On the Performance of MC-CDMA Systems with Partial Combining and Multiple Antennas in Fading Channels

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Abstract—In this paper we analytically evaluate the downlink performance of a multi-carrier code division multiple access (MC-CDMA) system by employing partial combining when multiple antennas at the transmitter or at the receiver are considered. It is known that for single reception/transmission, the partial combining technique depends on a parameter which can be optimized as a function of the number of subcarrier, the number of active users and the SNR improving the performance with respect to other classical techniques (such as maximal ratio combining, equal gain combining, or orthogonality restoring combining). In this paper we extend the analysis considering also multiple antennas aiming at showing how the spatial diversity affects the performance as a function of the partial combining parameter.

I. INTRODUCTION

Multi-carrier code division multiple access (MC-CDMA) consists in a combination of orthogonal frequency division multiplexing (OFDM) and spreading. Systems based on MC-CDMA are widely considered to counteract the frequency selectivity of the wireless channel, to avoid inter-symbol and inter-carrier interference and achieve high spectral efficiency. The basic principle of MC-CDMA is to spread each data symbol over several subcarriers and, hence, to efficiently exploit frequency diversity of the channel [1]. In this work we consider the MC-CDMA architecture presented in [2] and [3], where the spreading is performed in the frequency-domain and Walsh-Hadamard (WH) codes are used with spreading factor equal to the number of subcarriers. We focus on the downlink, hence the system is assumed synchronous and different users experience the same channel. However, in spite of this, in frequency selective fading channels, the orthogonality between spreading sequences is corrupted by fading, thus the performance at the receiver side is affected by multi-user-interference (MUI) [2], [3].

Focusing on linear combining techniques and assuming perfect channel state information (CSI), many techniques are known in the literature trying to minimize the bit error probability (BEP). It is readily understandable that when $\beta = 0$, $-1$ and $1$, then (1) coincides with EGC, MRC and ORC, respectively. In [4] the optimal value of $\beta$ for SISO systems was derived as a function of the number of subcarrier $M$, the number of active users $N_a$ and the mean signal-to-noise-ratio (SNR) averaged over small-scale fading, $\bar{\gamma}$. In this work we consider multiple antennas at the transmitter or at the receiver side in order to extend the analytical framework to jointly consider PC on both subcarriers and branches when multiple antennas are employed and to verify the improvement achieved by adding spatial diversity. For a deep investigation on how antenna diversity affects MC-CDMA systems performance, see, at instance, [5]–[7].

II. SYSTEM MODEL

Following the MC-CDMA architecture presented in [4], each data-symbol is copied over all sub-carriers and multiplied by the chip assigned to each particular sub-carrier. Binary phase shift keying (BPSK) modulation is assumed (see Fig. 1 for details).

A. Multiple Transmitting Antennas

In Fig. 1(a) the transmitter block scheme for the case of multiple transmitting antennas is presented: the transmitted signal referred to the $k^{th}$ user on the $i^{th}$ antenna, can be written as:

$$s^{(i,k)}(t) = \sqrt{\frac{2E_b}{LM}} \sum_{i=-\infty}^{\infty} \sum_{m=0}^{M-1} c_m^{(k)} a^{(k)}[i] g(t-iT_b) \times \cos(2\pi f_m t + \phi_m),$$  

(2)

where $E_b$ is the energy per bit, $L$ the number of antennas, $M$ the number of subcarriers. Index $i$ denotes the data index, $m$ is the sub-carrier index, $c_m$ is the $m^{th}$ chip of the WH

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code sequence, \( a^{(k)}[i] \) is the data-symbol transmitted by user \( k \) during the \( i \)th symbol time, \( g(t) \) is a rectangular pulse waveform with duration \([0, T] \) and unitary energy, \( T_b \) is the bit-time, \( f_m \) is the \( m \)th sub-carrier-frequency and \( \phi_m \) is the random phase uniformly distributed within \([-\pi, \pi] \). In particular, \( T_b = T + T_g \) is the total OFDM symbol duration, increased with respect to \( T \) of a time-guard \( T_g \) (inserted between consecutive multi-carrier symbols to eliminate the residual inter symbol interference, ISI, due to the channel delay spread).

