MINIMAL SETS ON GRAPHS AND DENDRITES
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Abstract. The topological structure of minimal sets of continuous maps on graphs, dendrites and dendroids is studied. A full characterization of minimal sets on graphs and a partial characterization of minimal sets on dendrites are given. An example of a minimal set containing an interval on a dendroid is given.

1. Introduction

Let \((X, f)\) be a dynamical system given by a compact topological space and a continuous map \(f : X \to X\) (we also write \(f \in C(X)\)). A subset \(M\) of \(X\) is a (topologically) minimal set if \(M\) is nonempty, closed and invariant (i.e. \(f(M) \subseteq M\)) and if \(M\) has no proper subset with these three properties. Note that a nonempty closed set \(M \subseteq X\) is minimal if and only if the orbit of every point from \(M\) is dense in \(M\). A dynamical system \((X, f)\) is called minimal if the set \(X\) is minimal. In such a case we also say that the map \(f\) itself is minimal.

When we study a dynamical system \((X, f)\) we try, among others, to find its periodic orbits and, more generally, its minimal sets. It is therefore useful to know how minimal sets of continuous maps \(X \to X\) can look like in the sense of their topological structure. Of course, the answer depends on \(X\).

For instance, if \(X\) is a compact Hausdorff zero-dimensional space (equivalently, a compact Hausdorff totally disconnected space) it is a folklore that if \(f \in C(X)\) and \(M \subseteq X\) is its minimal set then \(M\) is either a finite set (in fact, a periodic orbit of \(f\)) or a Cantor set, i.e. a compact, perfect and totally disconnected set. Conversely, whenever \(M \subseteq X\) is a finite or a Cantor set then there is an \(f \in C(X)\) such that \(M\) is a minimal set of \(f\). To verify the last statement, take into account that any closed set \(M\) in a compact zero-dimensional space \(X\) is its retract. So the case when \(M\) is finite is trivial and if \(M\) is a Cantor set, realize that any two Cantor sets are homeomorphic and examples of minimal maps on Cantor sets on real line are well-known (see e.g. [3, pp. 133–134]).

Among one-dimensional spaces, the characterization of minimal sets is well known on the interval — these are periodic orbits and Cantor sets (see [3, pp. 92–93]). Further, it is a folklore that on the circle there is the third possibility for a minimal set — the whole circle itself.

In the present paper we study minimal sets on graphs and dendrites.

The interest in studying graph maps is due to the fact that for maps on manifolds with an invariant foliation of codimension one, the corresponding quotient map turns out to be defined in general on a graph. Furthermore, the dynamics of pseudo-Anosov homeomorphisms on a surface can be essentially reduced to the analysis of some special graph maps (see e.g. [6]). Finally, a graph map sometimes

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imitates the behaviour of a smooth map (flow) in a neighbourhood of a hyperbolic attractor (see for instance [11]).

Recent interest in dynamics on dendrites is motivated by the fact that dendrites appear as Julia sets in complex dynamics (see [10] or [2]).

Our main results are a full characterization of minimal sets on graphs (Theorem 3.1) and a partial characterization of minimal sets on dendrites (Theorem 4.1).

2. Definitions and auxiliary results

We adopt the terminology from [9]. A continuum is a nonempty, compact, connected metric space. An arc is any space which is homeomorphic to the closed interval [0,1]. A graph is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their endpoints (i.e., it is a one-dimensional compact connected polyhedron). A Peano continuum is a continuum which is locally connected at every point. Equivalently, it is a continuum which is a continuous image of the interval [0,1]. A dendrite is a Peano continuum which contains no simple closed curve. Every subcontinuum of a dendrite is a dendrite. Every dendrite has the fixed point property.

A topological space is arcwise connected if any two of its points can be joined by an arc. A connected topological space X is called unicoherent provided that whenever A and B are closed, connected subsets of X such that X = A ∪ B, then A ∩ B is connected. A connected topological space is hereditarily unicoherent provided that each of its closed, connected subsets is unicoherent. A dendroid is an arcwise connected, hereditarily unicoherent continuum. Every dendrite is a dendroid.

