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Collective Tree Spanners in Graphs with Bounded Parameters

Feodor F. Dragan · Chenyu Yan

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Abstract In this paper we study collective additive tree spanners for special families of graphs including planar graphs, graphs with bounded genus, graphs with bounded tree-width, graphs with bounded clique-width, and graphs with bounded chordality. We say that a graph $G = (V, E)$ *admits a system of* μ *collective additive tree* r*-spanners* if there is a system $\mathcal{T}(G)$ of at most μ spanning trees of G such that for any two vertices x, y of G a spanning tree $T \in \mathcal{T}(G)$ exists such that $d_T(x, y) \leq d_G(x, y) + r$. We describe a general method for constructing a "small" system of collective additive tree r-spanners with small values of r for "well" decomposable graphs, and as a byproduct show (among other results) that any weighted planar graph admits a system of $O(\sqrt{n})$ collective additive tree 0-spanners, any weighted graph with tree-width at most $k - 1$ admits a system of $k \log_2 n$ collective additive tree 0-spanners, any weighted graph with clique-width at most k admits a system of $k \log_{3/2} n$ collective additive tree $(2\mathsf{w})$-spanners, and any weighted graph with size of largest induced cycle at most c admits a system of $\log_2 n$ collective additive tree $(2\lfloor c/2 \rfloor \mathsf{w})$-spanners and a system of $4 \log_2 n$ collective additive tree $(2(\lfloor c/3 \rfloor + 1)\mathsf{w})$-spanners (here, w is the maximum edge weight in G). The latter result is refined for weighted weakly chordal graphs: any such graph admits a system of $4 \log_2 n$ collective additive tree $(2\mathsf{w})$-spanners. Furthermore, based on this collection of trees, we derive a compact and efficient routing scheme for those families of graphs.

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F.F. Dragan (✉) · C. Yan
Algorithmic Research Laboratory, Department of Computer Science, Kent State University, Kent, OH 44242, USA
e-mail: dragan@cs.kent.edu

C. Yan
e-mail: cyan@cs.kent.edu

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1 Introduction

Many combinatorial and algorithmic problems are concerned with the distance d_G on the vertices of a possibly weighted graph $G = (V, E)$. Approximating d_G by a simpler distance (in particular, by tree-distance d_T) is useful in many areas such as communication networks, data analysis, motion planning, image processing, network design, and phylogenetic analysis. An arbitrary metric space (in particular a finite metric defined by a general graph) might not have enough structure to exploit algorithmically. Given a graph $G = (V, E)$, a spanning subgraph H is called a *spanner* if H provides a "good" approximation of the distances in G. More formally, for $t \geq 1$, H is called a *multiplicative t-spanner* of G [31, 32] if $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$. If $r \geq 0$ and $d_H(u, v) \leq d_G(u, v) + r$ for all $u, v \in V$, then H is called an *additive r-spanner* of G [30]. The parameters t and r are called, respectively, the *multiplicative* and the *additive stretch factors*. When H is a tree one has a *multiplicative tree t-spanner* [8] and an *additive tree r-spanner* [33] of G, respectively. For some recent results on sparse spanners and tree spanners of graphs we refer the reader to [17–19].

In this paper, we continue the approach taken in [10, 15, 16, 26] of studying *collective tree spanners*. We say that a graph $G = (V, E)$ *admits a system of μ collective additive tree r-spanners* if there is a system $\mathcal{T}(G)$ of at most μ spanning trees of G such that for any two vertices x, y of G a spanning tree $T \in \mathcal{T}(G)$ exists such that $d_T(x, y) \leq d_G(x, y) + r$ (a multiplicative variant of this notion can be defined analogously). Clearly, if G admits a system of μ collective additive tree r-spanners, then G admits an additive r-spanner with at most μ $(n - 1)$ edges (take the union of all those trees), and if $\mu = 1$ then G admits an additive tree r-spanner. Note also that any graph on n vertices admits a system of at most $n - 1$ collective additive tree 0-spanners (take $n - 1$ Shortest-Path-trees rooted at different vertices of G). In particular, we examine the problem of finding "small" systems of collective additive tree r-spanners for small values of r on special classes of graphs such as *planar graphs, graphs with bounded genus, graphs with bounded tree-width, graphs with bounded clique-width, and graphs with bounded chordality*.

