Homogeneously orderable graphs

Andreas Brandstädt*,**, Feodor F. Draganb,1, Falk Nicolai*c,2

*Universität Rostock, Fachbereich Informatik, Lehrstuhl für Theoretische Informatik,
D 18051 Rostock, Germany
bDepartment of Mathematics and Cybernetics Moldova State University,
A. Mateevici str. 60 Chișinău 277009 Moldova
cGerhard-Mercator-Universität-GH-Duisburg FB Mathematik FG Informatik I D 47048 Duisburg, Germany

Received January 1994
Communicated by M. Nivat

Abstract

In this paper we introduce homogeneously orderable graphs which are a common generalization of distance-hereditary graphs, dually chordal graphs and homogeneous graphs. We present a characterization of the new class in terms of a tree structure of the closed neighborhoods of homogeneous sets in 2-graphs which is closely related to the defining elimination ordering.

Moreover, we characterize the hereditary homogeneously orderable graphs by forbidden induced subgraphs as the house-hole-domino-sun-free graphs.

The local structure of homogeneously orderable graphs implies a simple polynomial-time recognition algorithm for these graphs.

Finally, we give a polynomial-time solution for the Steiner tree problem on homogeneously orderable graphs which extends the efficient solutions of that problem on distance hereditary graphs, dually chordal graphs and homogeneous graphs.

1. Introduction

Several important graph classes have a certain kind of tree structure which can be formulated in terms of hypergraph (namely hypertree) properties. Among them are the well-known chordal graphs (dual hypertrees of maximal cliques), the dually chordal graphs (hypertrees of maximal cliques, dual hypertrees of closed neighbourhoods [6, 14]) and the distance-hereditary graphs (dual hypertrees of maximal cographs [23, 24]).
The tree structure of the last two classes turned out to be useful especially for some distance and domination-like problems (cf. e.g. [11, 7, 8, 5, 12, 13]). The characterization of distance-hereditary graphs as dual hypertrees of cographs in [24] is used in [25] for designing efficient algorithms solving various Hamiltonian problems.

In [10] homogeneous graphs as a generalization of distance-hereditary graphs are introduced which lead to a polynomial-time algorithm for the Steiner tree problem on these graphs.

In this paper we define a new class of graphs which is a common generalization of distance-hereditary graphs, dually chordal graphs and homogeneous graphs. (For a recent survey on special graph classes cf. [4]). We present a characterization of the new class in terms of a tree structure of the closed neighbourhoods of homogeneous sets in 2-graphs which is closely related to the defining elimination ordering.

Moreover, we characterize the hereditary homogeneously orderable graphs by forbidden induced subgraphs as the house-hole-domino-sun-free graphs.

Finally, we give a polynomial-time solution for the Steiner tree problem on homogeneously orderable graphs which extends the efficient solutions of that problem on distance-hereditary graphs, dually chordal graphs and homogeneous graphs.

2. Preliminaries

Throughout this paper all graphs $G = (V, E)$ are finite, undirected, simple (i.e. loop-free and without multiple edges) and connected.

The (open) neighbourhood of a vertex $v$ of $G$ is $N(v) := \{u \in V: uv \in E\}$. The closed neighbourhood of $v$ is $N[v] := N(v) \cup \{v\}$. For a vertex set $U \subseteq V$ let

$$N(U) := \bigcup_{u \in U} N(u) \setminus U$$

and

$$N[U] := \bigcup_{u \in U} N[u].$$

A nonempty set $U \subseteq V$ is homogeneous in $G = (V, E)$ iff all vertices of $U$ have the same neighbourhood in $V \setminus U$:

$$N(u) \cap (V \setminus U) = N(v) \cap (V \setminus U)$$

for all $u, v \in U$,

i.e. any vertex $w \in V \setminus U$ is adjacent to either all or none of the vertices from $U$.

A homogeneous set $H$ is proper iff $|H| < |V|$. Trivially, for each $v \in V$ the singleton $\{v\}$ is a proper homogeneous set. Note also that for a subset $V' \subseteq V$ if a set $H \subseteq V'$ is homogeneous in $G$ then it is homogeneous also in the induced subgraph $G(V')$ but not vice versa.

A path is a sequence of vertices $v_1, \ldots, v_k$ such that $v_iv_{i+1} \in E$ for $i = 1, \ldots, k - 1$; its length is $k$. A graph $G$ is connected iff for any pair of vertices of $G$ there is a path in $G$ joining both vertices. The maximal-induced connected subgraphs of $G$ are called connected components.
The distance \( d_G(u, v) \) of vertices \( u, v \) is the minimal length of any path connecting these vertices. Obviously, \( d_G \) is a metric on \( G \). If no confusion can arise we will omit the index \( G \).

The \( k \)-th neighbourhood \( N^k(v) \) of a vertex \( v \) of \( G \) is the set of all vertices of distance \( k \) to \( v \):

\[
N^k(v) := \{ u \in V : d_G(u, v) = k \}.
\]

For convenience we denote by \( N^k_F(v) \) the intersection \( N^k(v) \cap F \), where \( F \subseteq V \).

The disk of radius \( k \) centred at \( v \) is the set of all vertices of distance at most \( k \) to \( v \):

\[
D(v, k) := \{ u \in V : d_G(u, v) \leq k \} = \bigcup_{i=0}^{k} N^i(v).
\]

Note that \( N[v] = D(v, 1) \) — a simple identity frequently used in this paper. Analogously to neighbourhoods of sets we define for \( U \subseteq V \)

\[
D(U, k) := \bigcup_{u \in U} D(u, k).
\]

For convenience we denote by \( D_F(U, k) \) the intersection \( D(U, k) \cap F \), where \( F \subseteq V \).

The \( k \)-th power \( G^k \) of a graph \( G = (V, E) \) is the graph with vertex set \( V \) and edges between vertices \( u, v \) with distance \( d_G(u, v) \leq k \).

Let \( e(v) \) denote the eccentricity of vertex \( v \in V \):

\[
e(v) := \max \{ d(v, u) : u \in V \}.
\]

Then, the radius \( \text{rad}(G) \) of \( G \) is the minimum over all eccentricities \( e(v), v \in V \), whereas the diameter \( \text{diam}(G) \) of \( G \) is the maximum over all eccentricities \( e(v) \) for \( v \) in \( V \).

In the sequel a subset \( U \) of \( V \) is a \( k \)-set iff \( U \) induces a clique in the power \( G^k \), i.e. for any pair \( x, y \) of vertices of \( U \) we have \( d_G(x, y) \leq k \). A graph \( G \) is a \( k \)-graph iff \( \text{diam}(G) \leq k \). If \( U \) is a subset of \( V(G) \) then \( U \) is called \( k \)-graph of \( G \) iff the induced subgraph \( G_U \) is a \( k \)-graph (i.e. \( \text{diam}(G_U) \leq k \)) and for any set \( U' \supseteq U \) holds \( \text{diam}(G_{U'}) \geq k + 1 \), i.e. \( G_U \) is a maximal induced subgraph of diameter \( \leq k \) in \( G \). Thus each \( k \)-graph is a \( k \)-set but the converse is in general not true as pointed out in Fig. 1 for \( k = 2 \).

Let \( U_1, U_2 \) be disjoint subsets of \( V \). If every vertex of \( U_1 \) is adjacent to every vertex of \( U_2 \) then \( U_1 \) and \( U_2 \) form a join, denoted by \( U_1 \bowtie U_2 \).

Let \( H = (V, \mathcal{E}) \) be a hypergraph, i.e. \( \mathcal{E} \) is a set of subsets of \( V \). Throughout this paper all hypergraphs are assumed to be reduced, i.e. no hyperedge is properly contained in another one.

For every vertex \( v \in V \) let \( \mathcal{E}(v) := \{ e \in \mathcal{E} : v \in e \} \) be the set of hyperedges incident to vertex \( v \). Then the dual hypergraph \( H^* \) of \( H \) is the hypergraph with vertex set \( \mathcal{E} \) and hyperedges \( \mathcal{E}(v), v \in V \).

The line graph \( L(H) \) is the intersection graph of the hyperedges, the 2-section graph \( 2\text{SEC}(H) \) is the graph with vertex set \( V \), where two vertices are adjacent iff there is a hyperedge in \( H \) containing both.
The following properties are well-known.

**Proposition 2.1.** Let $H = (V, \mathcal{E})$ be a hypergraph. Then

(i) $(H^*)^*$ is isomorphic to $H$ and

(ii) $L(H)$ is isomorphic to $2\text{SEC}(H^*)$.

Let $\mathcal{N}(G) = \{N[v]: v \in V\}$ be the *neighbourhood hypergraph* of $G$ and let $\mathcal{C}(G) = \{C: C$ is a maximal clique in $G\}$ be the *clique hypergraph* of $G$.

Tree structure of a hypergraph can be defined as follows: A hypergraph $H = (V, \mathcal{E})$ is a *hypertree* iff there is a tree $T$ with vertex set $V$ such that any hyperedge $e$ of $H$ induces a subtree in $T$. A hypergraph $H = (V, \mathcal{E})$ is a *dual hypertree* iff $H^*$ is a hypertree.

