On Operations Preserving Primitivity of Partial Words with One Hole*

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Abstract

The notion of a primitive word, one that is not a power of another word, is useful in many areas including coding theory, combinatorics on words, formal language theory, and text algorithms. The proportion of such words turns out to be very high. We investigate operations that preserve the primitivity of partial words with one hole, or words that have an undefined position. As a result, all primitive binary partial words with one hole of length up to 11 can be generated.

1 Introduction

The fundamental concept of primitive words plays an important role in several research areas related to codes [2], combinatorics on words [7,12], formal languages [10,14], and text algorithms [8]; a word is primitive if it cannot be written as a power of another word. Since the proportion of words that are primitive is very high, it is expected that various operations do preserve primitivity. Some

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have been investigated such as substituting a letter for another one \[11, 15\], inserting or deleting letters \[15\], applying morphisms \[13, 15\], and taking prefixes \[15\]. In \[9\], Dassow et al. consider the word \(ww'\), where \(w\) is a given primitive word and \(w'\) is a modified copy of \(w\) or a modified mirror image of \(w\) so that the edit distance of \(w\) and \(w'\) is very small (at most 2) or very large (almost the length of \(w\)), and study whether \(ww'\) is primitive. As a result of their study, which is motivated in particular by dynamical systems, all primitive binary words of length at most 11 can be generated as well as a large number with length up to 20.

Motivated by a practical problem on gene comparison, Berstel and Boasson introduced the notion of partial words, which are sequences that may have some holes, or undefined positions, that match every letter of the alphabet \[1\]. (Full) words are just partial words without holes. Primitive partial words were introduced in \[3\]; a partial word is primitive if it is not contained in a non-trivial power of a full word, that is, if there is no way of filling in the holes with letters of the alphabet that produces a word that is not primitive. Testing whether or not a partial word is primitive can be done in linear time in the length of the word \[5\]. Moreover in \[6\], the problem of counting primitive partial words with \(h\) holes of length \(n\) over a \(k\)-letter alphabet was initiated based on a constructive method related to the well-known periodicity result of Fine and Wilf.

In this paper, we investigate operations that preserve primitivity of partial words with one hole. Here, if we begin with a partial word \(w\), having some properties, we can produce primitive partial words which have \(w\) as a prefix. Generally, we make various changes to \(w\) to create \(w'\); the concatenation \(ww'\) ends up being primitive. Using our operations, which extend those of Dassow et al., we can generate all primitive binary partial words with one hole of length up to and including 11, as well as a large number of primitive binary partial words with one hole of length from 12 to 21. Since this is an extended abstract, most of the proofs are omitted due to page restriction.

2 Preliminaries

A partial word of length \(n\) over an alphabet \(A\) can be defined by a function \(u : \{0, \ldots, n - 1\} \to A_\circ\), where \(A_\circ = A \cup \{\circ\}\). For \(0 \leq i < n\), if \(u(i) \in A\), then \(i \in D(u)\); if \(u(i) = \circ\), then \(i \in H(u)\). We treat the \(\circ\)'s, called holes, like wild cards that match any letter of \(A\). A (full) word is a partial word without holes. The set of all words over \(A\) (resp., of length \(n\)) is denoted by \(A^*\) (resp., \(A^n\)). For any partial word \(u\), \(|u|\) denotes its length. A partial word \(v\) is a factor of the partial word \(u\) if there exist \(x, y\) such that \(u = xvy\). The empty word is denoted by \(\varepsilon\). If \(x = \varepsilon\), then \(v\) is a prefix of \(u\); if \(y = \varepsilon\), \(v\) is a suffix of \(u\). The notation \(u[i..j]\) (resp., \(u[i..j]\)) refers to the factor \(u(i) \cdots u(j - 1)\) (resp., \(u(i) \cdots u(j)\)). We write \(\text{rev}(u)\) for the reversal of the partial word \(u\).
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If $u$ and $v$ are two partial words of equal length, then $u$ is contained in $v$, denoted $u \subset v$, if $u(i) = v(i)$ for all $i \in D(u)$; $u$ and $v$ are compatible, denoted $u \uparrow v$, if there exists a partial word $w$ such that $u \subset w$ and $v \subset w$. A partial word $u$ is primitive if there is no $v$ such that $u \subset v^p$ with $p \geq 2$.

The following lemmas are useful throughout the paper.

**Lemma 1.** Let $w$ be a partial word. If, for all words $u$ such that $w \subset u$, $u$ is primitive, then $w$ is primitive.

**Lemma 2.** Let $w, w'$ be partial words. If, for all words $u, u'$ such that $w \subset u$ and $w' \subset u'$, $uw'$ is primitive, then $wu'$ is primitive.

**Lemma 3.** [4, p. 188] Let $u, v$ be non-empty words, let $y, z$ be partial words, and let $w$ be a partial word with one hole satisfying $|w| \geq |u| + |v|$. If $wy \subset u^p$ and $wz \subset v^q$, for some integers $p, q$, then there exists a word $x$ of length not greater than $\gcd(|u|, |v|)$ such that $u = x^k$ and $v = x^l$ for some integers $k, l$.

**Lemma 4.** [4, p. 189] Let $u, v$ be partial words. If there exists a primitive word $x$ such that $wv \subset x^p$ for some positive integer $p$, then a primitive word $y$ exists such that $vy \subset y^p$. Moreover, if $uv$ is primitive, then so is $vu$.

### 3 Generating Primitive Words of Even Length

In this section, we examine operations which produce primitive partial words of even length. First, we deal with operations on the first, middle, and/or last letter. Theorems 9 and 18 of [9] investigate the primitivity of $ww'$, where $w'$ is defined by a substitution operation on the first, middle, and last letter of $w$. We extend them to partial words using a similar proof.