Previously, collective tree spanners of particular classes of graphs were considered in [10, 15, 16, 26]. Paper [16] showed that any unweighted chordal graph, chordal bipartite graph or cocomparability graph admits a system of at most $\log_2 n$ collective additive tree 2-spanners. These results were complemented by lower bounds, which say that any system of collective additive tree 1-spanners must have $\Omega(\sqrt{n})$ spanning trees for some chordal graphs and $\Omega(n)$ spanning trees for some chordal bipartite graphs and some cocomparability graphs. Furthermore, it was shown that any unweighted c-chordal graph admits a system of at most $\log_2 n$ collective additive tree $(2\lfloor c/2 \rfloor)$-spanners and any unweighted circular-arc graph admits a system of two collective additive tree 2-spanners. Paper [15] showed that any unweighted

AT-free graph (graph without asteroidal triples) admits a system of two collective additive tree 2-spanners, any unweighted graph having a dominating shortest path admits a system of two collective additive tree 3-spanners and a system of five collective additive tree 2-spanners, and any unweighted graph with asteroidal number $\mathsf{an}(G)$ admits a system of $\mathsf{an}(G)(\mathsf{an}(G)-1)/2$ collective additive tree 4-spanners and a system of $\mathsf{an}(G)(\mathsf{an}(G)-1)$ collective additive tree 3-spanners. In paper [10], it was shown that no system of constant number of collective additive tree 1-spanners can exist for unit interval graphs, no system of constant number of collective additive tree r-spanners can exist for chordal graphs and $r \leq 3$, and no system of constant number of collective additive tree r-spanners can exist for weakly chordal graphs and any constant r. On the other hand, [10] proved that any unweighted interval graph of diameter D admits an easily constructable system of $2\log(D-1)+4$ collective additive tree 1-spanners, and any unweighted House-Hole-Domino-free graph with n vertices admits an easily constructable system of at most $2\log_2 n$ collective additive tree 2-spanners. Only paper [26] has investigated (so far) collective (multiplicative) tree spanners in the *weighted graphs* (they were called *tree covers* there). It was shown that any weighted n-vertex planar graph admits a system of $O(\sqrt{n})$ collective multiplicative tree 1-spanners (equivalently, additive tree 0-spanners) and a system of at most $2\log_{3/2} n$ collective multiplicative tree 3-spanners.

One of the motivations to introduce this new concept stems from the problem of designing compact and efficient routing schemes in graphs. In [21, 35], a shortest path routing labeling scheme for trees of arbitrary degree and diameter is described that, in total $O(n\log n)$ time, assigns each vertex of an n-vertex tree a $O(\log^2 n/\log\log n)$-bit label. Given the label of a source vertex and the label of a destination, it is possible to compute in constant time, based solely on these two labels, the neighbor of the source that heads in the direction of the destination. Clearly, if an n-vertex graph G admits a system of μ collective additive tree r-spanners, then G admits a routing labeling scheme of deviation (i.e., additive stretch) r with addresses and routing tables of size $O(\mu\log^2 n/\log\log n)$ bits per vertex. Once computed by the sender in μ time (by choosing for a given destination an appropriate tree from the collection to perform routing), headers of messages never change, and the routing decision is made in constant time per vertex (for details see [15, 16]).

1.1 Our Results

In this paper we generalize and refine the method of [16] for constructing a "small" system of collective additive tree r-spanners with small values of r to weighted and larger families of "well" decomposable graphs. We define a large class of graphs, called (α,γ,r)*-decomposable*, and show that any weighted (α,γ,r)-decomposable graph G with n vertices admits a system of at most $\gamma\log_{1/\alpha} n$ collective additive tree $2r$-spanners. Then, we show that all weighted planar graphs are $(2/3, \sqrt{6n}, 0)$-decomposable, all weighted graphs with genus at most g are $(2/3, O(\sqrt{gn}), 0)$-decomposable, all weighted graphs with tree-width at most $k-1$ are $(1/2, k, 0)$-decomposable, all weighted graphs with clique-width at most k are $(2/3, k, \mathsf{w})$-decomposable, all weighted graphs with size of largest induced cycle at most c are $(1/2, 1, \lfloor c/2\rfloor\mathsf{w})$-decomposable, $(1/2, 5, \lfloor(c+2)/3\rfloor\mathsf{w})$-decomposable and $(1/2, 4, (\lfloor c/3 \rfloor + 1)\mathsf{w})$-decomposable, and all weighted weakly chordal graphs are $(1/2, 4, \mathsf{w})$-decomposable. Here and in what follows, w denotes the maximum edge weight in G, i.e., $\mathsf{w} := \max\{w(e) : e \in E(G)\}$.