We now assume to jointly combine both antennas and subcarriers contributions, by weighting the signal on each antenna and subcarrier with the partial combining coefficient, \( G_{m,t} \), defined as in (1) when jointly considering antenna diversity (see Fig. 1(a) as example):

\[
G_{m,t} = \frac{H^*_{m,t}}{|H_{m,t}|^{1+\beta}}, \quad \beta \in [-1, 1], \tag{3}
\]

Hence, assuming uncorrelated fading channels, the total diversity degree is in the range \([M, LM] \) depending on the correlation among the antennas. All users exploit all the subchannels (e.g., spreading factor equal to the number of subcarriers), thus, considering perfect state information (CSI) at the transmitter, the total transmitted signal on the \( LM \) antenna results in:

\[
s^{(l)}(t) = \sum_{k=0}^{N_u-1} s^{(k)}(t) = \sqrt{\frac{2E_b}{LM}} \sum_{k=0}^{N_u-1} \sum_{i=-\infty}^{+\infty} \sum_{m=0}^{M-1} c_m^{(k)} \times a^{(k)}[i] G_{m,t} g(t - iT_b) \cos(2\pi f_m t + \phi_m), \tag{4}
\]

where \( N_u \) is the number of active users and, because of the use of orthogonal codes, \( N_u \leq M \). Finally, the total transmitted signal over \( L \) antennas is given by:

\[
s(t) = \sum_{l=0}^{L-1} s^{(l)}(t). \tag{5}
\]

**B. Multiple Receiving Antennas**

Following assumptions made in Section II-A (i.e., uncorrelated fading among the subcarriers and perfect CSI), in the opposite case of single transmission and multiple receiving antennas (see Fig. 1(b) for details), the total transmitted signal can be written as:

\[
s(t) = \sqrt{\frac{2E_b}{LM}} \sum_{k=0}^{N_u-1} \sum_{i=-\infty}^{+\infty} \sum_{m=0}^{M-1} c_m^{(k)} \times a^{(k)}[i] g(t - iT_b) \cos(2\pi f_m t + \phi_m), \tag{6}
\]

**III. DECISION VARIABLE**

We consider the downlink of a MC-CDMA system, hence users are synchronous and different delays affecting each subcarrier are assumed perfectly compensated. We consider a frequency-domain channel model with transfer function for each subchannel, \( H(f) \), given by:

\[
H(f) \simeq H(f_m) = \alpha_m e^{j\psi_m}, \tag{7}
\]

where \( \alpha_m \) and \( \psi_m \) are the \( m \)th amplitude and phase coefficients, respectively. Each \( H(f_m) \) is considered independent identically distributed (i.i.d.) complex zero-mean Gaussian random variable (r.v.) (i.e., each subcarrier experiments flat fading, uncorrelated with the other subcarriers) with variance \( \sigma_{H_m}^2 \) defined as:

\[
\mathbb{E}\{\alpha_m^2]\} = 2\sigma_{H_m}^2 = 1, \tag{8}
\]

where \( \mathbb{E}\{\cdot\} \) is the statistical expectation. Hence, we can write:

\[
\sum_{m=0}^{M-1} \sum_{l=0}^{L-1} \mathbb{E}\{\alpha_m^2\} = LM, \quad \forall \, l, m. \tag{9}
\]

Each \( \alpha_{m,l} = |H_{m,l}| \) is assumed Rayleigh distributed, i.i.d. for each subcarrier and each antennas. In the following, the decision variable for both multiple transmission and single reception and single transmission and multiple reception is derived.

**A. Multiple Transmitting Antennas**

In this case, the received signal can be written as:

\[
r(t) = \sqrt{\frac{2E_b}{LM}} \sum_{k=0}^{N_u-1} \sum_{i=-\infty}^{+\infty} \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} G_{m,l}^\dagger a_m^{(k)} a^{(k)}[i] + g'(t - iT_{b,l}) \cos(2\pi f_m t + \vartheta_m) + n(t), \tag{10}
\]

where \( g'(t) \) is the response of the channel to \( g(t) \) with unitary energy and duration \( T' = T + T_d \), being \( T_d \) the time delay, \( n(t) \) is the additive white Gaussian noise with two-side power spectral density (PSD) \( N_0/2 \) and \( \vartheta_m \triangleq \phi_m + \psi_m \). Note that, since \( \vartheta_m \) can be considered uniformly distributed in \([-\pi, \pi] \), we can consider \( H(f_m) \) distributed as \( \vartheta_m \) in the following. The receiver performs the correlation of the received signal with \( c_m^{(n)} \sqrt{2} \cos(2\pi f_m t + \vartheta_m) \), thus, after some algebra, the decision variable results in:

\[
\vartheta(t) = \sqrt{\frac{2E_b}{LM}} \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} \alpha_m l a_{m,l} c_m^{(n)} a^{(k)} + \sum_{m=0}^{M-1} n_m \tag{11}
\]