Hypertrees and dual hypertrees are closely related to chordal graphs – the graphs which do not contain any chordless cycle of length $\geq 4$. Walter et al. (cf. [17]) have shown that a graph is chordal iff it is the intersection graph of subtrees of a tree. The constructive proof shows that we can use the maximal cliques as the vertex set of a representing tree model. Hence we can conclude

**Theorem 2.2** A graph $G$ is chordal iff its clique hypergraph is a dual hypertree.

Let $\mathcal{M}$ be a set system over a set $E$. The system $\mathcal{M}$ has the *Helly property* (or for short: $\mathcal{M}$ is Helly) iff each subsystem of pairwise intersecting sets of $\mathcal{M}$ has a nonempty common intersection. A hypergraph $H = (V, \mathcal{E})$ is Helly iff $\mathcal{E}$ has the Helly property.

A hypergraph $H$ is *conformal* iff any clique of $2\text{SEC}(H)$ is contained in some hyperedge. It is well-known that a hypergraph $H$ is conformal iff its dual $H^*$ has the Helly property.

**Theorem 2.3** (Duchet [15]; Flament [16]). (i) A hypergraph $H$ is a hypertree iff $H$ is Helly and $L(H)$ is chordal.

(ii) A hypergraph $H$ is a dual hypertree iff $H$ is conformal and $2\text{SEC}(H)$ is chordal.
Next we recall the definition and some characterizations of dually chordal graphs. A vertex \( u \) is a maximum neighbour of a vertex \( v \) iff \( D(u, 1) = D(v, 2) \). A maximum neighbourhood ordering of a graph \( G \) is a sequence \( (v_1, \ldots, v_n) \) such that for all \( i = 1, \ldots, n \) the vertex \( v_i \) has a maximum neighbour in \( G_i := G_{\{v_1, \ldots, v_{i-1}\}} \).

**Theorem 2.4** (Brandstädt et al. [6]). The following conditions are equivalent:

(i) \( G \) has maximum neighbourhood ordering.
(ii) The clique hypergraph \( \mathcal{C}(G) \) is a hypertree.
(iii) The neighbourhood hypergraph \( \mathcal{N}(G) \) is a hypertree.

Due to condition (ii) of Theorem 2.4 graphs with maximum neighbourhood ordering are called dually chordal. In [11, 7, 8] efficient algorithms for various distance and domination-like problems are given using this hypertree structure.

Finally, we recall the definition and some characterizations of distance-hereditary graphs. An induced subgraph \( H \) of \( G \) is isometric iff the distances \( d_H(u,v) \) of any vertices \( u,v \) in \( H \) are the same as in \( G \). A graph \( G \) is distance-hereditary iff each connected induced subgraph \( H \) is isometric. This graph class was introduced in [21]. Some characterizations and a linear-time recognition algorithm are given in [1, 9, 18].

The following characterizations are due to [24]: A vertex \( v \) is called 2-simplicial iff the disk \( D(v, 2) \) induces a cograph in \( G \). Hereby a cograph is a \( P_4 \)-free graph, i.e. a connected cograph is a hereditary 2-graph. An ordering \( \tau = (v_1, \ldots, v_n) \) of the vertices of \( G \) is a 2-simplicial ordering iff for every index \( i = 1, \ldots, n \) the vertex \( v_i \) is 2-simplicial in \( G_i := G_{\{v_1, \ldots, v_{i-1}\}} \). A 2-simplicial vertex \( v \) is \( d \)-extremal iff \( e(v) = \text{diam}(G) \). Analogously, we can define a \( d \)-extremal ordering. Let \( \mathcal{E}(G) \) denote the set of maximal connected cographs of \( G \).

**Theorem 2.5** (Nicolai [24]). Let \( G = (V, E) \) be a graph. Then the following conditions are equivalent:

(i) \( G \) is distance-hereditary.
(ii) The cograph-hypergraph \( \mathcal{E}(G) \) of \( G \) is a dual hypertree.
(iii) \( G \) has a 2-simplicial ordering.
(iv) \( G \) has a \( d \)-extremal ordering.

Moreover, \( d \)-extremal vertices have nice local properties:

**Proposition 2.6** (Nicolai [24]). Let \( G \) be a distance-hereditary graph and \( v \) be a \( d \)-extremal vertex. Then there is a set \( S \subseteq N(v) \) which is homogeneous (in \( G \)) and dominates \( D(v, 2) \).

Note, that in dually chordal graphs a maximum neighbour \( v \) of a vertex \( u \) dominates \( D(u, 2) \) too. Thus, we can generalize these properties in the following way:

A vertex \( v \) of \( G = (V, E) \) with \( |V| > 1 \) is \( k \)-extremal iff (the subgraph induced by) \( D(v, 2) \) contains a proper homogeneous dominating set. More exactly: There is
A proper subset \( H \subset D(v, 2) \) which is homogeneous in \( G \) and for which \( D(v, 2) \subseteq D(H, 1) \) holds. A sequence \( \sigma = (v_1, \ldots, v_n) \) is a \( h \)-extremal ordering iff for any \( i = 1, \ldots, n - 1 \) the vertex \( v_i \) is \( h \)-extremal in \( G_i := G_{\{v_1, \ldots, v_i\}} \). A graph \( G \) is homogeneously orderable iff \( G \) has a \( h \)-extremal ordering. Thus, we immediately obtain

**Corollary 2.7.** Dually chordal and distance-hereditary graphs are homogeneously orderable graphs.

Sometimes we will write \( \sigma = ((v_1, H_1), \ldots, (v_n, H_n)) \) to emphasize the homogeneous dominating sets \( H_j \) for \( v_j \) in \( G_j \).

Now we present two lemmata which will be used frequently in the sequel.

**Lemma 2.8.** If \( H \) is a proper homogeneous set in \( G \) and \( x, y \in D(H, 1) \) then \( d(x, y) \leq 2 \).

**Proof.** Since \( H \) is proper and \( G \) is connected there must be a vertex \( v_H \) of \( V \setminus H \) adjacent to any vertex of \( H \).

If both \( x \) and \( y \) are in \( N(H) \) then by the definition of a homogeneous set both vertices are adjacent to each vertex of \( H \). If both vertices are within \( H \) they are adjacent to \( v_H \). Finally, if one vertex is in \( H \) and the other one in \( N(H) \) they are adjacent. \( \square \)

We immediately conclude

**Corollary 2.9.** If \( v \) is \( h \)-extremal in \( G \) then \( D(v, 2) \) is a 2-graph.

**Lemma 2.10.** Let \( v \) be a vertex in a graph \( G = (V, E) \).

(i) If \( e(v) \geq 2 \) and \( v \) is \( h \)-extremal then there is a proper homogeneous set \( H \subset N(v) \) which dominates \( D(v, 2) \).

(ii) If \( e(v) = 1 \) then \( G \) is homogeneously orderable with \( h \)-extremal ordering \( \sigma = ((v_1, \{v\}), \ldots, (v_{n-1}, \{v\})) \), where \( V = \{v_1, \ldots, v_{n-1}, v\} \).

**Proof.** For point (ii) there is nothing to show. So let \( e(v) \geq 2 \) and let \( H \) be a proper homogeneous dominating set in \( D(v, 2) \). If \( v \notin H \) then \( v \) must be dominated by some \( h \in H \). Since \( H \) is homogeneous we immediately conclude \( H \subseteq N(v) \).

Now consider the case \( v \in H \). Assume first that \( N^3(v) \neq \emptyset \) and let \( u \in N^3(v) \). Since \( H \) is homogeneous, \( v \in H \) and \( v \notin E \) for any neighbour \( x \) of \( u \) in \( N^2(v) \) no one of these neighbours \( x \) is in \( N(H) \). But \( H \) dominates \( D(v, 2) \), hence \( N(u) \cap N^2(v) \subset H \). But now \( H \) is not homogeneous.

Finally, assume that \( N^3(v) = \emptyset \), i.e. \( G \) is a 2-graph. Since \( v \) is in the homogeneous set \( H \) and \( v \) is not adjacent to any vertex of \( N^2(v) \), but \( H \) dominates \( D(v, 2) \), the second neighbourhood \( N^2(v) \) must be completely contained in \( H \). But now, \( N(v) \setminus H \) is homogeneous in \( G \) and dominates \( D(v, 2) \), so we have the desired set. \( \square \)
Therefore, for a homogeneously orderable graph $G$ with $h$-extremal ordering $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$ we will assume $H_j \subseteq N_G(v_j)$ for $j = 1, \ldots, n - 1$ in the sequel.

3. Homogeneously orderable graphs and corresponding hypergraphs

Recall that $F \subseteq V$ is a $k$-graph iff $\text{diam}(G_F) \leq k$ and for any set $U \supseteq F \text{ diam}(G_U) > k$ holds. Denote by $2\mathcal{G}(G)$ the set of all 2-graphs of $G$ and by $\mathcal{H}(G)$ the set of all maximal proper homogeneous sets of $G$.

Let $\mathcal{D}\mathcal{H}(G) := \{D_F(H, 1) : F \in 2\mathcal{G}(G) \text{ and } H \in \mathcal{H}(F)\}$. We will show that a graph $G$ is homogeneously orderable iff the hypergraph $\mathcal{D}\mathcal{H}(G)$ is a dual hypertree. Note that this equivalence does not hold for the 2-graph hypergraph $2\mathcal{G}(G)$ instead of $\mathcal{D}\mathcal{H}(G)$. Indeed, consider the chordless 5-cycle $C_5$ which is a $a$-graph. Thus, the (reduced) 2-graph hypergraph contains only one hyperedge and hence is a dual hypertree. On the other hand, any proper homogeneous set of a $C_5$ is a singleton, and any pair of vertices is contained in the neighbourhood of a homogeneous set. Thus, the 2-section graph $2\text{SEC}(\mathcal{D}\mathcal{H}(C_5))$ is complete, but there is no proper homogeneous set dominating the whole cycle. Consequently, $\mathcal{D}\mathcal{H}(C_5)$ is not conformal and thus it is not a dual hypertree. Moreover, no vertex of a $C_5$ is $h$-extremal.