**Theorem 1.** Let $n \geq 5$ be odd, let $w = a_0a_1 \cdots a_{n-1}$ be a primitive partial word of length $n$, and let $h : A \rightarrow A$ be a mapping such that $h(a) \neq a$ for all $a \in A$. Set $i = \frac{n-1}{2}$, $w_1 = a_1 \cdots a_{i-1}$ and $w_2 = a_{i+1} \cdots a_{n-2}$. Then $ww'$ is primitive if one of the following statements holds:

1. $0, \ldots, i, n-1 \in D(w)$ and $w' = h(a_0)w_1h(a_i)w_2h(a_{n-1})$;
2. $0, i, \ldots, n-1 \in D(w)$ and $w' = h(a_0)w_1h(a_i)w_2h(a_{n-1})$;
3. $0, i, \ldots, n-1 \in D(w)$ and $w' = h(a_{n-1})rev(w_2)h(a_i)rev(w_1)h(a_0)$;
4. $0, \ldots, i, n-1 \in D(w)$ and $w' = h(a_{n-1})rev(w_2)h(a_i)rev(w_1)h(a_0)$;
5. $0, \ldots, i, n-1 \in D(w)$ and $w' = h(a_0)w_1h(a_i)w_2a_{n-1}$.
6. \(0, i, \ldots, n - 1 \in D(w)\) and \(w' = h(a_0)w_1h(a_i)w_2a_{n-1}\);

7. \(0, \ldots, i, n - 1 \in D(w)\) and \(w' = a_0w_1h(a_i)w_2h(a_{n-1})\);

8. \(0, i, \ldots, n - 1 \in D(w)\) and \(w' = a_0w_1h(a_i)w_2h(a_{n-1})\).

We now construct \(w'\) by changing the first and last letters of a partial word with one hole, \(w\), and by substituting a letter for the hole in \(w\).

**Theorem 2.** Let \(n \geq 4\), let \(w = a_0a_1 \cdots a_{n-1}\) be a partial word of length \(n\) with one hole at position \(i\), where \(0 < i < n - 1\), and \(h : A \to A\) be a mapping such that \(h(a) \neq a\) for all \(a \in A\). Then \(ww'\) is primitive if \(w' = h(a_0)a_1 \cdots a_{i-1}b_{i+1} \cdots a_{n-2}h(a_{n-1})\), for some \(b \in A\).

**Proof.** Suppose there exist a word \(v\) and an integer \(p \geq 2\) such that \(ww' \subset v^p\). If \(p\) is even, then \(w \subset v^\frac{p}{2}\) and \(w' \subset v^\frac{p}{2}\); but \(w \neq w'\), a contradiction. So \(p\) is odd, and set \(p = 2m + 1\) for some \(m \geq 1\). The length of \(v\) is even, so let \(v = v_1v_2\) where \(|v_1| = |v_2|\). We have \(w \subset v^m v_1\) and \(w' \subset v_2 v^m\). Since \(a_0\) is the first letter of \(v\) and \(a_{n-1}\) is the last letter of \(w\), \(a_0\) is the first letter of \(v_1\) and \(a_{n-1}\) is the last letter of \(v_1\). Likewise, \(h(a_0)\) is the first letter of \(v_2\) and \(h(a_{n-1})\) is the last letter of \(v_2\). Set \(v_1 = a_0v'_1a_{n-1}\) and \(v_2 = h(a_0)v'_2h(a_{n-1})\).

First, suppose \(|v_1| = |v_2| = 1\), so \(a_0 = a_{n-1}\) and \(h(a_0) = h(a_{n-1})\). By substitution, \(w \subset (a_0h(a_0))a_0a_0w' \subset h(a_0)(a_0h(a_0))^m\). Since \(|w| \geq 4\), \(w \subset (a_0h(a_0))^m a_0h(a_0)a_0\) and \(w' \subset h(a_0)a_0h(a_0)(a_0h(a_0))^m - 1\). If \(i = 1\), the third letter of \(w\) is \(a_0\) and the third letter of \(w'\) is \(h(a_0)\), but \(a_0 \neq h(a_0)\). If \(i \geq 2\), the second letter of \(w\) is \(h(a_0)\) and the second letter of \(w'\) is \(a_0\), but \(a_0 \neq h(a_0)\). Now, suppose \(|v_1| = |v_2| \geq 2\). We have

\[
\begin{align*}
w & \subset (a_0v'_1a_{n-1}h(a_0)v'_2h(a_{n-1}))^m a_0v'_1a_{n-1} \\
w' & \subset h(a_0)v'_2h(a_{n-1})(a_0v'_1a_{n-1}h(a_0)v'_2h(a_{n-1}))^m
\end{align*}
\]

Assume \(|v_1| \leq i < n - 1\) (that is, the hole is not in the prefix of length \(|v_1|\) of \(w\)). Since \(|a_0v'_1a_{n-1}| = |h(a_0)v'_2h(a_{n-1})|\), the last letter of each segment coincide. So \(a_{n-1} = h(a_{n-1})\), a contradiction. Finally, assume \(0 < |v_1|\) (that is, the hole is in the prefix of length \(|v_1|\) of \(w\) but not in the suffix of length \(|v_1|\) of \(w\)). Since \(|a_0v'_1a_{n-1}| = |h(a_0)v'_2h(a_{n-1})|\), the first letter of each segment coincide. So \(a_0 = h(a_0)\), a contradiction.

