Robust Fuzzy Cascade Control Revised: Application to the Rotary Inverted Pendulum

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Abstract—Fuzzy controllers are used in many practical applications since they guarantee impressive robustness properties and the control design requires only simple rules which are based on heuristic knowledge. However, since the number of rules is increased with the number of control inputs and measurements the control design may become overloaded and the design efforts even require more time than with classical approaches. To overcome this drawback this paper revises a comprehensive robust fuzzy cascade control strategy where all controllers are based on a uniform fuzzy set with a minimum number of rules. This approach represents an interesting perspective in education since the individual controllers need to be only calibrated with the scaling on their inputs and outputs. To show the efficiency and the robustness properties of the proposed control framework this fuzzy cascade design method is applied to Quanser’s rotary inverted pendulum.

I. INTRODUCTION

During the past decades fuzzy control has emerged as one of the most active areas in control design and computational intelligence. Starting with Zadeh’s seminal papers [1]-[6] on the linguistic approach and system analysis based on fuzzy sets this novel control design approach has attract great interest among academic and industrial researchers [7], [8], [10], [11], [12], [13]. Today, fuzzy control provides an effective control design approach for complex ill-defined processes even if a skilled human operator has no fundamental knowledge of the underlying dynamics or even if the available sources of information are interpreted qualitatively, inexact, or uncertainly [9]. Here, the term effective control design approach means that the design efforts for the stabilization of the system are low while the system shows good robustness properties against model uncertainties and external disturbances. Thus fuzzy logic control may be viewed as a step toward a rapprochement between conventional precise mathematical control and human-like decision making, as indicated by [14]. Up to now fuzzy control has been successfully applied to a wide range of applications, i.e. water quality control [15], automatic train operation systems [16], elevator control [17], nuclear reactor control [18], automotive transmission control [19] and process control [20], [21]. However, it has to be remarked that up to now there exists no systematic design procedure for the corresponding fuzzy sets. Instead a skilled human operator defines linguistic control rules which are translated into a control law. Due to the dependency on a heuristic database fuzzy logic control (FLC) laws may become overloaded and the control design efforts may overcome the efforts for classical approaches in many practical applications. This fact represents a major obstacle for beginners in the field of fuzzy systems.

However, fuzzy logic control has nevertheless been incorporated within control education since several years. In many practical courses the well-known inverted pendulum is used for the first experimental studies. Here, an interesting control design task arises since this underactuated system should be stabilized around its unstable upright position. For this purpose several fuzzy control laws have been proposed so far [22], [23], [24], [25]. However, many fuzzy controllers suffer from a large number of linguistic rules since the inputs and outputs are divided into many sections to provide the best possible control performance. Additionally, all inputs and outputs are often incorporated within one single controller which leads to large and overloaded fuzzy sets [26].

In the early 1990s a comprehensive fuzzy cascade framework has been proposed by [27], [28], [29]. It was first applied to an inverted pendulum which was running on a straight path to balance the pole.

Thus, an inner loop controller (Fuzzy \( \dot{x}, \dot{x} \)) stabilizes the pole angle \( \phi \) around the upright position while the outer loop controller (Fuzzy \( x, \dot{x} \)) is able to hold the cart position \( x \) at its reference value \( x_r \). Both controllers require only the pole angle \( \phi \) and the cart position \( x \) with their corresponding velocities \( \dot{\phi} \) and \( \dot{x} \) as inputs while the output \( u = U_a \) is generated with a minimum fuzzy set including only 9 linguistic rules. The tuning of both controllers is very comprehensive and can be easily achieved with the scaling of the inputs and the outputs. In this contribution it will be shown that the same control framework can even be easily transferred from the original configuration to a rotary inverted pendulum while the control structure as well as the fuzzy sets of both controllers remain unchanged. Moreover, it will be emphasized that the proposed cascade controller
is suitable for educational purposes, especially for beginners in the field of fuzzy systems. Additionally, an experimental robustness analysis shows that the performance of the controller is superior to an LQR state feedback controller which has been designed as reference controller. Thus, this fuzzy cascade controller with its minimum number of fuzzy sets is also interesting for experienced control engineers. The remainder of the paper is organized as follows: In Section II a detailed nonlinear simulation model of the rotary pendulum is given and linearized around the upright position. The resulting linear time invariant model of the so-called inverted pendulum is used in Section III both for the design and controller. In Section IV experimental results are shown and a robustness analysis shows that the performance of the controller is superior to an LQR state feedback controller which has been designed as reference controller. Thus, this fuzzy cascade controller with its minimum number of fuzzy sets is also interesting for experienced control engineers.