To prove that $\mathcal{D}\mathcal{H}(G)$ of a homogeneously orderable graph $G$ is a dual hypertree we use Theorem 2.3 (ii), i.e. we show that $2\text{SEC}(\mathcal{D}\mathcal{H}(G))$ is chordal and $\mathcal{D}\mathcal{H}(G)$ is conformal.

First we prove the chordality of the 2-section graph.

Lemma 3.1. For any graph $G$ we have $2\text{SEC}(\mathcal{D}\mathcal{H}(G)) = G^2$.

Proof. (1) Let $xy$ be an edge in $2\text{SEC}(\mathcal{D}\mathcal{H}(G))$. Then by definition there must be a hyperedge $D_F(H, 1)$ containing both vertices. From Lemma 2.8 we obtain $d(x, y) \leq 2$, hence these vertices are adjacent in $G^2$.

(2) Let $xy$ be an edge in $G^2$, that is $d_G(x, y) \leq 2$. Consider a 2-graph $F$ of $G$ containing both vertices, and if $d_G(x, y) = 2$ a vertex $w$ which is adjacent to both. Obviously, there is a proper homogeneous set $H$ containing $x$ for $d_G(x, y) = 1$ and containing $w$ for $d_G(x, y) = 2$, respectively. In both cases $\{x, y\}$ is a subset of $D_F(H, 1)$.

The following straightforward lemma will be used frequently in the sequel.

Lemma 3.2. Let $G$ be graph and let $v$ be a $h$-extremal vertex of $G$ with $e(v) \geq 2$. Then $G\{v\}$ is an isometric subgraph of $G$. In particular, we have $G^2\{v\} = (G\{v\})^2$.

Proof. Since $e(v) \geq 2$ we can choose a homogeneous set $H \subseteq N(v)$ dominating $D(v, 2)$ due to Lemma 2.10. Thus, the distances in $G\{v\}$ are the same as in $G$.


Lemma 3.3. For any homogeneously orderable graph $G$ with $h$-extremal ordering $\sigma$ the square $G^2$ is chordal and $\sigma$ is a perfect elimination ordering of $G^2$.

Proof. Let $G$ be a homogeneously orderable graph with $h$-extremal ordering $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$. We prove that $v_1$ is simplicial in $G^2$. Let $x$, $y$ be neighbours of $v_1$ in $G^2$. Hence, $d_G(x, v_1) \leq 2$ and $d_G(y, v_1) \leq 2$, i.e. both $x$ and $y$ are contained in $D_G(v_1, 2)$ which is dominated by $H_1$. Thus, Lemma 2.8 implies $d_G(x, y) \leq 2$. Therefore, $x$ and $y$ are adjacent in $G^2$ and $D_{G^2}(v_1, 1)$ is complete, that is $v_1$ is simplicial in $G^2$.

If $e(v_1) \geq 2$ we can proceed by induction on the position in $\sigma$ due to the preceding lemma. Otherwise, $G = D(v_1, 1)$ and $G^2$ is complete. \qed

The following immediate consequence of Lemma 3.3 and Corollary 2.7 was known already from papers about distance-hereditary and dually chordal graphs.

Corollary 3.4. If $G$ is a distance-hereditary or dually chordal graph then $G^2$ is chordal.

Now we prove the conformality of $\mathcal{D}_H(G)$.

Lemma 3.5. If $G$ is homogeneously orderable then $\mathcal{D}_H(G)$ is conformal.

Proof. Let $\sigma = ((v_1, H_1), \ldots, (v_n, H_n))$ be a $h$-extremal ordering of $G$. Furthermore, let $C = \{c_1, \ldots, c_k\}$ be a maximal clique in $2SEC(\mathcal{D}_H(G)) = G^2$ such that $c_1 = v_1$ is the leftmost vertex of $C$ with respect to $\sigma$. Since $C$ is maximal in $G^2$ it cannot be completely contained in $D_G(v_i, 2)$ for $i = 1, \ldots, l - 1$ implying $e_G(v_i) \geq 2$ for $i = 1, \ldots, l - 1$. Thus, by Lemma 3.2 $C$ is a clique in $(G_1)^2$, i.e. for all $i, j = 1, \ldots, k$ we have $d_G(c_i, c_j) \leq 2$. Since $v_i = c_1$ is $h$-extremal in $G_i$ we immediately conclude $C \subseteq D(H_1, 1) = D_G(v_1, 2)$. Suppose $D_G(v_i, 2)$ is not a 2-graph of $G$. Then it must be properly contained in a 2-graph $F$ of $G$. But $F$ induces a clique in $G^2$ contradicting the maximality of $C$. Thus, $D_G(v_i, 2)$ is a 2-graph of $G$ and $C$ is contained in a hyperedge. \qed

Summarizing the above results we obtain

Corollary 3.6. If $G$ is homogeneously orderable then $\mathcal{D}_H(G)$ is a dual hypertree.

Lemma 3.7. Let $\mathcal{D}_H(G)$ be a dual hypertree with a hyperedge $D(H, 1) = G$. Then $G$ is homogeneously orderable and, with $H := \{u_1, \ldots, u_k\}$, $V \setminus H := \{v_1, \ldots, v_l\}$, both

$\sigma = ((u_1, V \setminus H), \ldots, (u_{k-1}, V \setminus H), (v_1, \{u_k\}), \ldots, (v_l, \{u_k\}))$

and

$\sigma' = ((v_1, 1), \ldots, (v_{l-1}, 1), (u_1, \{v_l\}), \ldots, (u_k, \{v_l\}))$

are $h$-extremal orderings of $G$. 
Proof. Follows immediately from \( D(H, 1) = G \). □

In order to prove the converse, i.e. if \( \mathcal{DH}(G) \) is a dual hypertree then \( G \) is homogeneously orderable, we introduce another hypergraph. Consider a 2-graph \( F \) which is dominated by some homogeneous (in \( F \)) set \( H \), i.e. \( F = D_F(H, 1) \). Then \( F \) is split into two joined sets, namely \( H \) and \( N_F(H) \): \( F = H \bowtie N_F(H) \). In general, a set \( U \subseteq V \) is join-split if \( U \) is the join of two nonempty sets, i.e. \( U = U_1 \bowtie U_2 \). Since any edge of a graph is a join-split set each connected graph can be covered by join-split sets. Thus, we can define the hypergraph \( \mathcal{DH}(G) \) of the maximal join-split sets of \( G \), and immediately obtain the following:

Lemma 3.8. For any graph \( G \) we have \( 2SEC(\mathcal{DH}(G)) = G^2 \).

So we get

Theorem 3.9. The following conditions are equivalent for a graph \( G \):

- (i) \( G \) is homogeneously orderable.
- (ii) \( \mathcal{DH}(G) \) is a dual hypertree.
- (iii) \( G^2 \) is chordal and every maximal 2-set of \( G \) is join-split (and hence a 2-graph).
- (iv) \( \mathcal{DH}(G) \) is a dual hypertree.

Proof. (i) \( \Rightarrow \) (ii) Follows from the preceding results.

(ii) \( \Rightarrow \) (iii) By Lemma 3.1 the square \( G^2 \) is chordal. Let \( S \) be a maximal 2-set in \( G \). Thus, \( S \) is a maximal clique in \( G^2 \), and the conformality of \( \mathcal{DH}(G) \) implies that \( S \subseteq D_F(H, 1) \) for some 2-graph \( F \) of \( G \) and some homogeneous set \( H \) of \( F \). From the maximality of \( S \) we conclude that \( S = D_F(H, 1) \) and hence \( S = F \). But now \( S = H \bowtie N_S(H) \), so we are done.

(iii) \( \Rightarrow \) (i) Let \( v \) be a simplicial vertex of the chordal graph \( G^2 \). If \( e(v) = 1 \) we are done by Lemma 2.10. So let \( e(v) \geq 2 \). We show that \( v \) is \( h \)-extremal in \( G \). Since \( v \) is simplicial in \( G^2 \) the disk \( D(v, 2) \) is complete in \( G^2 \). Thus that disk is a maximal 2-set in \( G \) and hence join-split, say \( D(v, 2) = X \bowtie Y \). W.l.o.g. assume \( v \in X \) implying \( Y \subseteq N(v) \). Therefore, \( Y \) is the desired homogeneous set dominating \( D(v, 2) \), and \( v \) is \( h \)-extremal. By Lemma 3.2 \( (G \setminus \{v\})^2 \) is chordal, and obviously each maximal 2-set of \( G \setminus \{v\} \) is join-split.

(iv) \( \Leftrightarrow \) (iii) By Lemma 3.8 and Theorem 2.3 statement (iv) is a reformulation of (iii).