Theorem 3 does reversal and substitution on a word \(w\) of odd length with one hole to create \(w'\). After reversing \(w\), we replace the hole with a letter and substitute for the middle letter to build a primitive word, \(ww'\).
Theorem 3. Let $n \geq 5$ be odd, let $w = a_0a_1 \cdots a_{n-1}$ be a partial word of length $n$ with one hole at position $j$, and let $h : A \rightarrow A$ be a mapping such that $h(a) \neq a$ for all $a \in A$. Set $i = \frac{n-1}{2}$. Then $ww'$ is primitive if one of the following statements holds, where $b \in A$:

1. $0 < j < i$ and $w' = a_{n-1} \cdots a_{i-1}ba_{i+1} \cdots a_{j}a_{j+1}ba_{j-1} \cdots a_0$;
2. $i < j < n-1$ and $w' = a_{n-1} \cdots a_{j-1}a_{j}a_{j+1}h(a_i)a_{i-1} \cdots a_0$;
3. $0 < j < i$ and $w' = a_{n-1} \cdots a_{i-1}a_{i+1}h(a_i)a_{i-1} \cdots a_{j}a_{j+1}ba_{j-1} \cdots a_0$;
4. $i < j < n-1$ and $w' = a_{n-1} \cdots a_{j-1}a_{j}a_{j+1}h(a_i)a_{i-1} \cdots a_0$.

Proof. For Statement 1, suppose there exist a word $v$ and integer $p \geq 2$ such that $ww' \subset v^p$. If $p$ is even, then $w \subset v^p/2$ and $w' \subset v^p/2$. However, since $h(a_i) \neq a_i$, it follows that $w \nparallel w'$, a contradiction. Thus, $p$ is odd and $p \geq 3$, and $|v|$ is even. Let $p = 2m + 1$ for some $m \geq 1$, and set $v = v_1v_2$, where $|v_1| = |v_2|$.

We have $w \subset v^m v_1 = (v_1v_2)^m v_1$ and $w' \subset v_2v^m = v_2(v_1v_2)^m$. Since $i$ is the middle position of $w$ and $|w| = (2m + 1)|v_1|$ is odd, it follows that $|v_1|$ is odd and $i = m|v_1| + \frac{|v_1| - 1}{2}$. In other words, $i$ is the middle position of some $v_1$ or some $v_2$. If $i$ corresponds to the middle position in some $v_1$ with respect to $w$, then it corresponds to the middle position in some $v_2$ with respect to $w'$, and vice versa. Because $w(i) = a_i \neq h(a_i) = w'(i)$, the letters in the middle positions of $v_1$ and $v_2$ are distinct. Since $w \subset (v_1v_2)^m v_1$ and $j < i$, the suffix of length $|v_1|$ of $w$ is $v_1$. Also, since $m \geq 1$, $i < n - |v_1|$. Because $w' \subset v_2(v_1v_2)^m$, the prefix of length $|v_2|$ of $w'$ is $v_2$, and so $v_1 = \text{rev}(v_2)$. However, we have a contradiction since the middle positions of $v_1$ and $v_2$ have different letters. □

Second, we deal with operations involving the first two and last two letters of a word $w$ with one hole.

Theorem 4. Let $n \geq 6$, let $w = a_0a_1 \cdots a_{n-1}$ be a partial word of length $n$ with one hole at position $i$, $0 < i < n-1$, and let $h : A^* \rightarrow A^*$ be a length preserving morphism such that $h(a) \neq a$ for all $a \in A$. Then $ww'$ is primitive if one of the following statements holds for some $b \in A$:

1. $w' = h(a_0a_1)a_2 \cdots a_{i-1}ba_{i+1} \cdots a_{n-3}h(a_{n-2}a_{n-1})$;
2. $w' = h(a_0a_1)a_2 \cdots a_{i-1}ba_{i+1} \cdots a_{n-1}$;
3. $w' = a_0a_1a_{i-1}ba_{i+1} \cdots a_{n-3}h(a_{n-2}a_{n-1})$;
4. $w' = a_0a_1h(a_2 \cdots a_{i-1}ba_{i+1} \cdots a_{n-3})a_{n-2}a_{n-1}$.

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Theorem 5. Let \( n \geq 3 \), let \( w \) be a primitive partial word of length \( n \) with one hole at position 1. If \( w' = w[2..n] \), then \( ww' \) is primitive.

**Proof.** Set \( w = a_0 \circ a_2 \cdots a_{n-1} \), and suppose there exist \( v \) and \( p \geq 2 \) such that \( ww' \subset v^p \). Note that \( w \) being primitive implies \(|v| \geq 2 \) and \( ww' = a_0 \circ v^p \). Consider the case where \( w(0) = a_0 \neq a_{n-1} = w(n-1) \) (the other case is similar). If \(|v| = 2 \), then \( v = a_0 a_{n-1} \) because the last letter of \( w' \) is equal to the last letter of \( w \). Moreover, if \(|w| \) is odd then \( w \subset v^p a_0 \) for some \( p' \), which contradicts \( w(0) \neq w(n-1) \). Thus, \(|v| \geq 3 \).

Suppose \( p \) is even, in other words, \( p = 2m \), for some \( m \geq 1 \). Since \( ww' \subset v^{2m} \), \( w[0..n-1] = a_0 \circ a_2 \cdots a_{n-2} \subset v^m \) and \( a_{n-1} w' \subset v^m \). It follows that \( a_0 = a_{n-1} \), a contradiction. Thus, \( p \) is odd, or \( p = 2m + 1 \) for some \( m \geq 1 \). Since the first letter of \( ww' \) is \( a_0 \) and the last letter of \( ww' \) is \( a_{n-1} \), set \( v = a_0 v_1 v_2 a_{n-1} \), where \(|v_1| = |v_2| + 2 \), \( w \subset (a_0 v_1 v_2 a_{n-1})^m a_0 v_1 \) and \( w' = v_2 a_{n-1} (a_0 v_1 v_2 a_{n-1})^m \). Since \( m \geq 1 \), the suffix of length \(|v_1| \) of \( w \) is full, and it follows that the last letter of \( v_1 \) is \( a_{n-1} \). Setting \( v_1 = v_1' a_{n-1} \), for some \( v_1' \), \( w \subset (a_0 v_1' a_{n-1} v_2 a_{n-1})^m a_0 v_1' a_{n-1} \) and \( w' = v_2 a_{n-1} (a_0 v_1' a_{n-1} v_2 a_{n-1})^m \). Since \( w' = w[2..n] \) and \( |v_1' a_{n-1}| = |v_2 v_1 a_{n-1}| \), we get \( v_1' a_{n-1} = a_{n-1} v_2 a_{n-1} \).