As a consequence, we obtain that any weighted planar graph admits a system of $O(\sqrt{n})$ collective additive tree 0-spanners, any weighted graph with genus at most g admits a system of $O(\sqrt{gn})$ collective additive tree 0-spanners, any weighted graph with tree-width at most $k-1$ admits a system of $k \log_2 n$ collective additive tree 0-spanners, any weighted graph with clique-width at most k admits a system of $k \log_{3/2} n$ collective additive tree $(2\mathsf{w})$-spanners, any weighted graph with size of largest induced cycle at most c admits a system of $\log_2 n$ ($5 \log_2 n$ and $4 \log_2 n$) collective additive tree $(2\lfloor c/2 \rfloor \mathsf{w})$-spanners (respectively, $(2\lfloor (c+2)/3 \rfloor \mathsf{w})$-spanners and $(2(\lfloor c/3 \rfloor + 1)\mathsf{w})$-spanners), and any weighted weakly chordal graph admits a system of $4 \log_2 n$ collective additive tree $(2\mathsf{w})$-spanners. Furthermore, based on this collection of trees, we derive compact and efficient routing schemes for those families of graphs.

1.2 Basic Notions and Notation

All graphs considered in this paper are connected, finite, undirected, loopless and without multiple edges. Our graphs can have (non-negative) weights on edges, $w(e)$, $e \in E$, unless specified otherwise. In a weighted graph $G = (V, E)$ the *distance* $d_G(u, v)$ between the vertices u and v is the length of a shortest path connecting u and v. If each edge of G has weight 1, then graph G is called *unweighted*.

The (*open*) *neighborhood* of a vertex u in G is $N(u) = \{v \in V : uv \in E\}$ and the *closed neighborhood* is $N[u] = N(u) \cup \{u\}$. Define the (hop-)layers of G with respect to a vertex u as follows: $L_i(u) = \{x \in V : x$ can be connected to u by a path with i edges but not by a path with $i-1$ edges$\}$, $i = 0, 1, 2, \ldots$. In a path $P = (v_0, v_1, \ldots, v_l)$ between vertices v_0 and v_l of G, vertices $v_1, \ldots, v_{l-1}$ are called *inner vertices*. Let r be a non-negative real number. A set $D \subseteq V$ is called an *r-dominating set* for a set $S \subseteq V$ of a graph G if $d_G(v, D) \leq r$ holds for any $v \in S$.

A tree-decomposition [34] of a graph G is a tree T whose nodes, called *bags*, are subsets of $V(G)$ such that: (1) $\bigcup_{X \in V(T)} X = V(G)$; (2) for all $\{u, v\} \in E(G)$, there exists $X \in V(T)$ such that $u, v \in X$; and (3) for all $X, Y, Z \in V(T)$, if Y is on the path from X to Z in T then $X \cap Z \subseteq Y$. The *width* of a tree-decomposition is one less than the maximum cardinality of a bag. Among all the tree-decompositions of G, let T be the one with minimum *width*. The width of T is called the *tree-width* of the graph G and is denoted by $tw(G)$. We say that G *has bounded tree-width* if $tw(G)$ is bounded by a constant. It is known that the tree-width of an outerplanar graph and of a series-parallel graph is at most 2 (see, e.g., [4, 28]).

A related notion to tree-width is *clique-width*. Based on the following operations on vertex-labeled graphs, namely (i) creation of a vertex labeled by integer l, (ii) disjoint union (i.e., co-join), (iii) join between all vertices with label i and all vertices with label j for $i \neq j$, and (iv) relabeling all vertices of label i by label j, the notion of *clique-width* $cw(G)$ of a graph G is defined in [20] as the minimum number of labels which are necessary to generate G by using these operations. Clique-width is a complexity measure on graphs somewhat similar to tree-width, but more powerful since every set of graphs with bounded tree-width has bounded clique-width

[11] but not conversely (cliques have clique-width 2 but unbounded tree-width). It is well-known that the clique-width of a cograph is at most 2 and the clique-width of a distance-hereditary graph is at most 3 (see [25]).

The *chordality* of a graph G is the size of the largest (in the number of edges) induced cycle of G. Define *c-chordal graphs* as the graphs with chordality at most c. Then, the well-known chordal graphs are exactly the 3-chordal graphs. An induced cycle of G of size at least 5 is called a *hole*. The complement of a hole is called an *anti-hole*. A graph G is *weakly chordal* if it has neither holes nor anti-holes as induced subgraphs. Clearly, weakly chordal graphs and their complements are 4-chordal. A *cograph* is a graph having no induced paths on 4 vertices (P_4s).