Corollary 3.10. If \( G \) is homogeneously orderable then for each perfect elimination ordering \( \sigma = (v_1, \ldots, v_q) \) of \( G^2 \) and \( k(\sigma) := \min \{i : G^2_{[v_1, \ldots, v_i]} \text{ is complete}\} \) there exists a \( h \)-extremal ordering \( \tau \) of \( G \) such that \( \tau(i) = \sigma(i) \) for \( i = 1, \ldots, k(\sigma) - 1 \).

Corollary 3.11. If \( G \) is a homogeneously orderable graph and \( v \) is an arbitrary vertex of \( G \) then there is a \( h \)-extremal ordering \( \sigma \) of \( G \) with \( v \) at the end.
Recall that in distance-hereditary graphs each maximal 2-set is a 2-graph and each 2-graph is a cograph, i.e. a hereditary 2-graph. Here, in homogeneously orderable graphs each maximal 2-set is a join-splitted 2-graph.

4. Homogeneous reductions and extensions

In [10] the authors generalize distance-hereditary graphs. Recall that any distance-hereditary graph can be generated from a single vertex by a sequence of the following three one-vertex extensions. Let $G' = (V', E')$ be a graph, $x' \in V'$ and $x \notin V'$. Define $G := (V' \cup \{x\}, E' \cup E_x)$ where $E_x$ is defined as follows:

- **PV $E_x := \{xx'\}$** - $x$ is a pendant vertex (leaf) to $x$,
- **FT $E_x := \{xy: y \in N_G(x')\} - x$ and $x'$ are false twins,
- **TT $E_x := \{xy: y \in D_G(x', 1)\} - x$ and $x'$ are true twins.

It is obvious that in the case of twin operations $\{x, x'\}$ forms a homogeneous set in $G$. Now, in [10] instead of twins (as a special kind of homogeneous sets) arbitrary homogeneous sets are used.

Let $H$ be a proper homogeneous set of $G$ containing at least two vertices and let $v_H \in H$. Then the graph $H \text{ Red}(G, H, v_H)$ obtained from $G$ by deleting $H \setminus \{v_H\}$, i.e. contracting $H$ to a representing vertex $v_H$, will be called the **homogeneous reduction** of $G$ (via $H$).

Conversely, the **homogeneous extension** $H \text{ Ext}(G, v, H)$ of $G$ via a graph $H$ in $v$ with $V(H) \cap V(G) = \emptyset$ is the graph obtained by substituting $v$ by $H$ such that the vertices of $H$ have the same neighbours outside of $H$ as $v$ had in $G$.

Thus, in distance-hereditary graphs the FT operation is the homogeneous extension of $G'$ in $x'$ via the non-edge $\{x, x'\}$. For a TT operation $G'$ is homogeneously extended in $x'$ via the edge $\{xx'\}$.

In what follows, we want to clarify the relations between some graph classes.

In the sense of [10] a graph $G$ is a **homogeneous graph** iff the iterated reduction via proper homogeneous sets of 2-connected components leads to a tree.

A more natural generalization of distance-hereditary graphs is the following: $G$ is in $\Gamma_{\{PV, HExt\}}(K_1)$ iff $G$ can be generated from a single vertex by a sequence of $PV$ operations and homogeneous extensions. Obviously, these graphs are homogeneous. Note that this inclusion is proper: Consider the graph in Fig. 2 which does neither contain a pendant vertex nor a nontrivial proper homogeneous set. Thus, that graph is not in $\Gamma_{\{PV, HExt\}}(K_1)$. On the other hand, there are two 2-connected components with cutvertex $x$. The vertex sets of the components minus $\{yi\}$ are homogeneous and hence this graph is reducible to a $P_3$. 

**Proof.** Recall that for chordal graphs each vertex can be placed at the end of a perfect elimination ordering. Let $\tau$ be such a perfect elimination ordering of $G^2$. Thus, the index of $v$ in $\tau$ is at least $k(\tau)$. By the above corollary and by Lemma 3.7 we are done. 

Recall that in distance-hereditary graphs each maximal 2-set is a 2-graph and each 2-graph is a cograph, i.e. a hereditary 2-graph. Here, in homogeneously orderable graphs each maximal 2-set is a join-splitted 2-graph.
Lemma 4.1. If $H \in \mathcal{H}(G)$ and $S$ is a maximal 2-set of $G$ such that $H \cap S \neq \emptyset$ then $H \subset S$.

Proof. First suppose $S \subseteq H$. Since $G$ is connected and $H$ is proper there must be a vertex $w$ in $V \setminus H$ such that $H \subseteq N(w)$ implying $S \subseteq N(w)$, a contradiction. Thus, $S \setminus H \neq \emptyset$. Next suppose $H \setminus S \neq \emptyset$. Define $S' := H \cup S$. We prove that $S'$ is a 2-set which contradicts the maximality of $S$. Let $w \in H \cap S$ and consider two vertices $x, y$ of $S'$. If both vertices are in $S$ then $d(x, y) \leq 2$. Now let $x \in S \setminus H$ and $y \in H \setminus S$. If $x$ is adjacent to $w$ then $x$ must be adjacent to $y$, since $H$ is homogeneous in $G$. Otherwise there must be a vertex $v$ adjacent to both $x$ and $w$. Note that $v$ cannot be in $H$. Thus, $vw \in E$ and $d(x, y) = 2$. Finally, let both $x$ and $y$ be in $H$ and choose a vertex $v \in S \setminus H$. If $wv \in E$ then both $x$ and $y$ must be adjacent to $v$. Otherwise there is a vertex $u$ in $V \setminus H$ adjacent to $w$ and $v$. Hence, $x$ and $y$ are adjacent to $u$ too. So we are done. 

Lemma 4.2. Let $G$ be a homogeneously orderable graph, $H$ be a nontrivial proper homogeneous set of $G$ and $v$ any vertex of $H$. Then the graph $G \setminus \{v\}$ is homogeneously orderable too.

Proof. It is easy to see that $G \setminus \{v\}$ is an isometric subgraph of $G$. Thus $(G \setminus \{v\})^2 = G^2 \setminus \{v\}$ which is chordal. Consider an arbitrary maximal 2-set $S$ in $G \setminus \{v\}$. If $S$ is not maximal in $G$ then $S' := S \cup \{v\}$ is a maximal 2-set in $G$. Therefore, $S'$ is join-splitted in $G$, say $S' = X \bowtie Y$ with $v \in Y$. If $Y$ contains at least two vertices we are done. So let $Y = \{v\}$. By the maximality of $S'$ the preceding lemma implies $H \subset S'$. Since $H$ is homogeneous and $X \subseteq N(v)$ we can split $S'$ by $X \setminus \{H\} \bowtie H$. Consequently, $S$ is join-splitted in $G \setminus \{v\}$ because $H$ is nontrivial. The assumption follows from Theorem 3.9. 

Lemma 4.3. If $G$ is chordal then so is the graph obtained from $G$ by adding a true twin $v$ to some vertex $x$ of $G$. 
Lemma 4.4. Let $G$ be a homogeneously orderable graph and $H$ be a proper homogeneous set of $G$. Then any graph $G + v := (V \cup \{v\}, E \cup \{vx: x \in N_G(H) \cup E'\}$, where $E' \subseteq \{vx: x \in H\}$, is homogeneously orderable too.

Proof. Since $N_{V \setminus H}(v) = N(H)$ it is easy to see that $G$ is an isometric subgraph of $G + v$, and $G^2 = (G + v)^2 \setminus \{v\}$. So $(G + v)^2$ can be obtained from the chordal graph $G^2$ by adding the true twin $v$ to some vertex $x$ of $H$ since $D_G(v, 2) = D_G(x, 2)$. Hence $(G + v)^2$ is chordal by Lemma 4.3.

Consider an arbitrary maximal 2-set $S$ in $G + v$. If $S$ does not contain $v$ then $S$ is a maximal 2-set in $G$ and hence is join-splitted. Otherwise $S' := S \setminus \{v\}$ is a maximal 2-set in $G$, and Lemma 4.1 implies $H \subseteq S$. Therefore, $S'$ is join-splitted, i.e. $S' = X \bowtie Y$. If $H$ is completely contained in one of the splitting sets we can add $v$ to this one to obtain a splitting for $S$ in $G + v$. So assume $H \cap X \neq \emptyset$ and $H \cap Y \neq \emptyset$. But in this case we can split $S$ into $X \setminus H$ and $Y \cup H$, thus we are again in the preceding case. By Theorem 3.9 we are done. \qed

Corollary 4.5. (1) If $H \in \mathcal{M}(G)$ and $G' := H \text{Red}(G, H, v_H)$ then $G'$ is homogeneously orderable too.

(2) If $v \in V(G)$, $H$ an arbitrary graph and $G' := H \text{Ext}(G, v, H)$ then $G'$ is homogeneously orderable too.

Proof. Follows immediately from the preceding two lemmata. \qed

Thus, we can summarize our results to

Corollary 4.6. Homogeneously orderable graphs are closed under homogeneous extensions, homogeneous reductions and under deleting and adding of a vertex with maximum neighbour.

Proof. The first two points follow directly from the above corollary. To show the third let $G$ be a homogeneously orderable graph and $\sigma = (v_1, \ldots, v_n)$ be a $h$-extremal ordering of $G$. Furthermore, let $y \notin G$ be a vertex with maximum neighbour $x \in G$. Obviously, $\{x\}$ is a homogeneous set dominating $D(y, 2)$. Thus, $\tau = (y, v_1, \ldots, v_n)$ is a $h$-extremal ordering of the new graph. \qed

Theorem 4.7. The homogeneously orderable graphs are exactly those graphs which can be generated from a single vertex by adding a vertex with maximum neighbour and by homogeneous extensions, i.e. $\Gamma_{[\text{MN, HExt}]}(K_1)$ is the class of homogeneously orderable graphs.