Further recall that for a given integer \( n \geq 3 \), a simple n-od is a space which is homeomorphic to the cone over an n-point discrete space (equivalently, homeomorphic to the set \( \{ z \in \mathbb{C} \colon z^n \in [0,1] \} \) with the standard topology). If \( Z \) is a simple n-od, then the unique point of \( Z \) which is of order \( \geq 3 \) in \( Z \) is called the vertex of \( Z \). A simple 3-od we also call a simple triod.

Lemma 1. [8] Let \( X \) be a compact Hausdorff space and \( f \in C(X) \). Assume that there is an open set \( B \neq \emptyset \) in \( X \) with \( f(B) \subseteq f(X \setminus B) \). Then \( f \) is not minimal.

Recall that a map \( f : X \to X \) is called feebly open if for every nonempty open subset \( U \) of \( X \), the set \( f(U) \) has nonempty interior (notice that then \( f(U) \) cannot be a singleton provided \( X \) has no isolated point).

Lemma 2. [8] Let \( X \) be a compact Hausdorff space and let \( f \in C(X) \) be a minimal map. Then \( f \) is feebly open.

Lemma 3. Let \( G \) be a graph, \( f \in C(G) \), \( M \) a minimal set of \( f \). Then \( M \) does not contain any simple triod.

Proof. If \( M \) contains an isolated point then \( M \) is just the periodic orbit of this point and there is nothing to prove. So assume that \( M \) is dense in itself.

Suppose that \( Y \subseteq M \) is a simple triod. Denote the branching point of \( Y \) by \( b \) and put \( g = f|_M \). We are going to show that there exists a set \( B \subseteq M \), open in \( M \), such that \( g(B) \subseteq g(M \setminus B) \) which will contradict the minimality of \( g \) by Lemma 2.1.

For any \( n \in \mathbb{N} \), \( g^n(Y) \) is a subgraph of \( G \) (nondegenerate because \( Y \) has nonempty interior and \( g \) is feebly open) therefore any branching point of \( g^n(Y) \) is also a branching point of \( G \). The orbit of \( b \) is dense in \( M \), so it cannot be a subset of the set of branching points of \( G \). Let \( k \) be the first positive integer such that \( g^k(b) \) is not a branching point of \( g^k(Y) \). Then there is a closed neighbourhood of \( g^k(b) \) in \( g^k(Y) \) such that it is an arc (\( g^k(b) \) being either its endpoint or an interior point) and
$g^{k-1}(b)$ is a branching point of $g^{k-1}(Y)$. Then $g^{k-1}(Y)$ contains (at least) three nonoverlapping arcs $A_1$, $A_2$, $A_3$ which form a simple triod with branching point $g^{k-1}(b)$ and which do not contain any other branching point of $G$. The $g$-image of each of these arcs is due to feeble openness of $g$ nondegenerate and so contains an arc with the endpoint $g^k(b)$. Hence there are two of the three arcs, say $A_1$ and $A_2$, such that $g(A_1)$ and $g(A_2)$ overlap, i.e. there is an arc $C \subseteq g(A_1) \cap g(A_2)$. Then we can find a set $B \subseteq A_1$ which is open in $M$ and $g(B) \subseteq g(A_2) \subseteq g(M \setminus B)$. □

Lemma 4. (Extension lemma) Let $X$ be a graph or a dendrite, $M \subseteq X$ closed, and $f : M \to X$ continuous. Then there is a map $F \in C(X)$ which is an extension of $f$.

Proof. Graph case. First, let $X$ be a graph. Put $BE = \{x \in X; x$ is a branching point or an endpoint of $X\}$ and $N = M \cup BE$. Now define $g : N \to X$ as follows: $g(x) = f(x)$ if $x \in M$ and $g(x) = x$ if $x \in BE \setminus M$. Obviously, $g$ is continuous. The set $X \setminus N$ is a union of countably many open arcs whose endpoints belong to $N$. By an open arc we mean here and below always an arc without its endpoints which is simultaneously an open set in $X$, its endpoints belong to $N$ and which is disjoint with $N$.