where \( \delta_d = 1/(1 + T_d/T) \) represents the loss of energy caused by time-spreading and the gains \( g_{m,l} \) are normalized as:

\[
g_{m,l} = G a_{m,l}^{-\beta}, \tag{12}
\]

being

\[
G = \sqrt{\frac{LM}{\sum_{m=0}^{M-1} \sum_{l=0}^{L-1} a_{m,l}^{-2\beta}}}. \tag{13}
\]
Multiplying each term of (11) by $G$, we can write:

\[
U(n) = \sqrt{\frac{E_0 \delta_d}{LM}} \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} \alpha_{m,l}^{1-\beta} a^{(n)} + \sqrt{\frac{E_0 \delta_d}{LM}} \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} \sum_{k=0}^{N_u-1} \alpha_{m,l}^{1-\beta} c_m^{(n)} c_m^{(k)} a^{(k)}
\]

\[
+ \sqrt{\frac{E_0 \delta_d}{LM}} \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} \sum_{k=0}^{N_u-1} \sum_{n=0}^{N-1} \alpha_{m,l}^{1-\beta} c_m^{(n)} c_m^{(k)} a^{(k)}
\]

Assuming to have a number of subcarriers sufficiently high to justify the use of the law of large numbers (LLN) and exploiting the independence among the subcarriers, by applying the Kolmogorov’s Law, we can write:

\[
\frac{1}{M} \sum_{m=0}^{M-1} \left( \sum_{l=0}^{L-1} \alpha_{m,l}^{1-\beta} \right) = \mathbb{E} \left\{ \sum_{l=0}^{L-1} \alpha_{m,l}^{1-\beta} \right\}
\]

\[
\frac{1}{M} \sum_{m=0}^{M-1} \left( \sum_{l=0}^{L-1} \alpha_{m,l}^{2-\beta} \right) = \mathbb{E} \left\{ \sum_{l=0}^{L-1} \alpha_{m,l}^{2-\beta} \right\}
\]

Since the channel coefficients are Rayleigh distributed:

\[
\mathbb{E} \left\{ \alpha_{m,l}^{1-\beta} \right\} = (2\sigma_H^2)^{1-\beta} \Gamma \left( \frac{3-\beta}{2} \right), \forall \ m, l
\]

\[
\mathbb{E} \left\{ \alpha_{m,l}^{2-\beta} \right\} = (2\sigma_H^2)^{-\beta} \Gamma \left( 1 - \beta \right), \forall \ m, l
\]

being $\Gamma(\cdot)$ the Euler-Gamma function. By substituting (17) and (18) in (15) and (16), respectively, and exchanging the sum operation with the statistical expectation, we have:

\[
\frac{1}{M} \sum_{m=0}^{M-1} \left( \sum_{l=0}^{L-1} \alpha_{m,l}^{1-\beta} \right) = L(2\sigma_H^2)^{1-\beta} \Gamma \left( \frac{3-\beta}{2} \right)
\]

\[
\frac{1}{M} \sum_{m=0}^{M-1} \left( \sum_{l=0}^{L-1} \alpha_{m,l}^{2-\beta} \right) = L(2\sigma_H^2)^{-\beta} \Gamma \left( 1 - \beta \right)
\]

Therefore, we can write:

\[
U = \sqrt{E_0 \delta_d LM (2\sigma_H^2)^{1-\beta} \Gamma \left( \frac{3-\beta}{2} \right)}
\]

\[
N = \sqrt{(2\sigma_H^2)^{-\beta} \Gamma \left( 1 - \beta \right)} \sum_{m=0}^{M-1} n_m
\]

Since $n_m$ are independent Gaussian zero-mean random variables with variance $N_0/2$, the noise term $N$ is gaussian distributed as:

\[
N \sim \mathcal{N} \left( 0, \sigma_N^2 \right)
\]

where

\[
\sigma_N^2 = (2\sigma_H^2)^{-\beta} \Gamma \left( 1 - \beta \right) \frac{MN_0}{2}
\]

For what concern the interference term $I$, we can concatenate the sums in $m$ and $l$ in one vector of dimension $LM$ and, by exploiting the independence between the coefficients and applying the methodology proposed in [4], we find out:

\[
I \sim \mathcal{N} \left( 0, \sigma_I^2 \right)
\]

where

\[
\sigma_I^2 = E_0 \delta_d (N_u - 1) \zeta_\beta(\alpha)
\]

with

\[
\zeta_\beta(\alpha) = (2\sigma_H^2)^{1-\beta} \left[ \Gamma(2 - \beta) - \Gamma^2 \left( \frac{3-\beta}{2} \right) \right]
\]

\[
\text{B. Multiple Receiving Antennas}
\]