The *genus* of a graph G is the smallest integer g such that G embeds in a surface of genus g without edge crossings. Planar graphs can be embedded on a sphere, hence $g = 0$ for them. A planar graph is *outerplanar* if all its vertices belong to its outerface.

2 (α, γ, r)-Decomposable Graphs and Their Collective Tree Spanners

Let α be a positive real number smaller than 1, γ be a positive integer and r be a non-negative real number. We say that an n-vertex graph $G = (V, E)$ is (α, γ, r)-*decomposable* if there is a separator $S \subseteq V$, such that the following three conditions hold:

Balanced Separator condition: the removal of S from G leaves no connected component with more than αn vertices;

Bounded r-Dominating Set condition: there exists a subset $D \subseteq V$ such that $|D| \leq \gamma$ and for any vertex $u \in S$, $d_G(u, D) \leq r$ (we say that D *r-dominates* S);

Hereditary Family condition: each connected component of the graph, obtained from G by removing vertices of S, is also an (α, γ, r)-decomposable graph.

Note that, by definition, any graph $G = (V, E)$ having an r-dominating set (for V) of size at most γ is (α, γ, r)-decomposable, for any positive $\alpha < 1$. In many cases, in what follows, D will be chosen to be S, resulting in 0-domination.

Using these three conditions, one can construct for any (α, γ, r)-decomposable graph G a *(rooted) balanced decomposition tree* $\mathcal{BT}(G)$ as follows. If G has an r-dominating set of size at most γ, then $\mathcal{BT}(G)$ is a one node tree. Otherwise, find a balanced separator S with bounded r-dominating set in G, which exists according to the first and second conditions. Let $G_1, G_2, \ldots, G_p$ be the connected components of the graph $G \setminus S$ obtained from G by removing vertices of S. For each graph G_i ($i = 1, \ldots, p$), which is (α, γ, r)-decomposable by the Hereditary Family condition, construct a balanced decomposition tree $\mathcal{BT}(G_i)$ recursively, and build $\mathcal{BT}(G)$ by taking S to be the root and connecting the root of each tree $\mathcal{BT}(G_i)$ as a child of S. Clearly, the nodes of $\mathcal{BT}(G)$ represent a partition of the vertex set V of G into *clusters* $S_1, S_2, \ldots, S_q$, each of them having in G an r-dominating set of size at most γ. For a node X of $\mathcal{BT}(G)$, denote by $G(\downarrow X)$ the (connected) subgraph of G induced by vertices $\cup\{Y : Y$ is a descendent of X in $\mathcal{BT}(G)\}$ (here we assume that X is a descendent of itself). See Fig. 1 for an illustration.

It is easy to see that a balanced decomposition tree $\mathcal{BT}(G)$ of a graph G with n vertices and m edges has depth at most $\log_{1/\alpha} n$, which is $O(\log_2 n)$ if α is a

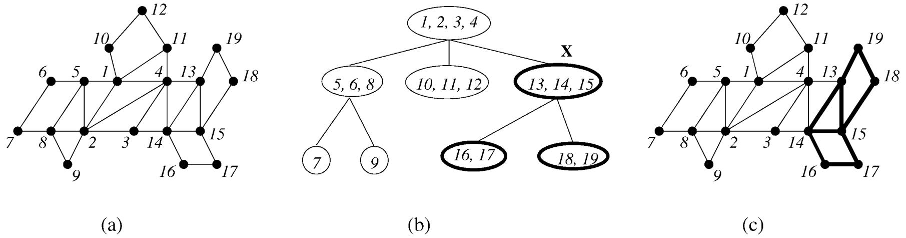

Fig. 1 (**a**) A graph G, (**b**) its balanced decomposition tree $\mathcal{BT}(G)$ and (**c**) an induced subgraph $G(\downarrow X)$ of G

constant. Moreover, assuming that a special balanced separator (mentioned above) can be found in polynomial, say $p(n)$, time, the tree $\mathcal{BT}(G)$ can be constructed in $O((p(n)+m)\log_{1/\alpha} n)$ total time.

Consider now two arbitrary vertices x and y of an (α,γ,r)-decomposable graph G and let $S(x)$ and $S(y)$ be the nodes of $\mathcal{BT}(G)$ containing x and y, respectively. Let also $NCA_{\mathcal{BT}(G)}(S(x),\ S(y))$ be the nearest common ancestor of nodes $S(x)$ and $S(y)$ in $\mathcal{BT}(G)$ and $(X_0, X_1, \dots, X_t)$ be the path of $\mathcal{BT}(G)$ connecting the root X_0 of $\mathcal{BT}(G)$ with $NCA_{\mathcal{BT}(G)}(S(x), S(y)) = X_t$ (in other words, $X_0, X_1, \dots, X_t$ are the common ancestors of $S(x)$ and $S(y)$). The following lemmata are crucial to our subsequent results.