Proof. By Corollary 4.6 any graph from $\Gamma_{[\text{MN, HExt}]}(K_1)$ is homogeneously orderable. To prove the converse note that every $h$-extremal vertex $v$ either has a maximum
neighbour or contains a proper nontrivial homogeneous set $H$ in its neighbourhood. After homogeneously reducing $H$ to $v_H$ vertex $v$ has a maximum neighbour $v_H$.

**Lemma 4.8.** A graph $G$ is homogeneously orderable iff each 2-connected component of $G$ is homogeneously orderable.

**Proof.** Let $G$ be a homogeneously orderable graph with $h$-extremal ordering $\sigma = (v_1, \ldots, v_n)$. Denote by $\sigma|_K$ the ordering of vertices of a 2-connected component $K$ of $G$ induced by $\sigma$. We show that $\sigma|_K = (v_i, \ldots, v_l)$ is a $h$-extremal ordering for $K$. Suppose the contrary and let $i, j < l$, be the smallest index such that $v_i$ is not $h$-extremal in $K_{ij} := K_{v_i, \ldots, v_l}$. By Lemma 3.2 the graph $K_{ij}$ is an isometric subgraph of $K$ and hence connected. Since $v_i$ is $h$-extremal in $G_{ij}$ there is a proper homogeneous set $H_{ij}$ dominating $D_{G_{ij}}(v_i, 2)$. Obviously, if $H' := H_{ij} \cap K_{ij}$ is nonempty then $H'$ dominates $D_{K_{ij}}(v_i, 2)$. Thus, $H'$ is empty, i.e., $H_{ij} \cap K = \emptyset$. This implies, together with $H_{ij} \subseteq N(v_i)$ and $D_{G_{ij}}(v_i, 2) = D_{G_{ij}}(H_{ij}, 1)$, that $G_{ij} \cap K = \{v_i\}$ and hence $j = l$, a contradiction.

In order to prove the converse consider the tree $T(G)$ defined by the 2-connected components of $G$. Let $K$ be a leaf of $T(G)$ and $v$ be the only cutvertex of $G$ in $K$. Then by Corollary 3.11 we have a $h$-extremal ordering $\sigma_K = (v_i, \ldots, v_l, v)$ of $K$ with vertex $v$ at the end. By induction hypothesis $G \setminus (K \setminus \{v\})$ possesses a $h$-extremal ordering $\tau = (v_1, \ldots, v_l)$. Obviously, $\sigma := (v_i, \ldots, v_l, v_1, \ldots, v_l)$ is a $h$-extremal ordering of $G$.

Remark that a graph $G$ is homogeneous iff each 2-connected component of $G$ is a homogeneous graph (cf. [10]). Thus we can prove

**Corollary 4.9.** Homogeneous graphs are homogeneously orderable.

**Proof.** By Lemma 4.8 and the remark it is sufficient to show that every 2-connected homogeneous graph is homogeneously orderable. If there is no nontrivial homogeneous set then $G$ is a tree and we are done. Otherwise we proceed by induction. Let $H$ be a nontrivial proper homogeneous set of a 2-connected homogeneous graph $G$. By the definition of homogeneous graphs $G' := H \text{ Red}(G, H, v_H)$ is homogeneous too. Thus, by induction hypothesis $G'$ is homogeneously orderable. Since $G = H \text{ Ext}(G', v_H, H)$ the assertion follows from Corollary 4.6.

As usual we denote by $\text{Ext}^*(G)$ the transitive closure of the graph class $G$ with respect to homogeneous extensions.

**Theorem 4.10.** (1) $\text{Ext}^*(\text{tree}) \subset \text{Ext}^*(\text{dually chordal gr.}) \subset \text{homogeneously orderable gr.}$

(2) $\text{Ext}^*(\text{tree}) \subset \text{homogeneous gr.}$

(3) distance-hereditary gr. $\subset$ homogeneous gr. $\subset$ homogeneously orderable gr (see Fig. 3).
Proof. The inclusions follow from above lemmata or are trivial. It remains to show that any of these inclusions is proper.

- A $C_4$ with a pendant vertex on each of its vertices is distance hereditary, hence homogeneous and homogeneously orderable but neither dually chordal nor in $Ext^*$ (dually chordal graphs) not in $Ext^*$ (tree).
- A $C_k$, $k \geq 5$, dominated by some vertex is in $Ext^*$ (tree) and is dually chordal but not distance-hereditary.
- The $C_4$ is in $Ext^*$ (tree) but not dually chordal.
- The graph shown in Fig. 4 is dually chordal but not homogeneous.

5. Hereditary homogeneously orderable graphs

In this section we will characterize hereditary homogeneously orderable graphs (i.e. those graphs $G$ for which each induced subgraph $G'$ is also homogeneously orderable) in terms of forbidden subgraphs. Since distance-hereditary graphs are homogeneously orderable,
orderable and have the property that their induced subgraphs are also distance-hereditary they are hereditary homogeneously orderable graphs. For our characterization house-hole-domino-free (HHD-free) graphs are important. A graph is HHD-free iff it does not contain an induced subgraph isomorphic to a $k$-cycle for $k \geq 5$ (the holes), the house and the domino (see Fig. 5).

HHD-free graphs are also characterized by an elimination ordering: A vertex $v$ is called semi-simplicial iff $v$ is not an inner point (midpoint) of any $P_4$ in $G$. Then, a semi-simplicial ordering is an ordering $\sigma = (v_1, \ldots, v_n)$ of the vertices of $G$ such that for every index $i = 1, \ldots, n$ the vertex $v_i$ is semi-simplicial in $G_i := G_{[v_1, \ldots, v_i]}$. In [22] the authors proved that a graph is HHD-free iff Lexicographic Breadth-First Search always generates a semi-simplicial ordering for every induced subgraph.

A class containing all HHD-free graphs is the class of pseudo-modular graphs (cf. [2]). A graph is pseudo-modular iff for any vertices $v_1, v_2, v_3$ there are vertices $x_1, x_2, x_3$ such that

$$d(v_i, v_j) = d(v_i, x_i) + d(x_i, x_j) + d(x_j, v_j) \quad \text{for all } i \neq j = 1, 2, 3,$$

and

$$d(x_1, x_2) = d(x_1, x_3) = d(x_2, x_3) \in \{0, 1\}.$$

We will prove that hereditary homogeneously orderable graphs are exactly the sun-free HHD-free graphs (in the sequel we call this call HHDS-free), where as usual a $k$-sun is a graph $S = (U \cup W, E)$ such that

1. $|U| = |W| = k$,
2. $U = \{u_1, \ldots, u_k\}$ is independent, $W = \{w_1, \ldots, w_k\}$ is a cycle (not necessary chordless),
3. $E = E(W) \cup \{u_iw_j; i = j \text{ or } i = j - 1 \mod k, i, j \in \{1, \ldots, k\}\}$, where $E(W)$ is a set of edges only between $W$-vertices.

If $W$ is complete then $S$ is called complete sun, otherwise an incomplete sun.

**Lemma 5.1.** Let $G$ be a HHDS-free graph. Then every 2-graph $D$ of $G$ contains a proper homogeneous set dominating $D$.

**Proof.** Let $D$ be a 2-graph of $G$ and $v$ be semi-simplicial in $D$. In the following all neighbourhoods are restricted to $D$.

![The house. The domino.](image)

Fig. 5. The house and the domino.
(1) For any two vertices $u, w$ of $N^2(v)$ there is a common neighbour in $N(v)$: If not then let $x, y$ be vertices of $N(v)$ such that $xu \in E$, $xw \notin E$, $yw \in E$ and $yw \notin E$. Since $v$ is semi-simplicial $xy \in E$ holds. If $u$ and $w$ are adjacent we obtain a house, a contradiction. Hence, $uw \notin E$. But $u, w \in D$, i.e. $d_D(u, w) = 2$. Thus there is a vertex $z \in N^2(v)$ adjacent to both $u$ and $w$. The following three cases can arise:

- $zx \notin E$ and $zy \notin E$ - we obtain a $C_5$,  
- $zx \in E$ and $zy \notin E$ or vice versa - we obtain a house,  
- $zx \in E$ and $zy \in E$ - we obtain a 3-sun.

In any of the above cases we get a forbidden induced subgraph, so $u, w$ have a common neighbour in $N(v)$.

(2) Let $H$ be the set of vertices of $N(v)$ dominating $N^2(v)$:

\[ H = \{ x \in N(v) : N^2(v) \subseteq N(x) \} . \]

We claim that $H$ is nonempty.