We find

\[
w \subset a_0 a_{n-1} v_2 a_{n-1} \quad v_2 a_{n-1} (a_0 a_{n-1} v_2 a_{n-1} v_2 a_{n-1})^m a_0 a_{n-1} v_2 a_{n-1}^{-1}
\]

\[
w' = v_2 a_{n-1} (a_0 a_{n-1} v_2 a_{n-1} v_2 a_{n-1})^m a_0 a_{n-1} v_2 a_{n-1}^{-1} v_2 a_{n-1}^{-1}
\]

Since \( w' = w[2..n] \), \( w'[v_2 a_{n-1} \cdot n - 2 - |v_2 a_{n-1}|] = w[2 + |v_2 a_{n-1}| \cdot n - |v_2 a_{n-1}|] \) and

\[
(a_0 a_{n-1} v_2 a_{n-1} v_2 a_{n-1})^m a_0 a_{n-1} v_2 a_{n-1} = v_2 a_{n-1} (a_0 a_{n-1} v_2 a_{n-1} v_2 a_{n-1})^m a_0 a_{n-1}
\]

Now, \( v_2 a_{n-1} a_0 a_{n-1} = a_0 a_{n-1} v_2 a_{n-1} \), and so there exists a word \( z \) such that \( v_2 a_{n-1} \) and \( a_0 a_{n-1} \) are powers of \( z \). By substitution, \( w \) is contained in a power of \( z \), which contradicts the fact that \( w \) is primitive.

Third, we deal with operations including a morphism on most letters.

**Theorem 6.** Let \( n \geq 4 \), let \( w = a_0 a_1 \cdots a_{n-1} \) be a word of length \( n \) with one hole at position \( i \), where \( a_0, a_{n-1} \) are distinct letters of \( A \), and \( h : A^* \to A^* \) be a length preserving morphism such that \( h(a) \neq a \) for all \( a \in A \). Then \( w w' \) is primitive if one of the following statements holds, where \( b \in A \):

1. \( w' = a_0 h(a_1 \cdots a_{i-1} b a_{i+1} \cdots a_{n-2}) a_{n-1} \);
2. \( h(h(a)) = a \), for all \( a \in A \), \( w' = a_{n-1} h(a_{n-2} \cdots a_{i+1} b a_{i-1} \cdots a_1) a_0 \).
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Proof. For Statement 1, suppose a word \( w \) and integer \( p \geq 2 \) exist such that \( ww^p \subset v^p \). If \( p \) is even, then \( w \subset v^p \) and \( w' \subset v^p \). But \( w' \not\subset w' \), a contradiction. So suppose \( p \) is odd, and set \( p = 2m + 1 \) for some \( m \geq 1 \). The length of \( v \) being even, let \( v_1, v_2 \) be two words such that \( v = v_1v_2, \) \( |v_1| = |v_2|, \) \( w \subset v^m v_1 \) and \( w' \subset v_2 v^m \). Since \( a_0 \) is the first letter of \( w \) and \( a_{n-1} \) is the last letter of \( w \), \( a_0 \) is the first letter of \( v_1 \) and \( a_{n-1} \) is the last letter of \( v_1 \). Likewise \( a_0 \) is the first letter of \( v_2 \) and \( a_{n-1} \) is the last letter of \( v_2 \).

Suppose \( |v_1| = 1 \), so \( a_0 = a_{n-1} \), a contradiction. So suppose \( |v_1| \geq 2 \), and set \( v_1 = a_0v_1' a_{n-1}, v_2 = a_0v_2' a_{n-1} \), for some \( v_1', v_2' \). We get

\[
\begin{align*}
w & \subset (a_0v_1'a_{n-1}a_0v_2'a_{n-1})^m a_0v_1' a_{n-1} \\
w' & \subset a_0v_2'a_{n-1}(a_0v_1'a_{n-1}a_0v_2'a_{n-1})^m
\end{align*}
\]

Assume \( |v_1| \leq i < n - 1 \). Since \( |a_0v_1'| = |a_0v_2'| = l \) for some \( l \in [1..n-1) \), the letter in position \( l \) of \( w \) and the letter in position \( l \) of \( w' \) are both \( a_{n-1} \). But since \( 1 \leq l < n - 1, a_i \neq h(a_i) \), a contradiction. A similar argument holds when \( 0 < i < |v_1| \).

Theorems 23 and 24 of [9] explore the primitivity of \( ww' \), where \( w' \) is built by applying a morphism on \( w \) excluding only the first or last letter, then reversing this sequence. We extend them to one-hole words.

**Theorem 7.** Let \( n \geq 2 \), let \( w = a_0 \cdots a_{n-1} \) be a primitive partial word of length \( n \) with one hole at position \( i \), \( 0 < i < n - 1 \), and let \( h : A^* \rightarrow A^* \) be a length preserving morphism such that \( h(a) \neq a \) for all \( a \in A \). If \( w' = a_{n-1}h(a_{n-2} \cdots a_{i+1}ba_{i-1} \cdots a_0) \) or \( w' = h(a_{n-1} \cdots a_{i+1}ba_{i-1} \cdots a_1)a_0 \), where \( b \in A \), then \( ww' \) is primitive.

Fourth, we deal with powers of two. The results are different from any of our previous results; instead of changing specific letters of \( w \) to create \( w' \), we instead change a certain number of letters of \( w \). To prove Theorem 8 the following useful result involves the Hamming distance, \( d(w, w') \), between two words \( w \) and \( w' \), or the number of positions \( i \) such that \( w(i) \neq w'(i) \). It extends Theorems 7 and 19 of [9] in terms of full words, using a similar proof, which shows that \( w \) being primitive is not a necessary condition.