Lemma 1 *Any path $P^G_{x,y}$, connecting vertices x and y in G, contains a vertex from $X_0 \cup X_1 \cup \dots \cup X_t$.*

Let $SP^G_{x,y}$ be a shortest path of G connecting vertices x and y, and let X_i be the node of the path $(X_0, X_1, \dots, X_t)$ with the smallest index such that $SP^G_{x,y} \cap X_i \neq \emptyset$ in G. Then, the following lemma holds.

Lemma 2 *We have $d_G(x,y) = d_{G'}(x,y)$, where $G' := G(\downarrow X_i)$.*

Let D_i be an r-dominating set of X_i in $G' = G(\downarrow X_i)$ of size at most γ. For the graph G', consider a set of $|D_i|$ *Shortest-Path-trees (SP-trees)* $\mathcal{T}(D_i)$, each rooted at a (different) vertex from D_i. Then, there is a tree $T' \in \mathcal{T}(D_i)$ which has the following distance property with respect to those vertices x and y.

Lemma 3 *For those vertices $x, y \in G(\downarrow X_i)$, there exits a tree $T' \in \mathcal{T}(D_i)$ such that $d_{T'}(x,y) \le d_G(x,y) + 2r$.*

Proof We know, by Lemma 2, that a shortest path $SP^G_{x,y}$, intersecting X_i and not intersecting any X_l $(l < i)$, lies entirely in $G' = G(\downarrow X_i)$. Let x' be a vertex of $SP^G_{x,y} \cap X_i$, and denote by l_1 the distance in $SP^G_{x,y}$ between x and x' and by l_2 the distance in $SP^G_{x,y}$ between x' and y. Since $SP^G_{x,y}$ is a shortest path of G, we have

$$d_G(x,y) = d_{G'}(x,y) = l_1 + l_2. \tag{1}$$

Since D_i is an r-dominating set of X_i in G', there exists a vertex $c \in D_i$ such that $d_{G'}(c, x') \le r$. Consider any Shortest-Path-tree T' of G' rooted at c. We have $d_{T'}(c, x) = d_{G'}(c, x) \le d_{G'}(c, x') + d_G(x', x) \le r + l_1$. Similarly, $d_{T'}(c, y) \le r + l_2$. By triangle inequality, we have

$$d_{T'}(x, y) \le d_{T'}(c, x) + d_{T'}(c, y) \le (r + l_1) + (r + l_2). \tag{2}$$

Combining (1) and (2), we obtain $d_{T'}(x, y) \le d_G(x, y) + 2r$. □

Let now $B_1^i, \ldots, B_{p_i}^i$ be the nodes on depth i of the tree $\mathcal{BT}(G)$ and let $D_1^i, \ldots, D_{p_i}^i$ be the corresponding r-dominating sets. For each subgraph $G_j^i := G(\downarrow B_j^i)$ of G ($i = 0, 1, \ldots, depth(\mathcal{BT}(G))$, $j = 1, 2, \ldots, p_i$), denote by $\mathcal{T}_j^i$ the set of *SP*-trees of graph G_j^i rooted at the vertices of D_j^i. Thus, for each G_j^i, we construct at most γ Shortest-Path-trees. We call them *local subtrees* of G. Lemma 3 implies

Theorem 1 *Let G be an (α, γ, r)-decomposable graph, $\mathcal{BT}(G)$ be its balanced decomposition tree and $\mathcal{LT}(G) = \{T \in \mathcal{T}_j^i : i = 0, 1, \ldots, depth(\mathcal{BT}(G)), j = 1, 2, \ldots, p_i\}$ be its set of local subtrees. Then, for any two vertices x and y of G, there exists a local subtree $T' \in \mathcal{T}_{j'}^{i'} \subseteq \mathcal{LT}(G)$ such that*

$$d_{T'}(x, y) \le d_G(x, y) + 2r.$$

This theorem leads to two import results for the class of (α, γ, r)-decomposable graphs. Let G be an (α, γ, r)-decomposable graph with n vertices and m edges, $\mathcal{BT}(G)$ be its balanced decomposition tree and $\mathcal{LT}(G)$ be the family of its local subtrees (defined above). Consider a graph H obtained by taking the union of all local subtrees of G (by putting all of them together), i.e.,

$$H := \bigcup \{T : T \in \mathcal{T}_j^i \subseteq \mathcal{LT}(G)\} = \Big(V, \bigcup \{E(T) : T \in \mathcal{T}_j^i \subseteq \mathcal{LT}(G)\}\Big).$$