II. MODELING

In this Section a mathematical model for the simulation of the rotary inverted pendulum is introduced. The experimental setup in Figure 2 consists of a vertical pendulum and a horizontal arm which is attached to an electric drive module in point O. The pendulum and the arm angular positions $\alpha$ and $\theta$ are measured with incremental encoders.

Fig. 2. Rotary inverted pendulum system

For the mechanical part of the nonlinear simulation model it is assumed that the pendulum mass is lumped in point B which is located at its geometric center. The coordinate frame is body-fixed to the arm in point A. Since the arm is rotating with $\dot{\theta}$ the absolute velocity and acceleration of the pendulum at point B can be expressed as

\[
\dot{x}_B = r \dot{\theta} - L \cos(\alpha) \dot{\alpha},
\]

\[
\ddot{y}_B = -L \sin(\alpha) \ddot{\alpha},
\]

(1)

and

\[
\dot{\dot{x}}_B = r \ddot{\theta} + L \sin(\alpha) \dot{\alpha}^2 - L \cos(\alpha) \dddot{\alpha},
\]

\[
\dot{\dot{y}}_B = -L \cos(\alpha) \dot{\alpha}^2 - L \sin(\alpha) \dddot{\alpha},
\]

(2)

respectively. Applying Newton’s second law to the pendulum in x and y direction, the resulting forces $F_x$ and $F_y$ at point B are given with

\[
F_x = mr \ddot{\theta} + mL \sin(\alpha) \dot{\alpha}^2 - mL \cos(\alpha) \dddot{\alpha},
\]

\[
F_y = mg - mL \cos(\alpha) \dot{\alpha}^2 - mL \sin(\alpha) \dddot{\alpha},
\]

(3)

where $m$ and $g$ denote the pendulum mass and the gravitational constant, respectively. For the rotational motion of the pendulum and the arm, Euler’s equation has to be applied about point B and $O$:

\[
J_p \ddot{\alpha} = F_L \cos(\alpha) - F_L \sin(\alpha),
\]

\[
J_b \ddot{\theta} = T_d - b_0 \dot{\theta} - F_x.
\]

(4)

Here, $J_p$ and $J_b$ represent the lumped moments of inertia at points B and O while the variables $T_d$ and $b_0 \dot{\theta}$ denote the drive torque and the viscous friction coefficient in the drive module. Substituting equations (3) into (4) the nonlinear simulation model of the inverted pendulum’s mechanical part is given with

\[
(J_b + mr^2) \ddot{\theta} - mL \cos(\alpha) \dot{\alpha} + mL \sin(\alpha) \dddot{\alpha} = T_d - b_0 \dot{\theta},
\]

\[-mL \cos(\alpha) \dddot{\alpha} - (J_p + mL^2) \ddot{\alpha} - mgL \sin(\alpha) = 0.
\]

(5)

For the simulation of the electric drive module it is assumed that the drive torque $T_d$ depends mainly on the motor’s armature current $i_a$:

\[
T_d = K_p K_i i_a,
\]

(6)

where $K_p$ and $K_i$ denote the conversion ratio of the drive’s transmission and the torque constant, respectively. Since the armature current $i_a = \frac{U_d}{K_p R_a} - \frac{K_m K_i K_r^2}{K_r} \dot{\theta}$ is a function of the input voltage $U_d$, the armature resistance $R_a$, the dc-motor constant $K_m$ and the arm’s angular velocity $\dot{\theta}$ the drive torque $T_d$ is given with

\[
T_d = \frac{K_p K_i U_d}{R_a} - \frac{K_m K_i K_r^2}{R_a} \dot{\theta}.
\]

(7)