**Theorem 8.** Let \( w, w' \) be words of length \( n \).

1. If \( d(w, w') \) is a power of 2, then \( ww' \) is primitive;
2. If \( n - d(w, w') \) is a power of 2, then \( ww' \) is primitive.

We extend Theorem 8 to partial words.
Theorem 9. Let \( w \) be a partial word with one hole of length \( n \).

1. If \( w' \) is a full word of length \( n \) such that there is exactly one defined position \( i \) of \( w \) such that \( w(i) \neq w'(i) \), then \( ww' \) is primitive;

2. If \( n = 2^q \), for some \( q \geq 0 \), and \( w' \) is a full word of length \( n \) such that, for every \( i \in D(w) \), \( w(i) \neq w'(i) \), then \( ww' \) is primitive.

Proof. For Statement 2, let \( w \) be a word with one hole at position \( j \) such that \( |w| = n \) is a power of 2. Let \( w' \) be a full word of length \( n \) such that, for every defined position \( i \) of \( w \), \( w(i) \neq w'(i) \). Consider a word \( u \) such that \( w \subset u \). If \( u(j) \neq w'(j) \), then \( d(u, w') = n \), which is a power of 2, so by Theorem 8 \( uw' \) is primitive. If \( u(j) = w'(j) \), then \( d(u, w') = n - 1 \). Now, \( |u| - d(u, w') = n - (n - 1) = 1 \), which is a power of 2, so by Theorem 8 \( uw' \) is primitive. Thus, for all words \( u \) such that \( w \subset u \), \( uw' \) is primitive and the result follows by Lemma 1.

4 Generating Primitive Words of Odd Length

In this section, we examine operations which produce primitive partial words of odd length. First, we deal with an operation involving one letter. The primitivity of \( ww' \), where \( w' \) is defined by a single insertion operation of one letter on \( w \), is studied in Theorem 13 of [9]. We can expand on it with the following theorem and similar proof. Here, we consider \( w \) with one hole. When forming \( w' \), we replace the hole in \( w \) with any letter \( a \in A \) and insert a letter \( b \in A \).

Theorem 10. 1. Let \( u_1, u_2, v \) be non-empty words, and let \( w = u_1 \circ u_2 v \) be primitive. If \( w' = u_1 \circ u_2 bv \), for some \( a, b \in A \), then \( ww' \) is primitive;

2. Let \( u, v_1, v_2 \) be non-empty words, and let \( w = uv_1 \circ v_2 \) be primitive. If \( w' = ubv_1 \circ v_2 \), for some \( a, b \in A \), then \( ww' \) is primitive.

Corollary 1. 1. Let \( u_1, u_2, v \) be non-empty words, and let be \( u_1 \circ u_2 v \) primitive. Then \( u_1 \circ u_2 vu_1 \circ u_2 av \), for any \( a \in A \), is primitive;

2. Let \( u, v_1, v_2 \) be non-empty words, and let \( uv_1 \circ v_2 \) be primitive. Then \( uv_1 \circ v_2 avu_1 \circ v_2 \), for any \( a \in A \), is primitive.

The primitivity of \( ww' \), where \( w' \) is defined by a deletion of one letter of \( w \), is studied in Theorem 11 of [9]. We extend it to partial words by deleting a letter of \( w \) and replacing the hole in \( w \) with a letter to create \( w' \).

Theorem 11. 1. If \( w = u_1 \circ u_2 av \) is a primitive word with one hole, \( a \in A \), then \( ww' = u_1 \circ u_2 avu_1 \circ u_2 bu_2 v \), for any \( b \in A \), is primitive;
Proof. Suppose there exist a word \( v \) and integer \( p \geq 2 \) such that \( vw' = v^p \). Since \( |vw'| = 2n - 1 \) is odd, both \( |v| \) and \( p \) are odd, and \( p \geq 3 \). Since \( w \) is primitive, \( |v| \geq 3 \). Let \( p = 2m + 1 \) and \( |v| = 2q + 1 \), where \( m, q \geq 1 \). It is convenient to refer to \( v^p \) as \( v_1v_2 \cdots v_p \), where \( v = v_1 = \cdots = v_p \), that is, \( v \)
is the $l$th $v$ block. The first letter of $v$ is the first one of $ww'$, so for each $l$, we have $v = v_l = a_0 v'_l v''_l$, with $|v'_l| = |v''_l| = q$. Note that $w = v^m a_0 v_{m+1}'$ and $w' = v''_{m+1}'$. Set $l' = l \mod 2q + 1$, $0 \leq l < n$. The case when neither $a_i$ nor $a_j$ falls in $v_{m+1}'$ is easy to see, so let us prove the remaining cases.

First, suppose $a_j$ falls in $v_{m+1}$ and $p = 3$ (or $m = 1$), and so $a_j = v_2(j')$. If $j' = 0$, then $a_j = v_2(0) = a_0$ and $h(a_j) = v_3(0) = a_0$, which contradicts the fact that $h(a_j) \neq a_j$. Thus, $j' \geq 1$ and $a_j$ must fall in $v'_2$. Note that $v''_2 = \text{rev}(v'_2)$ and we have $v''_2(j') = a_j$, so $v''_2(q - 1 - j') = h(a_j)$. We see that $|v''_1(j' + 1)q v''_2(a_0 v_{2}')| = |v''_2 a_0 v''_2[0..q - 1 - j']|$, and $v''_2(j')$ corresponds to $v''_2(q - 1 - j')$. If $i \neq j'$, we get a contradiction because $v''_2(j') = v_2(j') = a_j \neq h(a_j) = v''_2(q - 1 - j') = v''_2(q - 1 - j')$. Thus, $i = j'$. Now, $|v''_1(q - j'..q) a_0 v'_2| = |v''_2 a_0 v''_2[0..j'])|$. Since $a_i$ falls in $v''_2$ and $a_j$ falls in $v''_2$, $v''_2(q - 1 - j'') = v''_2(j')$. This again gives a contradiction since $v''_2(q - 1 - j') = v''_2(q - 1 - j') = h(a_i)$, $v''_2(j') = v''_2(j') = a_j$ and $h(a_j) \neq a_j$.