Clearly, H is a spanning subgraph of G and for any two vertices x and y of G, $d_H(x, y) \le d_G(x, y) + 2r$ holds. Also, since for any level i ($i = 0, 1, \ldots, depth(\mathcal{BT}(G))$) of balanced decomposition tree $\mathcal{BT}(G)$, the corresponding graphs $G_1^i, \ldots, G_{p_i}^i$ are pairwise vertex-disjoint and $|\mathcal{T}_j^i| \le \gamma$ ($j = 1, 2, \ldots, p_i$), the union $\bigcup \{T : T \in \mathcal{T}_j^i, j = 1, 2, \ldots, p_i\}$ has at most $\gamma(n - 1)$ edges. Therefore, H has at most $\gamma(n - 1)\log_{1/\alpha} n$ edges in total. Thus, we have proven the following result.

Theorem 2 *Any (α, γ, r)-decomposable graph G with n vertices admits an additive $2r$-spanner with at most $\gamma(n - 1)\log_{1/\alpha} n$ edges.*

Let $\mathcal{T}_j^i := \{T_j^i(1), T_j^i(2), \ldots, T_j^i(\gamma - 1), T_j^i(\gamma)\}$ be the set of *SP*-trees of graph G_j^i rooted at the vertices of D_j^i. Here, if $p := |D_j^i| < \gamma$ then we can set $T_j^i(k) := T_j^i(p)$ for any k, $p + 1 \le k \le \gamma$. By arbitrarily extending each forest $\{T_1^i(q), T_2^i(q), \ldots, T_{p_i}^i(q)\}$ ($q \in \{1, \ldots, \gamma\}$) to a spanning tree $T^i(q)$ of the graph G we can construct at most γ spanning trees of G for each level i ($i = 0, 1, \ldots, depth(\mathcal{BT}(G))$) of the decomposition tree $\mathcal{BT}(G)$. Totally, this will result into at most $\gamma \times depth(\mathcal{BT}(G))$ spanning trees $\mathcal{T}(G) := \{T^i(q) : i = 0, 1, \ldots, depth(\mathcal{BT}(G)), q = 1, \ldots, \gamma\}$ of the original graph G. Thus, from Theorem 1, we have the following.

Theorem 3 *Any (α, γ, r)-decomposable graph G with n vertices and m edges admits a system $\mathcal{T}(G)$ of at most $\gamma \log_{1/\alpha} n$ collective additive tree $2r$-spanners. Moreover, such a system $\mathcal{T}(G)$ can be constructed in $O((p(n) + \gamma(m + n \log n)) \log_{1/\alpha} n)$ time, where $p(n)$ is the time needed to find a balanced separator S and its r-dominating set D ($|D| \le \gamma$) in an (α, γ, r)-decomposable graph.*

From Theorem 3, results of [21, 35] and [16] we conclude.

Corollary 1 *Every (α, γ, r)-decomposable graph G with n vertices admits a routing labeling scheme of deviation $2r$ with addresses and routing tables of size $O(\gamma \log_{1/\alpha} n \log^2 n / \log\log n)$ bits per vertex. Once computed by the sender in $\gamma \log_{1/\alpha} n$ time, headers never change, and the routing decision is made in constant time per vertex.*

3 Particular Classes of (α, γ, r)-Decomposable Graphs

3.1 Graphs Having Balanced Separators of Bounded Size

In this section we consider graphs that have balanced separators of bounded size. To see that planar graphs are $(2/3, \sqrt{6n}, 0)$-decomposable, we recall the following theorem from [29].

Theorem 4 [29] *Let G be any n-vertex planar graph. Then the vertices of G can be partitioned into three sets A, B, C, such that no edge joins a vertex in A with a vertex in B, neither A nor B has more than $2/3n$ vertices, and C contains no more than $2\sqrt{2n}$ vertices. Furthermore A, B, C can be found in $O(n)$ time.*

Later, Djidjev [12] improved the constant $2\sqrt{2}$ to $\sqrt{6}$. Obviously, every connected component of $G \setminus C$ is still a planar graph. This theorem was extended in [2, 13, 22] to bounded genus graphs: a graph G with genus at most g admits a separator C of size $O(\sqrt{gn})$ such that any connected component of $G \setminus C$ contains at most $2n/3$ vertices. Moreover, such a balanced separator C can be found in $O(n + g)$ time [2]. Evidently, each connected component of $G \setminus C$ has genus bounded by g, too. Hence, the following results follow.