Define $F : X \to X$ as follows. If $x \in N$ put $F(x) = g(x)$. If $x \in A$ where $A$ is an open arc with endpoints $a, b \in N$ (note that $A$ is an open set in $X$), consider all arcs in $X$ whose endpoints are $F(a), F(b)$. There are only finitely many such arcs. Let $B$ be one with minimal diameter. Denote by $h$ a homeomorphism $A \to B$ sending $a$ to $F(a)$ and $b$ to $F(b)$. Finally put $F(x) = h(x)$ on $A$.

We need to prove that $F$ is continuous. Let $x \in X \setminus N$. Since $X \setminus N$ is open in $X$, from the definition of $F$ on $X \setminus N$ we immediately obtain that $F$ is continuous at $x$.

Now let $x \in N$. Take an arbitrary $\varepsilon > 0$. To prove the continuity of $F$ at $x$ we are going to find a neighbourhood $U$ of $x$ with $F(U) \subseteq B = B(F(x), \varepsilon)$. Due to the local arcwise connectivity of the graph $X$ we can find an arcwise connected neighbourhood $V$ of $F(x)$ such that $V \subseteq B(F(x), \varepsilon/3)$. Note that then any two points from $V$ can be joined by an arc with diameter less than $2\varepsilon/3$ and therefore any arc joining these two points and having minimal diameter lies in $B$. Further, since $X$ is a graph, there exists a closed neighbourhood $W$ of $x$ which is an arc or a simple $n$-od with $x$ as its vertex (see [9], p. 160). Thus, for some $k \geq 1$ we have $W = \bigcup_{i=1}^{k} I_i$ where, for any $i$, $I_i$ is an arc containing $x$ as its endpoint and, for any $i \neq j$, $I_i \cap I_j = \{x\}$. For any $i$ denote $I_i^* = I_i \setminus \{x\}$ and fix $i \in \{1, \ldots, k\}$. If $x \notin \text{clo}(I_i^* \cap N)$ let $J_i \subseteq I_i$ be an arc such that $x$ is its endpoint and $F(J_i) \subseteq V$ (such an arc $J_i$ exists because of continuity of $F|_A$ for any open arc $A \subseteq X \setminus N$). If $x \in \text{clo}(I_i^* \cap N)$ let $J_i \subseteq I_i$ be an arc such that one of its endpoints is $x$, the other endpoint is a point from $N$ and $F(J_i \cap N) \subseteq V$ (such an arc $J_i$ exists because of continuity of $F|_N = g$). Then $U = \bigcup_{i=1}^{k} J_i$ is a closed neighbourhood of the point $x$. To prove that $F(U) \subseteq B$ we need only to show that if $i \in \{1, \ldots, k\}$ and $P \subseteq J_i$ is an arc with endpoints $p, q \in N$ and containing no other point from $N$ then $F(P) \subseteq B$. But this is easy to see. In fact, by the definition of $F$, $F(P)$ is an arc whose endpoints are $F(p), F(q)$ and has minimal diameter among all such arcs. But $F(P), F(q) \in V$ and so, by what was said above, the arc $F(P)$ lies in $B$. This finishes the proof of continuity of $F$.

Dendrite case. Now, let $X$ be a dendrite. Then $X$ can be embedded into the square $I^2$ where $I = [0, 1]$. So, without loss of generality we may assume that $X \subseteq I^2$. Then for $x \in M$ we can write $f(x) = (f_1(x), f_2(x))$ where $f_1, f_2 : M \to I$ are continuous. By Tietze Theorem we can continuously extend $f_1, f_2$ to $F_1, F_2 : X \to I$ and we obtain a continuous map $G = (F_1, F_2) : X \to I^2$ such that $G|_M = f$.  