The claim is shown by induction on the number $k$ of vertices in $N^2(v)$. For $k = 1$ there is nothing to show and for $k = 2$ we are done by (1). So let $k \geq 3$, $N^2(v) = \{ y_0, \ldots, y_{k-1} \}$. By the induction hypothesis for any of the three sets $N^2(v) \setminus \{ y_i \}$, $i = 0, 1, 2$, there is a vertex $x_i$ dominating these sets. If for some $i \in \{ 0, 1, 2 \}$ vertex $x_i$ is adjacent to $y_i$ we are done. So assume $x_iy_i \notin E$ for $i = 0, 1, 2$. Consider the path $x_i - v - x_{i+1} - y_i$ where addition is taken modulo 3. If $x_ix_{i+1}y \notin E$ vertex $v$ is not semi-simplicial, a contradiction. Thus $\{ x_0, x_1, x_2 \}$ is a clique. By considering the subgraph induced by the vertices $\{ v, x_i, y_i, x_j, y_j \}$ for $i \neq j$ we conclude that $\{ y_0, y_1, y_2 \}$ must be independent, for otherwise we obtain a forbidden house. But now the vertices $\{ x_i, y_i : i = 0, 1, 2 \}$ form a 3-sun, a contradiction. Therefore, there is at least one vertex in $N(v)$ dominating $N^2(v)$.

(3) $H$ is homogeneous and dominates $D$:

By the definition of $H$ we have only to consider vertices of $N(v) \setminus H$. Suppose there are nonadjacent vertices $w \in N(v) \setminus H$ and $x \in H$. Since $w$ is not in $H$ there must be a vertex $y$ in $N^2(v)$ which is not adjacent to $w$. But now, $v$ is mid-point of the $P_4y - x - v - w$, a contradiction. □

Lemma 5.2. Let $G$ be a HHDS-free graph. Then each clique in $G^2$ is contained in some 2-graph in $G$.

**Proof.** Let $C$ be a maximal clique in $G^2$. We will show that $C$ is a 2-graph in $G$. Note that for any set $U \supseteq C$ we have $\text{diam}(G_U) \geq 3$ for otherwise $U$ would be complete in $G^2$. Thus it suffices to show $\text{diam}_C(C) \leq 2$. Assume $\text{diam}_C(C) \geq 3$.

Note that for any pair $c, c'$ of vertices of $C$ we have $d_G(c, c') \leq 2$. Thus there must be a set $U \subseteq V \setminus C$ such that $\text{diam}_{G_{U \cup C}}(C) \leq 2$ and for each $U' \subseteq U$ $\text{diam}_{G_{U' \cup C}}(C) > 2$, i.e. $U$ is minimal. Define $F := G_{U \cup C}$. Therefore for each $u \in U$ there are personal neighbours of $u$, i.e. nonadjacent vertices $u_1, u_2 \in F$ such that $u$ is the only common neighbour of $u_1, u_2$ in $F$. Furthermore $\text{diam}_F(F) \geq 3$ since $C$ is maximal in $G^2$. 
Since $F$ is an induced subgraph of $G$ it must be HHDS-free. Hence there is a semi-simplicial vertex $v$ in $F$.

Case 1: $v \in U$.

Let $v_1, v_2$ be personal neighbours of $v$. Note that $N^2_F(v) \neq \emptyset$ since otherwise $F = D_F(v, 1)$ is a clique in $G^2$. Let $x \in N^2_F(v)$ and $y \in N_F(v)$ be a neighbour of $x$. If $y$ is one of $v_1$, $v_2$, say $v_1$, then $xv_2 \notin E$. Thus, $v$ is midpoint of the $P_4 v_2 - v - v_1 - x$, a contradiction. So let $y$ be distinct from $v_1$ and $v_2$. Note that neither $y$ is adjacent to both $v_1, v_2$ nor $x$ is. W.l.o.g. assume $v_1y \notin E$. If $v_1x \notin E$ $v$ is midpoint of the $P_4 v_1 - v - y - x$. If $v_1x \in E$ then $v_3x \notin E$. But now, either $v_3y \notin E$ implying the $P_4 v_2 - v - y - x$, or $v_2y \in E$ yielding a house induced by $\{v_1, v_2, v, x, y\}$. In any case we obtain a contradiction.

Case 2: $v \in C$.

Then $C \subseteq D_F(v, 2)$. Since $diam(C) \geq 3$ there are vertices $c_1, c_2$ such that $d_C(c_1, c_2) \geq 3$. Thus not both vertices can be contained in $N(v)$. W.l.o.g. let $c_1 \in N^2_F(v)$ and let $x \in F$ be a neighbour of $v$ and $c_1$.

Case 2.1: $c_2 \in N_F(v)$.

If $xc_2 \notin E$ we obtain the $P_4 c_2 - v - x - c_1$, a contradiction. Otherwise $x$ must be a vertex of $U$. Since $C \cup \{x\}$ is not a clique in $G^2$ there must be a vertex $c_3 \in C$ such that $d_G(x, c_3) \geq 3$. Thus, $c_3 \in N^2(v)$ implying $d_G(x, c_3) = 3$. Let $y$ be a common neighbour of $v$ and $c_3$ in $F$. By distance requirements we obtain $xy \notin E$ and $xc_3 \notin E$. Therefore, $v$ is midpoint of the $P_4 x - v - y - c_3$, a contradiction.

Case 2.2: $c_2 \in N^2_F(v)$.

If $x \in U$ we can proceed as in Case 2.1. So assume that $c_1$ has no neighbour in $U \cap N_F(v)$. But $d_F(c_1, c_2) = 2$. Thus there is a vertex $u \in U \setminus N_F(v)$ adjacent to both $c_1, c_2$. Let $y \in F$ be a common neighbour of $v$ and $c_2$. Note that $c_1y \notin E$. Thus, if $xy \notin E$ then $v$ is midpoint of the $P_4 c_1 - x - v - y$, a contradiction. So let $xy \in E$. Now, $u$ must be in $N^2_F(v) \cup N^3_F(v)$. If $u \in N_F(v)$ then the vertices $c_1, x, y, c_2, u$ induce either a $C_5$ (for $ux$, $uy \notin E$), a house (for $ux \in E$ and $uy \notin E$ or vice versa) or a 3-sun (for $ux, uy \notin E$), contradicting that $F$ is HHDS-free.

Otherwise, i.e. if $u \in N^3_F(v)$, then the vertices $c_1, x, y, c_2, u$ induce a $C_5$, again a contradiction. So we are done. 

**Corollary 5.3.** Let $G$ be a HHDS-free graph. Then $\mathcal{D}H(G)$ is conformal.

**Proof.** We have to show that every clique $C$ of $2SEC(\mathcal{D}H(G))$ is contained in some hyperedge. By Lemma 3.1 we have $2SEC(\mathcal{D}H(G)) = G^2$. Thus the above lemma implies that $C$ is contained in a 2-graph $F$ of $G$. From Lemma 5.1 the existence of a homogeneous set $H$ in $F$ dominating $F$ follows. Hence, $C$ is contained in the hyperedge $D_F(H, 1)$ of $\mathcal{D}H(G)$.

**Lemma 5.4.** Let $G$ be a house-free graph and $v$ be a semi-simplicial vertex in $G$ with $rad(G) \geq 2$. Then $G^2 \setminus \{v\} = (G \setminus \{v\})^2$. 


Proof. At first we show that \( v \) is not a cut vertex using \( \text{rad}(G) \geq 2 \) and the semi-simplicity of \( v \). Assume to the contrary that there are at least two connected components \( K_1, K_2 \) of \( G \setminus \{ v \} \). Let \( x \in K_1 \) and \( y \in K_2 \). If one of these vertices is not in \( D_G(v, 1) \) any shortest path connecting \( x \) and \( y \) induces a \( P_4 \) in \( D(v, 2) \) such that \( v \) is midpoint. Hence, \( \{ x, y \} \subseteq N(v) \) and thus \( e(v) = 1 \), a contradiction.

Next, note that any edge in \((G \setminus \{ v \})^2\) is an edge in \( G^2 \setminus \{ v \} \). Now suppose there are vertices \( x, y \) such that \( xy \) is an edge in \( G^2 \setminus \{ v \} \) but not in \((G \setminus \{ v \})^2\). Thus \( d_G(x, y) = 2 \) and \( N_G(x) \cap N_G(y) = \{ v \} \). Since \( e(v) \geq 2 \) there must be a vertex \( w \) in \( N^2 \setminus \{ v \} \). Let \( u \) be one of its neighbors in \( N(v) \). If \( u = x \) or \( u = y \) the semi-simplicial vertex \( v \) is midpoint of the \( P_4 \) either \( w - u - v - y \text{ or } w - u - v - x \). By similar arguments \( w \) must be adjacent to exactly one of the vertices \( x, y \), say \( x \). This forces \( uy \in E \) and \( xu \notin E \). But now we have an induced house. \( \square \)

Lemma 5.5. If \( G \) is a HHDS-free graph then \( G^2 \) is chordal.