Theorem 5 *Every n-vertex planar graph is $(2/3, \sqrt{6n}, 0)$-decomposable. Every n-vertex graph with genus at most g is $(2/3, O(\sqrt{gn}), 0)$-decomposable.*

There is another extension of Theorem 4, namely, to the graphs with an excluded minor [3]. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. By an H-minor one means a minor of G isomorphic to H. Thus the Pontryagin-Kuratowski-Wagner Theorem asserts that planar graphs are those without K_5 and $K_{3,3}$ minors. The following result was proven in [3].

Theorem 6 [3] *Let G be an n-vertex graph and H be an h-vertex graph. If G has no H-minor, then the vertices of G can be partitioned into three sets A, B, C, such that no edge joins a vertex in A with a vertex in B, neither A nor B has more than $2/3n$ vertices, and C contains no more than $\sqrt{h^3 n}$ vertices. Furthermore A, B, C can be found in $O(\sqrt{hn}(n+m))$ time, where m is the number of edges in G.*

Since induced subgraphs of an H-minor free graph are H-minor free, we conclude.

Theorem 7 *Let G be an n-vertex graph and H be an h-vertex graph. If G has no H-minor, then G is $(2/3, \sqrt{h^3 n}, 0)$-decomposable.*

Now we turn to graphs with bounded tree-width. The following theorem is true.

Theorem 8 *Every graph with tree-width at most k is $(1/2, k+1, 0)$-decomposable.*

Proof It is well known that if $tw(G) = k$ for a graph $G = (V, E)$, then G can be transformed, by adding new edges, to a chordal graph $G^+ = (V, E^+)$ such that the maximum clique of G^+ is of size $k+1$ (see, e.g., [4, 28]). Moreover, if k is a constant, then the chordal graph G^+ can be constructed in at most $O(|V| + |E^+|)$ time [4, 5]. In [23] it was shown that every n-vertex chordal graph Γ contains a maximal clique C such that if the vertices in C are deleted from Γ, every connected component in the graph induced by any remaining vertices is of size at most $n/2$. Moreover, according to [23], for any chordal graph on n vertices and m edges, such a separating clique C can be found in $O(n+m)$ time. Applying this result to an n-vertex chordal graph G^+, we obtain a set $S \subseteq V$ of at most $k+1$ vertices such that each connected component of $G^+ \setminus S$ will have at most $n/2$ vertices. Since G is a spanning subgraph of G^+, any connected component of $G \setminus S$ will have at most $n/2$ vertices, too.

Thus, any graph G with $tw(G) = k$ has a balanced separator consisting of at most $k+1$ vertices. Since induced subgraphs of a graph with tree-width at most k have also tree-width at most k (see, e.g., [4, 28]), the result follows. □

Table 1 summarizes the results on collective additive tree spanners of graphs having balanced separators of bounded size. The results are obtained by combining Theorem 3 with Theorems 5, 7 and 8. Note that, for planar graphs, the number of trees in the collection is at most $O(\sqrt{n})$ (not $\sqrt{6n}\log_{3/2} n$ as would follow from Theorem 3). This number can be obtained by solving the recurrent formula $\mu(n) = \sqrt{6n} + \mu(2/3n)$. Similar argument works for other two families of graphs.

Those systems of collective tree spanners described in Table 1 can be constructed in $O((n+\sqrt{n}(m+n\log n))\log n) = O(n^{3/2}\log^2 n)$ time for planar graphs, in $O((n+g+\sqrt{gn}(m+n\log n))\log n) = O(n^{3/2}g^{1/2}\log^2 n)$ time for graphs with genus g, in $O((\sqrt{hn}m + \sqrt{h^3 n}(m+n\log n))\log n) = O(h^{3/2}n^{1/2}(m\log n + n\log^2 n))$ time for graphs without an h-vertex minor, and in $O((n^2+km+kn\log n)\log n)$ time for graphs with tree-width at most $k-1$ (for any constant $k \geq 2$).