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3. Characterization of Minimal Sets on Graphs

**Theorem 1.** Let $G$ be a graph, $f \in C(G)$, and $M \subseteq G$ a minimal set of $f$. Then $M$ is

(i) a finite set (in fact a periodic orbit of $f$), or
(ii) a Cantor set, or
(iii) a union of (finitely many) pairwise disjoint circles.

Conversely, whenever $M \subseteq G$ is of one of the above forms then there is a map $f \in C(G)$ such that $M$ is a minimal set of $f$.

**Remark.** In (i) and (iii), $f|_M$ is a homeomorphism. In (ii), $f|_M$ may or may not be a homeomorphism.

**Proof.** $M$ is closed. If $M$ contains a point isolated in $M$ then, from the minimality of $M$, we have that this point has to be periodic and, hence, $M$ is just the periodic orbit of this point.

So, suppose that every point of $M$ is its limit point, i.e. $M$ is perfect. If $M$ is nowhere dense in $G$ then $M$ is a Cantor set.

Finally let $M$ be a perfect set which is dense in some open set in $G$. Then $M$, being closed, contains an arc and hence also an arc $A$ which does not contain any branching point of the graph $G$. Considering a point from $A$ and taking into account that its orbit has to hit $A$ infinitely many times we get that $f^k(A) \cap A \neq \emptyset$ for some $k \geq 1$. Then the sets $A_j = \bigcup_{r=0}^{\infty} f^{rk+j}(A)$, $j = 0, 1, \ldots, k-1$ are connected and their union is $\text{Orb}_f(A)$ which is a dense subset of $M$. Then $M = \text{clos} A_0 \cup \cdots \cup \text{clos} A_{k-1}$. Since each of the sets $\text{clos} A_j$ is compact and connected, $M = K_1 \cup \cdots \cup K_p$ where $1 \leq p \leq k$ and the sets $K_i$ are compact, connected, pairwise disjoint (hence components of $M$) and nondegenerate (note that $M$ is perfect and $f|_M$ is feebly open). We are going to show that each $K_i$ is a circle.

Suppose that some $K_i$ is not a circle. Due to Lemma 2.3 we have that $K_i$ is an arc. But then $f|_{K_i}: K_i \to K_i$ has a fixed point $z \in K_i$ and $\text{Orb}_f(z)$ is not dense in $M$ which is a contradiction.

Now, conversely, let $M \subseteq G$ be of the form (i), (ii) or (iii). We are going to show that there exists a continuous minimal map $g: M \to M$. If $M$ is a finite set, this is trivial. If $M$ is a Cantor set, use the facts that there are a homeomorphism $h: M \to C$ where $C$ is the middle third Cantor set and a continuous minimal map $\varphi: C \to C$ and put $g = h^{-1} \circ \varphi \circ h$. Now let $M = K_1 \cup \cdots \cup K_n$ where $K_i$, $i = 1, \ldots, n$ are disjoint circles. For $i = 1, \ldots, n$ let $h_i$ be a homeomorphism $K_i \to K_{i+1(\text{mod} n)}$ such that $h_n \circ \cdots \circ h_1$ is the identity $K_1 \to K_1$. Further, let $\psi: K_1 \to K_1$ be topologically conjugate with an irrational rotation of the circle. Now define $f: M \to M$ as follows. For $i = 1, 2, \ldots, n - 1$ let $f|_{K_i} = h_i$ and $f|_{K_n} = \psi \circ h_n$. Then $f$ is continuous and minimal.

In each of the three cases we have defined a minimal map $g \in C(M)$. To find a continuous extension of $f$ to the whole graph $G$, use Lemma 2A.

**Remark.** Let $G^*$ be a disjoint union of graphs $G_1, G_2, \ldots, G_n$, $f \in C(G^*)$ and $M \subseteq G^*$ a minimal set of $f$. Using the previous theorem it is easy to prove that then

(i) $M$ is a finite set (a periodic orbit of $f$) and every $G_i$ containing points of $M$ contains the same number of them, or

(ii) $M$ is a Cantor set, or
Theorem 2. Let \( M \) be a dendrite, \( f \in C(D) \), and \( M \subseteq D \) a minimal set of \( f \). Then \( M \) is

(i) a finite set (in fact a periodic orbit of \( f \)), or

(ii) a perfect nowhere dense set with uncountably many components, at most countably many of them being nondegenerate (i.e. dendrites).