Proof. We proceed by induction on \( |V| \). Let \( v \) be a semi-simplicial vertex of \( G \) and suppose \( G^2 \) is not chordal. By the induction hypothesis \((G \setminus \{ v \})^2 = G^2 \setminus \{ v \} \) is chordal. Hence, any chordless cycle of length \( k \geq 4 \) in \( G^2 \) must contain \( v \). Let \( C = v - v_1 - \cdots - v_{k-1} - v \) such a cycle. Since \( d_G(v_1, v_{k-1}) \geq 3 \) at least one of the distances \( d_G(v, v_1) \) and \( d_G(v, v_{k-1}) \) must be 2. Suppose \( d_G(v, v_{k-1}) = 1 \) and let \( x \) be a vertex adjacent to both \( v \) and \( v_1 \) in \( G \). Due to the distance requirements \( v_{k-1} \) cannot be adjacent to \( x \) or \( v_1 \) in \( G \), thus \( v \) is midpoint of the \( P_4 \) either \( v_{k-1} - v - x - v_1 \), a contradiction. Therefore, \( d_G(v, v_1) = d_G(v, v_{k-1}) = 2 \). Let \( w_{k-1} \) be adjacent to \( v \) and \( v_{k-1} \) in \( G \). The semi-simplicity of \( v \) then implies \( xw_{k-1} \in E \). Hence, \( C' = x - v_1 - \cdots - v_{k-1} - x \) is a cycle in \( G^2 \setminus \{ v \} \). By the induction hypothesis that graph is chordal, thus \( x \) must be adjacent to each vertex \( v_i \) in \( G^2 \), \( i = 1, \ldots, k - 1 \). Since for any \( i = 2, \ldots, k - 2 \) we have \( d_G(x, v_i) = d_G(x, v_{i+1}) = 2 \geq d_G(v_i, v_{i+1}) \) the pseudo-modularity of HHD free graphs implies the existence of a neighbour \( w_i \) of \( x \) which is adjacent to both \( v_i \) and \( v_{i+1} \) in \( G \). Obviously, for \( i \neq j \) we have \( w_i \neq w_j \).

At first consider the case \( k = 4 \). The subgraph induced by \( \{ v, x, w_2, v_3, w_3 \} \) implies the edge \( w_2w_3 \), since \( G \) is house-free. If \( v_1 \) and \( v_2 \) are adjacent we obtain a house induced by \( \{ x, v_1, v_2, w_2, w_3 \} \), a contradiction. Otherwise (i.e. \( v_1v_2 \notin E \)) by pseudo-modularity of \( G \) we have a vertex \( w_1 \) adjacent to \( x, v_1 \), and \( v_2 \). If \( w_1w_2 \notin E \) we get a house induced by \( \{ v_1, x, w_2, v_2, w_1 \} \). If \( w_1w_2 \in E \) then we get a 3-sun induced by \( \{ v, v_3, w_1 \} \cup \{ x, w_3, w_2 \} \).

For the sequel let \( k \geq 5 \). We consider the subgraph induced by the vertices \( \{ x, w_i-1, w_i, w_{i+1}, v_i, v_{i+1} \} \) for \( i = 3, \ldots, k - 2 \). We will prove \( w_{i-1}w_i \in E \) and \( w_iw_{i+1} \in E \). First note that \( xw_i \notin E \), \( xv_{i+1} \notin E \) and \( w_{i-1}v_{i+1} \notin E \), \( w_{i+1}v_i \notin E \).

Suppose \( v_iu_{i+1} \in E \). Since \( G \) does not contain a house the edges \( w_{i-1}w_i \) and \( w_iw_{i+1} \) must exist. So the vertices \( \{ x, w_i-1, v_i, v_{i+1}, w_i, w_{i+1} \} \) induce either a \( C_5 \) or a house depending on whether \( w_{i-1} \) is adjacent to \( w_{i+1} \) or not, a contradiction in both cases. Hence, \( v_iu_{i+1} \notin E \).
Now by assuming $w_i-1w_i \notin E$ we get the chordless 4-cycle $x-w_i-1-v_i-w_i$. The only possible edges are $w_iw_{i+1}$ and $w_i-1w_{i+1}$. If both edges do not exist we obtain a domino, in all other cases we get a house.

Consequently, $w_i-1w_i \in E$, $w_iw_{i+1} \in E$ and $v_iv_{i+1} \notin E$ for $i = 3, \ldots, k-2$. Now consider $v_1$ and $v_2$. If these vertices are adjacent we obtain a house induced by $\{v_1, v_2, w_2, w_3, x\}$. If $v_1$ is not adjacent to $v_2$ then by pseudo-modularity of $G$ there must be a vertex $w_1$ adjacent to $\{x, v_1, v_2\}$. Obviously, $v_1w_2 \notin E$. Thus, we obtain the edge $w_1w_2$ since the subgraph induced by $\{x, v_1, v_2, w_1, w_2\}$ must be house-free. If $w_1$ is not adjacent to $w_3$ we get a 3-sun induced by $\{v_1, w_3, v_2, v_1, w_2\}$. So $w_1w_3 \in E$. By assuming $v_2v_3 \in E$ we obtain a house induced by $\{x, w_3, v_3, v_2, w_1\}$. If $v_2$ is not adjacent to $v_3$ then the vertices $\{x, w_1, w_2, \ldots, w_{k-1}\} \cup \{v, v_1, v_2, v_3, \ldots, v_{k-1}\}$ induce a $k$-sun. This completes the proof.

In preparing our main result of this section we finally use another graph class containing all HHD-free graphs. A graph is called weakly chordal iff it does not contain any induced cycles of length greater than four or their complements. Since each complement of a cycle of length greater than five contains an induced house HHD-free graphs are weakly chordal. These graphs were introduced in [19] and characterized in [20] in terms of 2-pairs. Hereby, a 2-pair is a pair of nonadjacent vertices such that each induced path joining these vertices is of length 2. In [20] the authors proved, that a graph is weakly chordal iff each induced subgraph is either complete or contains a 2-pair. Using this characterization we show

**Lemma 5.6.** In weakly chordal house-free graphs any incomplete sun contains a complete 3-sun.

**Proof.** Let $\{w_0, \ldots, w_{k-1}\} \cup \{u_0, \ldots, u_{k-1}\}$ induce an incomplete $k$-sun in a weakly chordal house-free graph $G$. Since the sun is incomplete the connected subgraph induced by the cycle $w_0 - \cdots - w_{k-1}$ is not a clique and hence must contain a 2-pair $w_i, w_j, i < j, |i-j| \geq 2$. We conclude that any vertex $w_l, l \neq i, j, |l-i| = 1$ or $|l-j| = 1$ modulo $k$ must be adjacent to both vertices $w_i, w_j$. Now consider the subgraph induced by $\{w_{i-1}, w_i, w_{i+1}, w_j, u_l\}$. Since $G$ is house-free the vertices $w_{i-1}$ and $w_{i+1}$ must be adjacent. But now, $\{w_{i-1}, w_i, w_{i+1}\} \cup \{w_j, u_l, u_{l+1}\}$ induces a 3-sun. □

Now to the main result of this section:

**Theorem 5.7.** A graph $G$ is hereditary homogeneously orderable iff it does not contain a $C_k$ for $k \geq 5$, a house, a domino, or a complete $k$-sun for $k \geq 3$ as an induced subgraph.

**Proof.** To verify that the stated graphs are not homogeneously orderable is straightforward. For the converse we have to show by Theorem 3.9 that for any induced subgraph $H$ of a HHDS-free graph $\mathcal{D}(H)$ is a dual hypertree. Since any induced
subgraph of a HHDS-free graph is HHDS-free again the assertion follows from the above lemmata. □

**Corollary 5.8.** Hereditary homogeneously orderable graphs are hereditary pseudo-modular.

### 6. The recognition algorithm

Our polynomial-time recognition algorithm of homogeneously orderable graphs bases on Lemma 2.10, Corollary 3.10 and Lemma 3.7. Assume that \( u \) is \( h \)-extremal with \( e(u) \geq 2 \) and let \( \bar{G} = (V, \bar{E}) \) be the complement of \( G \). By Lemma 2.10 there is a proper homogeneous dominating set \( H \subseteq N(v) \) dominating \( D(v, 2) \). Thus, in \( \bar{G} \) no vertex of \( H \) is adjacent to some vertex of \( D(v, 2) \). It suffices to consider the complement of the graph induced by the disk \( D(v, 2) \). Let \( C_v \) denote the connected component of this graph which contains \( v \) (and hence \( N^2(v) \)).

**Lemma 6.1.** A vertex \( v \) such that \( e(v) \geq 2 \) is \( h \)-extremal iff \( H := N(v) \setminus C_v \) is a homogeneous set.

**Proof.** Let \( C_v \) be the connected component of the graph induced by \( D(v, 2) \) such that \( v \in C_v \). If \( v \) is \( h \)-extremal then by Lemma 2.10 there is a homogeneous set \( H' \subseteq N(v) \) dominating \( D(v, 2) \). Obviously, \( H' \cap C_v = \emptyset \). Since \( H' \subseteq H \) the set \( H \) is nonempty. So it remains to show that \( H \) is homogeneous, but this is trivial. The other direction is obvious. □

In the following algorithm all neighbourhoods are restricted to the rest graph \( G_i := G \setminus \{v_1, \ldots, v_i\} \).

**Algorithm RecHom:**

**Input:** A connected graph \( G = (V, E) \).
**Output:** A \( h \)-extremal ordering \( \sigma = ((v_1, H_1), \ldots, (v_m, H_m)) \) or answer 'NO'.

1. Compute \( G^2 \).
2. if \( G^2 \) is not chordal then STOP. NO
3. else let \( \tau = (v_1, \ldots, v_n) \) be a perfect elimination ordering of \( G^2 \) and \( k(\tau) := \min \{ i : G^2_{v_1, \ldots, v_i} \text{ is complete} \} \);  
   for \( i := 1 \) to \( k(\tau) - 1 \) do
5.   BFS\((v_i)\); 
6.   compute a connected component \( C_{v_i} \) generated by \( D(v_i, 2) \) in \( G_i \); 
7.   determine \( H := N(v_i) \setminus C_{v_i} \); 
8.   if \( H = \emptyset \) then STOP. NO else \( H_i := H \);  
9. endfor; 
10. \( \sigma(i) := (v_i, H_i) \); 
11. endfor;
Theorem 6.2. The algorithm RecHom is correct and works within $O(n^3)$ steps.