Now, suppose $a_j$ falls in $v_{m+1}$ and $p \geq 5$. Some $v_l$ in $w$ exists such that neither $a_i$ nor $a_j$ falls in it. Set $w = x_1 v_l x_2 a_{i-1}$, $w' = x_3 x_4$, where $|x_1| = |x_4|$ and $|x_2| = |x_3|$. If $v = a_0 v' v''$, with $|v'| = |v''|$, we get $ww' = x_1 a_0 v' v'' x_2 a_{i-1} x_3 a_0 v'' x_4$. Since $|x_2 a_{i-1}| = |x_3 a_0|$ and neither $a_i$ nor $a_j$ falls in $v''_2$, it follows that $v'' = \text{rev}(v')$. So, $v = a_0 v' \text{rev}(v')$. As we have just shown, $v''_{m+1}' = \text{rev}(v''_{m+1})$. By a similar argument to that above, $a_j \neq v_{m+1}(0)$, and thus $a_j$ falls in $v''_{m+1}'$. But this implies $v''_{m+1}' \neq \text{rev}(v''_{m+1})$.

**Corollary 3.** Let $w = a_0 a_1 \cdots a_{n-1}$ be a primitive partial word of length $n$ with one hole at position $i$, $1 \leq i < n$, let $w' = a_{n-1} \cdots a_{i+1} h(a_j) a_{i-1} \cdots a_1$, $1 \leq j < n$ and $j \neq i$, and let $h : A \to A$ be a mapping such that $h(a) \neq a$ for all $a \in A$. Then $ww'$ is primitive.

Third, we deal with operations including a morphism on most letters.

**Theorem 15.** Let $n \geq 4$, let $w = a_0 a_1 \cdots a_{n-1}$ be a partial word of length $n$ with one hole at position $i$, $0 \leq i < n$, and let $h : A^* \to A^*$ be a length preserving morphism such that $h(a) \neq a$ and $h(h(a)) = a$ for all $a \in A$. If $w' = h(a_0 \cdots a_{i-1} a_{i+1} \cdots a_{n-1})$, then $ww'$ is primitive.

**Proof.** Suppose there exist a word $v$ and integer $p \geq 2$ such that $ww' \subset v^p$. Since $|w| = q$ and $p$ are odd. Let $p = 2m + 1$ for some $m \geq 1$, and set $v = v_1 v_2$. Since $|v_1| = |v_2| + 1$. So $w \subset (v_1 v_2)^m v_1$ and $w' \subset (v_1 v_2)^m$. Let $v_1 = v'_1 a$ for some $a \in A$, so $|v'_1| = |v_2|$, and $w \subset (v'_1 a v_2)^m v'_1 a$ and $w' \subset (v'_1 a v_2)^m$. If $|v'_1| \leq i < n$ (that is, the hole is not in the prefix of length $|v'_1|$ of $w$), then $h(v'_1) = v_2$. Similarly, if $0 \leq i < |v'_1|$, then $h(v_2) = v'_1$ or $v_2 = h(v'_1)$. In either case, $v_2 = h(v'_1)$, and so $w \subset (v'_1 a h(v'_1))^m v'_1 a$ and $w' \subset h(v'_1)(v'_1 a h(v'_1))^m$. If
Proof. Suppose that \(|v'_1| = 0\), then \(w \subset a^n\) and \(w' \subset a^{n-1}\); but \(h(a) \neq a\), a contradiction. Thus \(|v'_1| \geq 1\). First, suppose \(|v_1v_2| \leq i < n\). So \(h(v'_1 ah(v'_1)) = h(v'_1)v'_1a\) or \(h(v'_1)h(a)v'_1 = h(v'_1)v'_1a\). Thus \(h(a)v'_1 = v'_1a\), and \(h(a) = a\), a contradiction. Now, suppose \(0 \leq i < |v_1v_2|\). So \(h(v'_1a) = ah(v'_1)\), or \(v_2h(a) = av_2\), and \(h(a) = a\). \(\square\)

Theorem 16 defines \(w'\) by deleting the hole from \(w\) and then performing a morphism on the remaining sequence excluding only the last letter.

Theorem 16. Let \(n \geq 6\), let \(w = a_0a_1 \cdots a_{n-1}\) be a partial word of length \(n\) with one hole at position \(i\), \(0 \leq i < n\), and let \(h : A^* \to A^*\) be a length preserving morphism such that \(h(a) \neq a\) and \(h(h(a)) = a\) for all \(a \in A\). Suppose that \(w \not\subset a^n\) for any \(a \in A\).

1. If \(w' = h(a_0 \cdots a_{i-1}a_{i+1} \cdots a_{n-2})a_{n-1}\) and the first two defined letters are distinct, then \(ww'\) is primitive;

2. If \(w' = a_0h(a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n-1})\) and the last two defined letters are distinct, then \(ww'\) is primitive.

We now define \(w'\) by substituting a letter for the hole in a word with one hole, \(w\), then perform a morphism on the sequence remaining after we substitute a letter for the hole. We then insert a letter at the end of \(w'\).

Theorem 17. Let \(n \geq 5\), \(w = a_0a_1 \cdots a_{n-1}\) be a word of length \(n\) with one hole at position \(i\), \(i = 0\) or \(i = 1\), and let \(h : A^* \to A^*\) be a length preserving morphism such that \(h(a) \neq a\) and \(h(h(a)) = a\) for all \(a \in A\). If \(w' = h(a_0a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n-1})b\) for some \(a, b \in A\), then \(ww'\) is primitive.