Substituting equation (7) into (5) the overall nonlinear simulation model of the inverted pendulum is given by

\[
\alpha \dddot{\alpha} - b \cos(\alpha) \dddot{\alpha} + b \sin(\alpha) \dot{\alpha}^2 + c \theta = f U_d,
\]

\[-b \cos(\alpha) \dddot{\alpha} + c \ddot{\alpha} - d \sin(\alpha) = 0,
\]

(8)

where $a = J_b + mr^2$, $b = mL$, $c = J_p + mL^2$, $d = mgL$, $e = \frac{K_m K_i K_r^2}{K_r}$ and $f = \frac{K_p K_i}{K_r}$. However, for implementation issues the nonlinear simulation model (8) should be solved for $\dot{\alpha}$ and $\theta$. Then, the resulting simulation model is given with

\[
\dot{\alpha} = \frac{1}{ac - b^2 \cos^2(\alpha)} (adsin(\alpha) - b^2 \sin(\alpha) \cos(\alpha) \dot{\alpha}^2 - be \cos(\alpha) \ddot{\alpha} + b \cos(\alpha) U_a),
\]

\[
\dot{\theta} = \frac{1}{ac - b^2 \cos^2(\alpha)} (-b \cos(\alpha) \dot{\alpha}^2 + b \ddot{\alpha} \cos(\alpha) - ce \dot{\theta} + c f U_a).
\]

(9)
For control design purposes the nonlinear model in (9) is linearized at the so-called upright position, i.e. \( \alpha_0 = \theta_0 = 0 \). Then, the linear time invariant model is given with

\[
\begin{bmatrix}
\alpha' \\
\theta' \\
\alpha'' \\
\theta''
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -b e & a c
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\theta \\
\dot{\alpha} \\
\dot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
-1\frac{c}{a c - b e}
\end{bmatrix}
U_d,
\]

and the input vector \( \mathbf{b} \) has been calculated to

\[
\mathbf{A} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
55.7 & 0 & 0 & -13.89 \\
50.41 & 0 & 0 & -23.9
\end{bmatrix},
\mathbf{b} = \begin{bmatrix} 0 \\
0 \\
24.76 \\
42.6 \end{bmatrix}.
\] (11)

Here, the unknown model parameters have been identified experimentally.

III. CONTROL DESIGN

The control design for the rotary inverted pendulum system includes the swing-up of the pendulum and the stabilization in its upright position. Swing-up controllers are often designed using energy based design methods or even heuristically. In the remainder of this work we focus on the design of control laws for the stabilization of the pendulum. Of course, there exists still a wide variety of stabilizing control laws ranging from integrator backstepping [30] to sliding mode control [32]. Often a linear state feedback controller

\[
u = -k^T \mathbf{x}
\] (12)

is designed in the first steps and serves as a reference control law. Using the Linear Quadratic Regulator (LQR) design method the saturation on the control input can be easily incorporated into the control law [33] as well. For this purpose a cost function

\[
J = \int_0^\infty (\mathbf{x}^T \mathbf{Q} \mathbf{x} + uR \mathbf{u}) d\tau
\] (13)

is minimized while \( \mathbf{Q} \) and \( \mathbf{R} \) serve as weightings on the system state \( \mathbf{x} = [\alpha, \theta, \dot{\alpha}, \dot{\theta}]^T \) and the control input \( u = U_d \). To calculate the feedback gain \( \mathbf{k}^T \) the algebraic Riccati equation

\[
\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} - \mathbf{P} \mathbf{R}^{-1} \mathbf{b}^T \mathbf{P} + \mathbf{Q} = 0
\] (14)

is solved offline using MATLAB. The resulting feedback gain \( \mathbf{k}^T \) is given with \( \mathbf{k}^T = \mathbf{R}^{-1} \mathbf{b}^T \mathbf{P} \) where the positive definite matrix \( \mathbf{P} \) represents the unique solution of (14).