Proof. The correctness of the algorithm immediately follows from Theorem 3.9, Lemma 2.10, Corollary 3.10, Lemma 3.7 and Lemma 6.1.

Time bound: The square $G^2$ of a graph $G$ can be computed in $O(n^2)$. The chordality can be tested in linear time in the size of $G^2$ and if the graph is chordal one gets a perfect elimination ordering $\tau$ in the same time. Thus, (1)–(3) are done in $O(n^2)$ steps. Lines (5)–(10) are iterated $k(\tau)$ times. BFS takes $O(\mid E \mid)$ steps and finding the connected components in the complement graph $\overline{G}$ takes $O(\mid E \mid)$ steps. Thus, the total amount of time is bounded by $O(n^3)$. □

7. The Steiner tree problem

In this section, we present an algorithm solving the Steiner tree problem on homogeneously orderable graphs in time $O(|E(G^2)|)$ provided a $h$-extremal ordering is given. Recall that given a Steiner set $T \subseteq V$ we have to compute a minimal set $S \subseteq V$ such that $T \subseteq S$ and $G_S$ is connected.

We may assume that $G_T$ is disconnected for otherwise there is nothing to do, and connectedness can be tested in linear time using Depth First Search (DFS).

At first some technical lemmata.

Lemma 7.1. Let $v$ be a $h$-extremal vertex of a graph $G = (V, F)$ with homogeneous dominating set $H \subseteq N(v)$ and $u, w \in N(v)$. Define $G' := (V, E \cup \{uw\})$. Then only the distance between $u$ and $w$ is changed in $G'$, i.e. for any vertex $x$ of $V$ and any vertex $y$ of $V \setminus \{u, w\}$ we have $d_G(x, y) = d_{G'}(x, y).

Proof. Follows immediately from the properties of $H$. □

Lemma 7.2. Let $G$ be a homogeneously orderable graph, $\sigma = ((v_1, H_1), \ldots, (v_m, H_m))$ be a $h$-extremal ordering of $G$, $u, w \in N(v_1) \setminus H_1$, $G'$ defined as above with $v = v_1$. Then $G'$ is homogeneously orderable and $\sigma$ is a $h$-extremal ordering of $G'$.
Proof. Since \( G^2 = (G')^2 \) by Corollary 3.10 it suffices to show that \( H_i \) remains homogeneous in \( G_i \). Suppose the contrary and let \( i (i \geq 2) \) be the smallest index such that the set \( H_i \) is not homogeneous in \( G_i \). Then, \( u, w \) are \( G_i \) and, say, \( u \in H_i, w \notin H_i \). Since \( H_i \) dominates \( D_{G_i}(v_i, 2) \) we conclude \( d_{G_i}(v_i, w) \geq 3 \). Lemma 3.2 implies \( d_{G}(v_i, w) \geq 3 \). But then \( d_{G}(v_i, w) = 2 \) contradicts Lemma 7.1. □

Now we are ready to formulate the algorithm. To make the algorithm clear we consider at first a homogeneously orderable graph \( G \) with a \( h \)-extremal vertex \( v \) and a homogeneous set \( H \subseteq N(v) \) dominating \( D(v, 2) \). Furthermore, let \( T \subset V \) be given. For the sequel define \( G' := G \setminus \{v\} \) and let \( S' \) be an optimal solution of the Steiner tree problem in \( G' \) with respect to a Steiner set \( T' \) defined in the different cases:

Case 1: \( T \subseteq D(v, 1) \).
This is a trivial case. With \( S := T \cup \{v\} \) we are done.

Case 2: \( v \in T \) and \( T \cap N(v) = \emptyset \) but \( T \setminus D(v, 1) \neq \emptyset \).
Define \( T' := (T \setminus \{v\}) \cup \{h\} \) for some vertex \( h \in H \) and \( S := S' \cup \{v\} \). We claim that \( S \) is optimal for \( G \) with respect to \( T \).

Suppose to the contrary that there is a set \( F \) containing \( T \) such that \( G_F \) is connected and \( |F| < |S| \). Since \( v \in T \) we have \( v \in F \). Define \( F' := F \setminus \{v\} \). For \( |F'| = |F| - 1 < |S| - 1 = |S'| \) and by the optimality of \( S' \) the set \( F' \) cannot be a solution in \( G' \) for \( T' \). First assume that \( F' \) is not connected. Then \( v \) is a cutvertex in \( G_F \).
Let \( x_1, \ldots, x_k, k \geq 2, \) be the neighbours of \( v \) in \( F \). Since \( H \) is homogeneous and a subset of \( N(v) \) dominating the whole disk \( D(v, 2) \) no \( x_i, i = 1, \ldots, k \) can belong to \( H \). But \( T \cap N(v) = \emptyset \). Thus we can replace the vertices \( x_1, \ldots, x_k \) by some vertex \( h \in H \) obtaining a smaller set, a contradiction. Thus \( F' \) is connected and hence it cannot contain \( h \). But \( v \in F \) and \( T \setminus D(v, 1) \neq \emptyset \). Thus, \( F' \) must contain some vertex from \( N(v) \) which can be replaced by \( h \), again a contradiction. Consequently, \( S \) is optimal.

Case 3: \( v \in T \) and \( T \cap N(v) \neq \emptyset \) and \( T \setminus D(v, 1) \neq \emptyset \).
Case 3.1: \( T \cap H \neq \emptyset \).
Here we have the problem that a solution \( S' \) of \( G' \) for \( T' := T \setminus \{v\} \) may contain some vertex of \( H \). We must decide whether this \( H \)-vertex is necessary or whether it can be replaced by \( v \) (see Fig. 6, the set \( T \) is formed by the filled circles).

Fig. 6. An example to explain the Steiner tree algorithm.
Thus, we make \( T \cap N(v) \) complete and define \( G' := (V \setminus \{v\}, (E \setminus \{ux: x \in N(v)\}) \cup \{xy: x, y \in T \cap N(v)\}) \) and \( S := S' \cup \{v\} \). By Lemma 7.2 \( G' \) has the same \( h \)-extremal ordering \( \sigma = ((v_1, H_1), \ldots, (v_n, H_n)) \). To verify the correctness note that for each solution \( F \) in \( G \) for \( T \) the set \( F \setminus \{v\} \) is a solution in \( G' \) for \( T' \) except the case \( F \cap (N(v) \setminus T) \neq \emptyset \). But now we consider \( F \setminus (D(v, 1) \setminus T) \cup \{h\} \) for some \( h \in H \) which is a solution in \( G' \) for \( T' \).

**Case 3.2:** \( T \cap H \neq \emptyset \).

We define \( T' := T \setminus \{v\} \) and \( S := S' \cup \{v\} \). The correctness is trivial.

**Case 4:** \( v \notin T \) and \( T \setminus D(v, 1) \neq \emptyset \).

With \( T' := T \) we define \( S := S' \). Assume there is a set \( F \subseteq V \) containing \( T \) such that \( G_F \) is connected and \( F \) has a smaller number of vertices than \( S \). Since \( S' \) is optimal in \( G' \) and \( v \) is not in \( T \) we conclude \( v \notin F \). Thus, \( F' := F \setminus \{v\} \) includes \( T \) and cannot be connected. We proceed as in Case 2.

**Theorem 7.3.** The Steiner tree problem on homogeneously orderable graphs can be solved in time \( O(|E(G^2)|) \) provided a \( h \)-extremal ordering is given.

**Proof.** The algorithm steps through the given \( h \)-extremal ordering and computes an optimal solution \( S \) recursively as described above. The time bound follows from the fact that all added edges are edges of the square of \( G \), see Lemma 7.1. \( \square \)

**8. Summary**

In this paper we defined a new class of graphs which is a common generalization of distance-hereditary graphs, dually chordal graphs and homogeneous graphs. We presented a characterization of the new class in terms of a tree structure of the closed neighbourhoods of homogeneous sets in 2-graphs which is closely related to the defining \( h \)-extremal ordering.

Moreover, we characterized the hereditary homogeneously orderable graphs by forbidden induced subgraphs as the house-hole-domino-sun-free graphs.

Finally, we gave a polynomial time solution for the recognition and the Steiner tree problem on homogeneously orderable graphs. Thus we obtain:

<table>
<thead>
<tr>
<th>Class</th>
<th>Recognition</th>
<th>Steiner Tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tree</td>
<td>( O(n) )</td>
<td>( O(n) )</td>
</tr>
<tr>
<td>( Ext^*(\text{tree}) )</td>
<td>( O(m) )</td>
<td>( O(m) )</td>
</tr>
<tr>
<td>Distance-hereditary graphs</td>
<td>( O(m) )</td>
<td>( O(m) )</td>
</tr>
<tr>
<td>Dually chordal graphs</td>
<td>( O(m) )</td>
<td>( O(</td>
</tr>
<tr>
<td>Homogeneous graphs</td>
<td>Polynomial</td>
<td>Polynomial</td>
</tr>
<tr>
<td>Homogeneously orderable graphs</td>
<td>( O(n^4) )</td>
<td>( O(</td>
</tr>
</tbody>
</table>

**Here:**\( O(m) \) instead of \( O(n + m) \) since any graph is connected in our paper.
References