The following theorem considers a partial word with one hole \(w\) and creates \(w'\) through operations of reversal, deletion, insertion, and morphism.

Theorem 18. Let \(n \geq 4\), \(w = a_0a_1 \cdots a_{n-1}\) be a partial word of length \(n\) with one hole at position \(i\) \(= n - 3, n - 2, \text{ or } n - 1\), and \(h : A \to A\) be a mapping such that \(h(a) \neq a\) and \(h(h(a)) = a\) for all \(a \in A\). If \(i = n - 1\) (resp., \(i = n - 2\), \(i = n - 3\)), let \(w' = h(a_{n-2}a_{n-3} \cdots a_0)\) (resp., \(w' = ah(a_{n-3}a_{n-4} \cdots a_0)\), \(w' = h(a_{n-2})ah(a_{n-4}a_{n-5} \cdots a_0)\)), for any \(a \in A\). Then \(ww'\) is primitive.

Proof. Suppose that \(ww' \subset v^p\) for some \(v\) and \(p \geq 2\). Since \(|ww'|\) is odd, \(|v|\) and \(p\) are odd, and let \(p = 2m + 1\). Since \(|w| \geq 4\), \(w(0) = a_0 \in A\). By the construction of \(w'\), \(w'(n - 2) \neq w(0)\), and thus \(v\) must contain at least two distinct letters, so \(|v| \geq 2\). Moreover, since \(|v|\) is odd, \(|v| \geq 3\). Set \(v = v_1v_2\), with \(|v_1| = |v_2| + 1\), so \(w \subset (v_1v_2)^m v_1\) and \(w' = v_2(v_1v_2)^m\).
First, suppose $|v| = 3$, $p = 3$, so $ww' \subset vvv = a_0bh(a_0)a_0bh(a_0)a_0bh(a_0)$, for some $b \in A$, and $w \subset a_0bh(a_0)a_0b$, $w' = h(a_0)a_0bh(a_0)$. By definition of $h$, $w(1) \neq w'(2)$, a contradiction. Now, suppose $|v| \geq 5$. Note that $|v_1| \geq 3$. Since $w \subset (v_1v_2)^m v_1$, the first $|v|$ positions of $w$ are defined. Furthermore, since $w' = v_2(v_1v_2)^m$, we have $h(\text{rev}(v_1v_2)) = v_1v_2$. Let $v_1 = v_1'b$ for some $b \in A$. Note that $|v_1'| = |v_2|$, and $h(\text{rev}(v_2))h(b)h(\text{rev}(v_1')) = v_1'b v_2$. However, this implies that $h(b) = b$, a contradiction. Finally, suppose $p \geq 5$. Since $m \geq 2$, the first $|v|$ positions of $w$ are defined, and we again have a contradiction by a similar argument to the case when $|v| \geq 5$.

Fourth, we deal with operations with a morphism on half the letters.

**Theorem 19.** Let $n \geq 2$, let $w = w_1w_2\diamond$ where $w_1$ and $w_2$ are full words of length $n$, and let $h : A^* \to A^*$ be a length preserving morphism such that $h(a) \neq a$ and $h(h(a)) = a$ for all $a \in A$. Set $w_1 = a_0a_1\ldots a_n$, and let $h(h(a_0)) = a_0h(a_0)$, then $ww'$ is primitive.

**Proof.** For $w' = h(w_1)w_2a_0h(a_0)$, let $v$ and $p \geq 2$ be such that $ww' \subset v^p$. Since $|ww'|$ is odd, $|v|$ and $p$ are odd. Let $v = v_1v_2$ such that $|v_1| = |v_2| - 1$, and $p = 2m + 1$. So $w_1w_2\diamond \subset (v_1v_2)^m v_1$ and $h(w_1)w_2a_0h(a_0) \subset v_2(v_1v_2)^m$, and $v_2 = v_2'h(a_0)$ where $|v_2'| = |v_1|$. By substitution, $w_1w_2\diamond \subset (v_1v_2'h(a_0))^m v_1$, $h(w_1)w_2a_0h(a_0) \subset v_2'h(a_0)(v_1v_2'h(a_0))^m$. If $|v_1| = 0$, then $w_1w_2\diamond \subset (h(a_0))^m$, so $h(a_0) = w_1(0)$, a contradiction. Thus $|v_1| \geq 1$. Note that $v_1b$, where $b = v_2'(0)$, is a prefix of $w_1$, and $v_2'h(a_0)$ is a prefix of $h(w_1)$. So $v_1b = h(h(v_1b)) = h(v_2'h(a_0))$, and thus $b = h(h(a_0)) = a_0$, implying that $a_0 = v_2'(0)$. But $h(a_0) = h(w_1)(0)$, and $h(a_0) \neq a_0$.

The following is concerned with $|w|$ odd and $w'$ defined by reversing $\diamond w$ and applying a morphism on the first half excluding the middle letter.

**Theorem 20.** Let $n \geq 3$ be odd, $w = a_0a_1\ldots a_n$ a full word of length $n$, $h : A^* \to A^*$ a length preserving morphism such that $h(a) \neq a$ for all $a \in A$. Set $i = \frac{n-1}{2}$. If $w' = h(a_{n-1}\ldots a_{i+1})a_i\ldots a_0\diamond$, then $ww'$ is primitive.

Fifth, we deal with concatenating a square.

**Theorem 21.** Let $m$ be a non-negative integer, let $w$ be a partial word with one hole of length $n = 4m + 3$, and let $w'$ be a full word of length $n + 1$. If $w \not\subseteq a^n$, for any $a \in A$, and $w[0..2] \not\subseteq w'[0..2]$, then $ww'w'$ is primitive.

