lambda_{i-1} \Gamma \frac{d}{dt} |z_{i-1} - z_{i-2}| \frac{d}{dt} (z_{i-1} - z_{i-2}) + z_i \\
\dot{z}_i &= \lambda_i \Gamma (z_i - z_{i-1})
\end{align*}
\]

(8)

The observer order is \( i = \alpha + k - 1 \), the values of the \( \lambda_i \)'s can be calculated as in [17], \( \Gamma \) is a Lipschitz constant of \( H^{(\alpha + k)}(t) \), which for our case can be calculated in the following way: starting from the fact that \( z^{k-1} = e(t) \) remains bounded by (5), the next derivative \( z^k = \dot{e}(t) \), will be also bounded \( \| \dot{e}(t) \| \leq \| A - LC \| e^+ + \| B \| w^+ \). In general \( e^\alpha(t) \) can be represented as a linear combination of \( \{ e^k, e^{k+1}, \ldots, e^{\alpha-1}, \dot{w}, \ldots, w^\alpha \} \) and it can be verified that

\[
\Gamma \geq \| A - LC \| e^+ + \sum_{j=0}^{\alpha-1} \| A - LC \|^{j+1} \| B \| w^+
\]

(9)

A. State variable observation

In [17] it was shown that with the proper choice of the constants \( \lambda_i \), there is a finite time \( T \) such that the identity \( z_j(t) = \frac{d^j}{dt^j} H(t) \) is achieved for every \( j = 0, \ldots, \alpha + k - 1 \). The vector \( e(t) \) can be reconstructed from the \((k - 1) - th\) order sliding dynamics. Thus, we achieve the identity \( z_{k-1}(t) = e(t) \), and consequently

\[
\hat{x}(t) := z_{k-1}(t) + \hat{x}(t) \quad \text{for all} \quad t \geq T
\]

where \( \hat{x} \) represents the estimated value of \( x \). Therefore, the identity \( \hat{x}(t) \equiv x(t) \), for all \( t \geq T \) is achieved.

B. Uncertainties Identification

Now, consider the system error dynamics

\[
\dot{e}(t) = (A - LC) e(t) + B w(t)
\]

(10)

We can recover \( \dot{e}(t) \) from the HOSM differentiator (8) in finite time, the equality \( z_k(t) = \dot{e}(t) \) is achieved for all \( t \geq T \) and the next equation holds

\[
\dot{w}(t) = -B^+ [(A - LC) z^{k-1}(t) - z^k]
\]

(11)

IV. ROBUST CONTROL DESIGN

The proposed control law is

\[
u(t) = -\rho(x) \frac{s(t)}{\| s(t) \|}
\]

(12)

where the proposed switching function is designed as

\[
s(t) = K x_1(t) + x_2(t) + G \dot{w}
\]

(13)

The matrix \( K \in \mathbb{R}^{m \times (n-m)} \) could be designed to prescribe the required performance of the reduced-order system. The term \( G \dot{w} \) is added to compensate the unmatched uncertainties.

First, it is necessary guarantee that the proposed control law (12) induces a sliding motion despite the presence of the uncertainties.

A. Ideal Sliding Mode Design

From (13), the time derivative of \( s(t) \) is given by

\[
\dot{s}(t) = \Phi x + (KD_1 + D_2) w + G \dot{w} + u(t)
\]

(14)

where matrix \( \Phi \in \mathbb{R}^{m \times n} \) is defined as \( \Phi := \left[ KA_{11} + A_{21} \right. \left. K A_{12} + A_{22} \right] \).

Choosing a Lyapunov candidate \( V(s) = \frac{s^T s}{2} \) and taking its derivative along the time yields:

\[
\dot{V}(s) = s^T \left( \Phi x + (KD_1 + D_2) w + G \dot{w} - \rho(x) \frac{s}{\| s \|} \right)
\]

\[
\leq -\| s \| (\rho(x) - \| \Phi \| \| x \| - \phi)
\]

(15)

the scalar gain \( \rho(x) \) satisfies the condition

\[
\rho(x) - \| \Phi \| \| x \| - \phi \geq \zeta > 0
\]

where \( \zeta \) is a constant and \( \phi := \| (KD_1 + D_2) \| w^+ + \| G \| w^+ \).

\[
\rho(x) > \| \Phi \| \| x \| + \phi + \zeta
\]

(16)

Combining inequalities (15) and (16), it follows that the derivative of the Lyapunov function satisfies \( \dot{V}(s) \leq -\zeta \dot{V}^2 \).

Gain \( \rho(x) \) will induce the sliding motion.