In the remainder of this Section a fuzzy control design strategy will be discussed. Similar to Section I a cascade structure with two identical fuzzy logic controllers is proposed for the stabilization of the pendulum and the arm in their upright and middle positions, respectively. That means the inner fuzzy logic controller (Fuzzy \( \alpha, \dot{\alpha} \)) calculates the control input \( u_d \) from the difference \( \alpha \) of the measured pendulum angle \( \alpha \) and its reference value \( \alpha_0 \) and from the pendulum velocity \( \dot{\alpha} \). The variable \( \alpha_0 \) is given from the outer loop controller (Fuzzy \( \theta, \dot{\theta} \)) which uses the measured arm position \( \theta \), its reference value \( \theta_0 \) and the arm velocity \( \dot{\theta} \) as inputs. Thus, the control gains \( k_d \) and \( k_b \) correspond to \( k_q \) and \( k_p \) in Section I and allow to tune the closed-loop behaviour of the inner and the outer loop system individually.

In the remainder of this Section the design of such a single FLC is explained. Here, we focus on the inner loop controller which includes a fuzzification interface, a knowledge base with its decision making logic (also known as inference engine) and a defuzzification interface. Moreover the inputs and the outputs of the controller have to be scaled as shown in Figure 3.

![Figure 3: Closed-loop system structure with fuzzy logic controller](image)

Firstly, the fuzzification and defuzzification interfaces are considered. Here, the behaviour of both FLC inputs \( \alpha \) and \( \dot{\alpha} \) and of the FLC output \( U_d \) has to be described with linguistic variables such that the corresponding signal range is divided into a positive part (P), a negative part (N) and a part in the neighbourhood of the zero position (NP). For this purpose three piecewise linear functions are defined each for the pendulum angle \( \alpha \) and its angular velocity \( \dot{\alpha} \) (see Figure 4a). All these membership functions are symmetric to the upright position of the rotary inverted pendulum and scaled to the angle and velocity limitations \( |\alpha_{max}| = 45 \text{ degrees} \) and \( |\dot{\alpha}_{max}| = 360 \text{ degrees/s} \), respectively. The membership
functions for the FLC output (see Figure 4b) are scaled to the maximum controller output $|U_{a,max}| = 15V$.

Of course this membership function design represents only a very simple and conservative approach. However, in the field of education it is not easy to give a general design strategy to beginners in this field. To reduce the complexity of the final fuzzy set it turned out that the number of membership functions should be as small as possible. This is in particular true for fuzzy logic controllers with more than two input variables. If it is possible to apply the aforementioned conservative FLC design approach the corresponding fuzzy set requires only 9 rules. Considering the FLC controller for the inner and outer loop of the inverted pendulum all these rules are summarized in a so-called Karnough table for consistency.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\bar{\alpha}$</th>
<th>$\theta$</th>
<th>$\bar{\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>NP</td>
<td>P</td>
<td>NP</td>
<td>N</td>
</tr>
<tr>
<td>P</td>
<td>N</td>
<td>P</td>
<td>NP</td>
</tr>
<tr>
<td>NP</td>
<td>P</td>
<td>NP</td>
<td>NP</td>
</tr>
</tbody>
</table>

Based on the rules of Table I the corresponding fuzzy output variable of the controller can be calculated with the Mamdani inference scheme. However, since the input variables are all linguistic variables the calculated output is linguistic, too. To generate a sharp controller output $u = U_a$ directly from this resulting variable the so-called center of gravity method is applied as shown in Figure 5. Finally, it has to be mentioned that the overall implementation efforts can be considerably minimized if a characteristic map of the controller is used instead of the complete software code with all its design rules. Since the inner loop controller (Fuzzy $\bar{a}$, $\bar{\alpha}$) for the pendulum’s motion has two inputs and one output this characteristic map can be also easily visualized as shown in Figure 6.

IV. EXPERIMENTAL RESULTS

In this Section some representative experimental results are given to show the efficiency and the robustness properties of the proposed fuzzy cascade controller. For this purpose the control design approaches of Section III have been implemented on a test rig which includes Quanser’s rotary inverted pendulum and a dSpace rapid control prototyping (RCP) system. The corresponding test rig assembly is shown in Figure 7.

With regards to the implementation issues it has to be noted that both control design approaches of Section III
require all system states to be available for feedback. Since the angular velocities $\dot{\alpha}$ and $\dot{\theta}$ are hard to measure directly they are estimated from the measured angular positions $\alpha$ and $\theta$. For this purpose a numerical differentiator turned out to be sufficient for sampling times $t_s$ around or greater than 10ms. If lower sampling times are required dedicated high-pass filtering techniques lead to better performance. In the following experimental study the parameter $T_s$ was however set to 10ms.