With multiple receiving antennas, the contributions of each antenna have to be weighted with the complex coefficients $G_{m,l}$ which in multiple transmission case has been applied at the transmitter side, as below:

\[
r(t) = \sum_{l=0}^{L-1} G_{l,m} r^{(l)}(t)
\]

being $r^{(l)}(t)$ the signal received on the $l$th antenna. Hence, we can write:

\[
r(t) = \sqrt{\frac{2E_b}{LM}} \sum_{k=0}^{N_u-1} \sum_{m=0}^{+\infty} \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} G_{m,l} \alpha_{m,l} c_m^{(k)} a^{(k)}[i]
\]

\[
\times g'[t - iT_b + \phi_m + \psi_m] + \sum_{l=0}^{L-1} n_l(t)
\]

being $n_l(t)$ the noise at the $l$-th antenna supposed to be Gaussian with two-side power spectral density (PSD) $N_0/2$. For what concern the evaluation of the decision variable, the procedure is similar to the one presented in [4], except for the further degree of freedom given by spatial diversity. Hence, the statistical distribution of $U N$ and $I$ results in:

\[
U \sim \mathcal{N} \left( \sqrt{E_0 \delta_d ML} \left\{ a_{1-\beta} \right\}, \sigma_U^2 \right)
\]

\[
N \sim \mathcal{N} \left( 0, \sigma_N^2 = ML \frac{N_0}{2} (2\sigma_H^2)^{-\beta} \Gamma(1 - \beta) \right)
\]

\[
I \sim \mathcal{N} \left( 0, \sigma_I^2 = E_0 \delta_d L (N_u - 1) \zeta_\beta(\alpha) \right)
\]

\[
\text{IV. PERFORMANCE EVALUATION}
\]

In this section we derive the BEP and the optimum PC parameter, $\beta$, defined as the value of $\beta$ minimizing the BEP.

\[
\text{A. Multiple Transmitting Antennas}
\]

Following the previously mentioned assumptions, the BEP can be evaluated as:

\[
P_b = \frac{1}{2} \text{erfc} \left( \frac{U}{\sqrt{2(\sigma_I^2 + \sigma_N^2)}} \right) = \frac{1}{2} \text{erfc} \left( \sqrt{L \cdot \text{SNIR}} \right)
\]

where

\[
\text{SNIR} = \frac{\gamma}{2} \frac{\Gamma^2 \left( \frac{3-\beta}{2} \right)}{\Gamma(2 - \beta) - \Gamma^2 \left( \frac{3-\beta}{2} \right) + \Gamma(1 - \beta)}
\]

\[
\frac{\Gamma^2 \left( \frac{3-\beta}{2} \right)}{\Gamma(2 - \beta) - \Gamma^2 \left( \frac{3-\beta}{2} \right) + \Gamma(1 - \beta)}
\]
represents the signal-to-noise plus interference ratio for the single transmission/reception scenario described in [4] and
\[ \gamma = \frac{2\sigma_eta^2E_b\delta_d}{N_0}, \] (30)
is the mean SNR averaged over small-scale fading. Note that, being SNIR equal to the single transmission/reception case, the optimum value of \( \beta \) minimizing the BEP does not change with multiple transmitting antennas, but the performance is improved by the spatial diversity.