B. Sliding mode dynamics

The equation representing the motion when confined to the sliding surface is obtained when \( s(t) = 0 \). When the system reaches the sliding surface \( s = 0 \), we have

\[
x_2 = -K x_1 - G \dot{w}
\]

(17)

\[
\dot{x}_1 = (A_{11} - A_{12} K) x_1 - A_{12} G \dot{w} + D_1 w
\]

(18)

As \((A, B)\) pair is controllable, it is well known that \((A_{11}, A_{12})\) pair will be controllable [11] so that, it is possible to design a matrix \( K \) in order to matrix \( A_2 \triangleq (A_{11} - A_{12} K) \) has stable eigenvalues. The \( G \) gain matrix should be selected in order to compensate the unmatched uncertainties. In order to compensate \( w \) from \( x_1 \), matrix \( D_1 \) must be matched with respect to \( A_{12} \); therefore, it will be assumed that:
A4. \( D_1 \in \text{span} (A_{12}) \)
Then there is a matrix \( G \in \mathbb{R}^{m \times p} \) such that
\[
A_{12}G = D_1 \quad (19)
\]
Then the equation (18) yields
\[
\dot{x}_1 (t) = (A_{11} - A_{12}K) x_1 (t) + D_1 (w - \hat{w}) \quad (20)
\]
so, in the ideal case,
\[
\dot{x}_1 (t) = A_s x_1 (t) \quad (21)
\]
In particular, when \( \text{rank}(A_{12}) = n - m \), matrix \( G = A_{12}^{+} D_1 \) where \( A_{12}^{+} \) is understood as the right inverse of \( A_{12} \), that is \( A_{12}^{+} A_{12} = A_{12} A_{12}^{+} = I \).
Since the eigenvalues of \( A_s \) have negative real part, equation (21) is exponentially stable. So, the unmatched uncertainties are compensated and coordinate \( x_1 \) is stabilized. The trajectories of the state \( x_1 \) will converge to a bounded region, i.e. there exist some constants \( a_1, a_2 > 0 \) such that
\[
\| x_1 (t) \| \leq a_1 \| x_1 (0) \| \exp^{-a_2 t} \quad \forall t > t_\sigma
\]
Furthermore, \( x_2 \) is bounded as well indeed during sliding motion. Taking the norm of equation (17) we have
\[
\| x_2 (t) \| \leq \| K \| \| x_1 (t) \| + \| G \| w^+ \quad \forall t > t_\sigma \quad (22)
\]
From the above equation, it is clear that the trajectories of \( x_2 \) are bounded.

V. SIMULATION EXAMPLE
Here, we present an academic example in order to show the feasibility of the proposed methodology. Consider the system
\[
\begin{pmatrix}
0 & 0 & 1 & 0.25 \\
0 & 1 & 0.25 & 0 \\
0 & -4 & 0.25 & 0 \\
1 & 0.25 & 0 & -0.5
\end{pmatrix}
\begin{bmatrix}
x \\ u \\
w \\ y
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-1 \\
0 \\
-1
\end{bmatrix} w = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} x \quad (23)
\]
where the unknown input \( w = 0.5 \sin (3t) + 0.3 \).

Observer design. It can be verified that system (23)-(24) does not have invariant zeros. The Luemberger-type observer is designed such that matrix \( A - LC \) has a set of eigenvalues given by \(-1, -2, -3, -4\). For triplet \((A, C, D)\), \( k = 2 \), i.e. we need to derive two times in order to reconstruct \( x \) and \( w \). The \( M_k \) matrices are:
\[
M_1 = C; \quad M_2 = \begin{bmatrix}
-3.5 & 0 & 0 & 0.25 \\
0 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]
The STA gains are \( \alpha = 5, \lambda = 2 \). The identification process has parameters \( \Gamma = 3.11, \Lambda = 1.45 \). The sampling step is \( \delta = 10(\mu s) \). First, transform the system (23) to the canonical form (3)-(4) with:
\[
T = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]
It is clear from the above equation that there are matched and unmatched perturbations. Selecting the sliding mode controller gain as suggested in (16) we have \( \rho (x) = 20 \| x (t) \| + 2 \). For this example matrix \( K \) was selected using the quadratic minimization approach [1], such that the reduced order system has a pair of complex eigenvalues \(-1.34 \pm i1.7015 \).

The simulation was carried comparing a conventional sliding mode controller design using a \( s = x_1 + K x_2 \) surface against the methodology proposed in this manuscript. Fig. (1) shows the states of the regularized system, column (A) shows the results when no compensation is carried: the perturbation effects are present in all the states. The column (B) shows the states when the compensation of unmatched uncertainties is done through the sliding surface, here the stabilization of state \( x_1 \) (solid-line plot) is achieved, while the trajectories of state \( x_2 \) (dotted-line) remains bounded.
VI. CONCLUSIONS

A output regulation for linear systems with unmatched uncertainties was presented. Based on the exact reconstruction of the unknown inputs (unmatched uncertainties), we propose designing a sliding mode control which allows compensating the uncertainties from the non actuated system dynamics and maintains the trajectories of the remained states bounded.

REFERENCES