For an objective analysis of the robustness properties of both controllers the pendulum is stabilized around its upright position while the arm moves to the middle of the considered range. After $t = 0.4$ s the measured pendulum angle $\alpha$ is modified with an additional impulse which represents a fault on the pendulum angle measurement device. In the following representative experiment the corresponding amplitude is set to $\Delta \alpha = 5$ degrees and the impulse width is given with $\Delta t = 250$ ms. Consequently, the pendulum angle $\alpha$ and the arm position $\theta$ move away from their initial position and the controller is tasked to minimize the effects of the sensor signal disturbance. These experiments are performed for both controllers of Section III and for two different pendulum configurations. Firstly, the standard pendulum configuration is considered. In the second case an extra mass $\Delta m = 50$ g is mounted on the outer tip of the pendulum. Figures 8 and 9 show representative results of the measured pendulum angle $\alpha$ and the arm position $\theta$ for the reference LQR controller and the fuzzy cascade controller, respectively. In both Figures the blue lines represent the standard pendulum configuration and the red lines show the measurements of the extra mass configuration.

From Figures 8 and 9 it can be clearly seen that both controllers are able to stabilize the pendulum in its initial position after the sensor fault has occurred. Regarding the arm position there exist some differences between the reference LQR and the fuzzy cascade controller. Here, a limit cycle arises in the arm position if the reference LQR controller is used. This limit cycle can be ascribed to the friction effects of the real plant which have been neglected on the control orientated LTI-model in (10) and (11). However, if the fuzzy cascade controller is used instead this effect vanishes clearly.

**TABLE II**

<table>
<thead>
<tr>
<th>Fuzzy</th>
<th>LQR</th>
</tr>
</thead>
</table>
| $|\Delta \alpha|_{\text{mean}}$ (degrees) | 0.0577 | 2.1317  
| $|\Delta \alpha|_{\text{max}}$ (degrees) | 2.7738 | 5.4727  
| $|\Delta \theta|_{\text{mean}}$ (degrees) | 5.7855 | 6.3066  

With regards to robust disturbance rejection properties it can be seen that the standard configuration and the extra mass configuration lead to similar results for both controllers. However if the deviation between the pendulum angles from both configurations is analyzed in detail it can be found that the absolute values of the corresponding mean and maximum difference $|\Delta \alpha|_{\text{mean}}$ and $|\Delta \alpha|_{\text{max}}$ is much smaller for the fuzzy cascade controller than for the reference LQR controller. The same is true for the absolute values of the mean and maximum difference in the arm position $|\Delta \theta|_{\text{mean}}$ and $|\Delta \theta|_{\text{max}}$. However, here the effect is not so evident like for the pendulum angle, but still present. Similar results
have been obtained for further experiments within the study. With regards to the overall experimental study it can be finally stated that the fuzzy cascade controller shows better robustness properties than the reference LQR controller if an extra mass is added at the outer tip of the pendulum.

V. CONCLUSIONS AND FUTURE WORKS

This paper deals with a fuzzy cascade control framework which has been designed in the early 1990s for an inverted pendulum that was running on a straight path to balance its pole. Recently this original framework has been revised and applied to a rotary inverted pendulum configuration. Here, it could be demonstrated that the original cascade control structure with both fuzzy logic controllers could be applied without any structural or design changes. The tuning is very intuitively since it is achieved with the input and output scaling of both individual controllers. Since the proposed control structure is simplified as far as possible its usage in any educational purposes (i.e. beginner tutorials) is highly recommended. Moreover, a representative experimental analysis shows impressively that the fuzzy logic controller leads to better robustness properties of the closed-loop system than the reference LQR controller. Thus, the cascade structure with its simple and conservative FLC design approach is not only interesting for beginners but also for experienced engineers and scientists in the field of fuzzy systems.

In the upcoming work the proposed fuzzy control design approach will be applied to industrial control design tasks in the field of automotive and aerospace engineering. Moreover the adaptive fuzzy control design approach of [29] will be revised and extended for further improvements to the robustness properties.

REFERENCES