On the other hand, if \( M \subseteq D \) is a finite set or a Cantor set then it is a minimal set for some map \( f \in C(D) \).

Proof. In the same way as in the proof of Theorem 3.1 we get that \( M \) is either a periodic orbit or a perfect set. Suppose \( M \) is a perfect set. We are going to prove that then it is nowhere dense.

Suppose, on the contrary, that \( M \) is dense in a nonempty open set \( A \) in \( D \). Since \( M \) is closed, it contains \( A \). Therefore, due to the local connectedness of \( D \), \( M \) contains an open connected set \( E \) in \( D \). Since \( E \) is open and \( f \) is minimal, \( f^k(E) \cap E \neq \emptyset \) for some \( k \geq 1 \). Then the set \( S = \text{clos} \left( \bigcup_{k=0}^{\infty} f^k(E) \right) \) is a subcontinuum of \( D \), hence a dendrite. Since \( f^k(S) \subseteq S \subseteq M \) and \( S \) has the fixed point property, \( M \) contains a periodic point. This is a contradiction, since \( M \) is minimal and infinite.

Now we are going to prove that \( M \) has uncountably many components. Decomposition of \( M \) into its components is upper semicontinuous (see [9, Lemma 13.2]). Then the corresponding factor space \( \tilde{M} \) is metrizable (see [9, Theorem 3.9]), compact (it is the image of the compact space \( M \) under the continuous natural projection) and totally disconnected (this follows from the definition of the considered decomposition). The induced map \( \tilde{f} \) on \( \tilde{M} \) is again minimal (this is a property of factors). Hence if \( \tilde{M} \) contains an isolated point then it is finite, otherwise it is a Cantor set.

Now, if \( M \) is a finite set or a Cantor set then we get by Lemma 2.4, similarly as in the graph case, that \( M \) is a minimal set for some map \( f \in C(D) \).

Remark. It is an open problem whether in case (ii) all the components are degenerate, i.e., whether an infinite minimal set on a dendrite is necessarily a Cantor set. We are going to show that on a dendroid this may not be the case. Start with a minimal dynamical system \((M, F)\) such that \( M \subseteq C \times [0, 1] \), \( C \) being a Cantor set on the real line and \( F : M \to M \) is a triangular map, i.e. a map of the form \( F(x, y) = (f(x), g(x, y)) \). Moreover, let \( M \) contain a vertical interval. For examples of such a minimal system see [4], [1, pp. 24–27] or [5]. Now let \( \Delta = (C \times [0, 2]) \cup (\text{conv } C \times \{2\}) \) where \( \text{conv } C \) denotes the convex hull of \( C \). Then \( \Delta \) is a dendroid and \( \Delta \supseteq M \). Similarly as in [7, Lemma 1] one can extend \( F \) to a continuous triangular map \( G : \Delta \to \Delta \). Clearly, \( M \) is a minimal set for \( G \).

Remark. Let \( D^* \) be a disjoint union of dendrites \( D_1, D_2, \ldots, D_n \), \( f \in C(D^*) \) and \( M \subseteq D^* \) a minimal set of \( f \). Using the previous theorem it is easy to prove that then

(i) \( M \) is a finite set (a periodic orbit of \( f \)) and every \( D_i \) containing points of \( M \) contains the same number of them, or
(ii) \( M \) is a perfect nowhere dense set with uncountably many components, at most countably many of them being nondegenerate (i. e. dendrites).

One can also show that if \( M \subseteq D^* \) is of the form described in (i) or is a Cantor set then there is a map \( f \in C(D^*) \) such that \( M \) is a minimal set of \( f \).

References