**Proof.** Suppose there exist a word $v$ and integer $p \geq 2$ such that $ww'w' \subset v^p$. Since $|ww'w'| = (4m + 3) + 2(4m + 4) = 12m + 11$ is odd, $|v|$ and $p$ are odd. Let $p = 2q + 1$ for some $q \geq 1$, and let $v = v_1v_2$ where $|v_1| = |v_2| - 1$. Since
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|w'| is even, set w' = w'₁w₂ such that |w'₁| = |w'₂|, and so ww'₁ ⊂ (v₁v₂)ⁿv₁ and w'₁w₂ ⊂ v₂(v₁v₂)ⁿ. Let v₂ = v₂ᵃ for some a ∈ A, so |v₂| = |v₁|, ww'₁ ⊂ (v₁v₂ᵃ)ⁿv₁ and w'₁w₂ ⊂ v₂ᵃ(v₁v₂ᵃ)ⁿ. Looking at the latter, since |w'₂| = |w'₁|, |v₂ᵃ(v₁v₂ᵃ)| is divisible by 3. If |v₁| = 0, then ww'₁ ⊂ aⁿ implies that w ⊂ aⁿ, a contradiction. So |v₁| ≥ 1, and we consider the case where q mod 3 = 2 (the others are similar).

Suppose q mod 3 = 2, so q = 3t + 2 for some t ≥ 0, thus |v₂ᵃ(v₁v₂ᵃ)| = (6t + 5)|v₁| + (3t + 3) is divisible by 3 only if |v₁| is. So let v₁ = v₁v₁⁺v₁⁺⁺ and v₂ = v₂⁺v₂⁺⁺ such that |v'₁| = |v₁| = |v'₂| = |v₂| = |v'⁺| = |v⁺| = |v'⁺⁺| = |v⁺⁺|. We get ww'₁ ⊂ (v₁v₁⁺v₂⁺v₂⁺⁺v₂⁺⁺)ⁿ−¹v₁v₁⁺v₁⁺⁺v₂⁺v₂⁺⁺v₂⁺⁺ av₁v₁⁺ and w'₁w₂ ⊂ v₂⁺v₂⁺⁺av₁v₁⁺v₂⁺⁺v₂⁺⁺v₂⁺⁺ av₁v₁⁺v₂⁺⁺v₂⁺⁺v₂⁺⁺v₂⁺⁺ v₂⁺⁺ v₂⁺⁺. Since |w'₁| = |w₂|,

\[
\begin{align*}
|w'₁| &< v₂⁺⁺ v₂⁺⁺ av₁v₁⁺v₁⁺⁺ (v₂⁺⁺ v₂⁺⁺ v₂⁺⁺ av₁v₁⁺) tubing \\
|w'₁| &< v₁⁺⁺ v₂⁺⁺ v₂⁺⁺ av₁v₁⁺⁺ (v₂⁺⁺ v₂⁺⁺ av₁v₁⁺⁺) tubing \\
|w'₂| &< v₂⁺⁺ v₂⁺⁺ av₁v₁⁺⁺ (v₂⁺⁺ v₂⁺⁺ v₂⁺⁺ av₁v₁⁺⁺) tubing \\
|w₂| &< v₁⁺⁺ av₁v₁⁺⁺ v₂⁺⁺ v₂⁺⁺ av₁v₁⁺⁺ tubing \\
\end{align*}
\]

Since w'₁ and w₂ are both full words, v₂⁺⁺ v₂⁺⁺ av₁v₁⁺⁺ v₁⁺⁺ = v₂⁺⁺ v₂⁺⁺ v₂⁺⁺ av₁⁺⁺ and v₂⁺⁺ v₂⁺⁺ av₁v₁⁺⁺ v₁⁺⁺ = v₁⁺⁺ v₁⁺⁺ v₂⁺⁺ av₁⁺⁺. From the former, v₁⁺⁺ = v₂⁺⁺ = v₁⁺⁺. Similarly from the latter, v₂⁺⁺ = v₂⁺⁺, and so v₁⁺⁺ v₁⁺⁺ = v₂⁺⁺ v₂⁺⁺. Since |v₂⁺⁺| ≥ 1 is divisible by 3, we have |v₁⁺⁺| ≥ 3. Thus |v₁⁺⁺| ≥ 1. We obtain w[0..2] ⊂ v₁⁺⁺ v₁⁺⁺ v₂⁺⁺ and w[0..2] = w₁⁺⁺ v₁⁺⁺ v₂⁺⁺ = w₁⁺⁺ v₁⁺⁺ v₂⁺⁺, and so w[0..2] ↑ w'[0..2].

5 Conclusion

In addition to exploring operations which produce primitive partial words of even or odd length exclusively, we include operations which yield primitive words of any length.

The following generalizes Theorem 27 of [9].

**Theorem 22.**

1. Let n ≥ 2, w be a primitive partial word of length n, and a, b ∈ A be distinct. Then waⁿ, waⁿ⁻¹ and wabⁿ⁻² are primitive;

2. Let n ≥ 4 be even, w be a primitive partial word of length n, and a, b ∈ A be distinct. Then w(ab)ⁿ⁻³⁺² a and w(ab)ⁿ⁻³⁺² b are primitive.

Furthermore, if a ∈ A and w is a partial word of length n such that w ⊄ aⁿ, then waⁿ and waⁿ⁻¹ are primitive.

A World Wide Web server interface has been established at

www.unmc.edu/cmp/research/primitive3

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for automated use of a program that when given as input a binary partial word with one hole, outputs the primitive partial words that can be generated using our operations.

When we perform the operations of our results along with Lemma 4 and the fact that if a partial word is primitive, then so is its reversal, we can generate all the primitive binary partial words with one hole of length \( n \), \( 1 \leq n \leq 11 \), as well as many of the ones in the range \( 12 \leq n \leq 21 \). The table below gives the number \( G_{1,2}(n) \) of primitive binary partial words with one hole of length \( n \) generated with our results, the number \( P_{1,2}(n) \) of primitive binary partial words with one hole of length \( n \), as well as their ratio.

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References


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