\section*{B. Multiple Receiving Antennas}

The unconditioned BEP with \( L \) receiving antennas is given by:
\[ P_b \simeq \frac{1}{2} \text{erfc} \left( \sum_{l=0}^{L-1} \sqrt{\frac{E_b\delta_d}{N_0L}} (2\sigma_eta^2)^{1-\beta} \frac{1}{2} \Gamma \left[ 3 - \frac{\beta}{2} \right] \right) \] (31)
\[
\times \frac{1}{\sqrt{\sum_{l=0}^{L-1} \frac{E_b\delta_d}{N_0L} 2N_u-1_{\beta}l + (2\sigma_eta^2)_{\beta}l} \Gamma \left[ 1 - \beta \right]} \]
being
\[ \sigma_eta^2(\beta, l) = (2\sigma_eta^2)^{1-\beta} \frac{1}{2} \Gamma \left[ 2 - \beta \right] - \Gamma \left[ 3 - \frac{\beta}{2} \right] \] (32)
\[ + \sum_{l=0,l \neq l}^{L-1} \left\{ \sigma_eta^2(\beta, l) - \frac{1}{2} (2\sigma_eta^2)^{1-\beta} (2\sigma_eta^2)^{1-\beta} \frac{1}{2} \Gamma \left[ 3 - \frac{\beta}{2} \right] \right\}, \]
where, due to the independence among the subcarriers, \( \sigma_eta^2(\beta, l) = 0.5 \cdot \mathbb{E} \{ \alpha_m, l \alpha_{m, l} \}, \) \( \forall \ m \). In case of uncorrelated antennas, the second term in (32) is zero, thus (31) reduces to the following expression:
\[ P_b \simeq \frac{1}{2} \text{erfc} \left( \sqrt{\text{SNIR}} \right), \] (33)
where
\[ \text{SNIR} = \frac{2M - 1}{N_u} \left[ \Gamma(2 - \beta) - \Gamma \left[ \frac{3 - \beta}{2} \right] \right] \] (34)

It is immediate to verify that when \( L = 1 \), (33) reduces to the expression found in [4]. Note that (32) is general, thus valid also for correlated fading among the antennas.

We now aim at evaluation the optimum value of \( \beta \) which minimizes the BEP. It can be found out that for spatial uncorrelated antennas:
\[ \beta_{\text{opt}} = \arg \min_{\beta} \{ P_b \} = \arg \max_{\beta} \{ \text{SNIR} \}. \] (35)

By defining the parameter
\[ \xi = \frac{2\gamma}{M}, \] (36)
we find that (36) leads to:
\[ \xi = L \left( \Psi \left( \frac{3-\beta}{2} \right) - \Psi(1-\beta) + \beta - 1 \right)^{-1}, \] (37)
where \( \Psi(x) \) is the logarithmic derivative of the Euler Gamma function. Note that \( \xi \) is a monotonic function of \( \beta \), thus invertible, and it confirms that if the system is noise-limited MRC is optimum, while if the system is interference limited, a choice close to ORC is required.

It is also possible to observe that for high values of mean SNR, \( \gamma \), the second term at the denominator of (30) is negligible, thus, for interference-limited systems the SNIR evaluated for multiple transmissions agrees with the one of multiple receptions.

\section*{V. NUMERICAL RESULTS}

The mean BEP as a function of \( \beta \) for various SNRs, \( \gamma \), and diversity degrees, \( L \), at the transmitter and at the receiver is shown in Fig. 2 in fully loaded system conditions (\( N_u = M = 1024 \)). In Fig. 2(a) the improvement on the BEP given by spatial diversity at the transmitter can be appreciated especially increasing the mean SNR, \( \gamma \). It is also noticeable that the performance strongly depends on \( \beta \). Same conclusions can be drawn by observing Fig. 2(b), where the BEP vs. \( \beta \) is plotted considering multiple receiving antennas. From a comparison between the two figures, it appears that the optimum value of \( \beta \) is in the range \([0.4, 0.7]\) and, on equal diversity degrees, multiple transmission performs better than multiple reception.\(^1\)

This is caused by summation of noise on all the antennas branches when multiple reception is assumed.

When multiple antennas at the receiver are considered, Fig. 3 shows the mean BEP as a function of the mean SNR \( \gamma \) in [dB] for \( \beta = 0.5 \) and various diversity degrees, \( L \). Fully loaded (\( N_u = M = 1024 \)) and half loaded (\( N_u = M/2 \)) system conditions are considered. It is possible to observe that, by fixing a target BEP, \( P_b = 10^{-3} \), it can be achieved only if the system is half loaded with single reception, whereas a fully loaded system can be considered by adopting two antennas at the receiver.

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\section*{REFERENCES}


\(^1\)Note that, being in the downlink, multiple transmission can be easily implemented at the base station.
(a) Multiple transmission and single reception.

(b) Single transmission and multiple reception.

Fig. 1. Multiple transmission and multiple reception block schemes.

\[ P_b \]

\[ \gamma = 5 \text{dB} \]

\[ \gamma = 10 \text{dB} \]

\[ \gamma = 15 \text{dB} \]

L = 1

L = 2

L = 4

Fig. 2. Mean BEP as a function of \( \beta \) for different SNRs, \( \gamma \), and diversity degrees, \( L \).

Fig. 3. Mean BEP as a function of mean SNR for \( \beta = 0.5 \), various diversity degrees, \( L \), at the receiver side and fully or half loaded conditions.