Note that, any shortest path routing labeling scheme in n-vertex planar graphs requires at least $\Omega(\sqrt{n})$-bit labels [1]. Hence, by Corollary 1, there must exist n-vertex planar graphs, for which any system of collective additive tree 0-spanners

Table 1 Collective additive tree spanners of n-vertex m-edge graphs having balanced separators of bounded size

Graph class	Number of trees in the collection, μ	Additive str. factor, r	Construction time
Planar graphs	$O(\sqrt{n})$	0	$O(n^{3/2}\log^2 n)$
Graphs with genus g	$O(\sqrt{gn})$	0	$O(n^{3/2}g^{1/2}\log^2 n)$
Graphs without an h-vertex minor	$O(\sqrt{h^3 n})$	0	$O(h^{3/2}n^{1/2}(m\log n + n\log^2 n))$
Graphs with tree-width $k-1$	$k\log_2 n$	0	$O((n^2 + km + kn\log n)\log n)$

needs to have at least $\Omega(\sqrt{n}\log\log n/\log^2 n)$ trees. We conclude this section with another lower bound, which follows from a result in [10]. Recall that all outerplanar graphs have tree-width at most 2.

Proposition 1 *No system of constant number of collective additive tree r-spanners can exist for outerplanar graphs, for any constant $r \geq 0$.*

3.2 Graphs with Bounded Clique-Width

In this section we will prove that each graph with clique-width at most k is $(2/3, k, \mathsf{w})$-decomposable. Recall that w denotes the maximum edge weight in a graph G, i.e., $\mathsf{w} := \max\{w(e) : e \in E(G)\}$.

Theorem 9 *Every graph with clique-width at most k is $(2/3, k, \mathsf{w})$-decomposable.*

Proof It was shown in [6] that the vertex set V of any graph $G = (V, E)$ with n vertices and clique-width $cw(G)$ at most k can be partitioned (in polynomial time) into two subsets A and $B := V \setminus A$ such that both A and B have no more than $2/3n$ vertices and A can be represented as the disjoint union of at most k subsets $A_1, \ldots, A_k$ (i.e., $A = A_1\dot{\cup}\cdots\dot{\cup}A_k$), where each A_i ($i \in \{1, \ldots, k\}$) has the property that any vertex from B is either adjacent to all $v \in A_i$ or to no vertex in A_i.

Using this, we form a balanced separator S of G as follows. Initially set $S := \emptyset$, and in each subset A_i, arbitrarily choose a vertex v_i. Then, if $N(v_i) \cap B \neq \emptyset$, put v_i and $N(v_i) \cap B$ into S. Since for any edge $ab \in E$ with $a \in A$ and $b \in B$, vertex b must belong to S, we conclude that S is a separator of G, separating $A \setminus S$ from $B \setminus S$. Moreover, each connected component of $G \setminus S$ lies entirely either in A or in B and therefore has at most $2/3n$ vertices. By construction of S, any vertex $u \in B \cap S$ is adjacent to a vertex from $A' := A \cap S$. As $|A'| \leq k$ and w is an upper bound on any edge weight, we deduce that A' w-dominates S in G.

Thus, S is a balanced separator of G and is w-dominated by a set A' of cardinality at most k. To conclude the proof, it remains to recall that induced subgraphs of a graph with clique-width at most k have clique-width at most k, too (see, e.g., [11]), and therefore, by induction, the connected components of $G \setminus S$ induce $(2/3, k, \mathsf{w})$-decomposable graphs. □

Combining Theorem 9 with the results of Sect. 2, we obtain the following corollary.

Corollary 2 *Any graph with n vertices and clique-width at most k admits a system of at most* $k \log_{3/2} n$ *collective additive tree* $2\mathsf{w}$*-spanners, and such a system of spanning trees can be found in polynomial time.*

To complement the above result, we give the following lower bound.

Proposition 2 *There are (infinitely many) unweighted n-vertex graphs with clique-width at most* 2 *for which any system of collective additive tree* 1*-spanners will need to have at least* $\Omega(n)$ *spanning trees.*

Proof Consider the complete bipartite graph $G = K_{n,n}$ on $2n$ vertices. Since G does not have any induced P_4, it is a cograph. It is known that any cograph has clique-width at most 2 (see. e.g., [25]). We show that G will require at least $\Omega(n)$ spanning trees in any system of collective additive tree 1-spanners. Let $\mathcal{T}(G)$ be a system of μ collective additive tree 1-spanners of G. Then, for any two adjacent vertices x and y of G there must exist a spanning tree T such that $d_T(x, y) \leq 2$. If $d_T(x, y) = 2$ then a common neighbor z of x and y in G would form a triangle with vertices x and y, which is impossible for $G = K_{n,n}$. Hence, $d_T(x, y) = 1$ must hold. Thus, every edge xy of G is an edge of some tree $T \in \mathcal{T}(G)$. Since there are n^2 graph edges to cover by spanning trees from $\mathcal{T}(G)$, we conclude $\mu \geq n^2/(2n-1) > n/2$. □

3.3 Graphs with Bounded Chordality

In this section, we consider the class of c-chordal graphs and its subclasses. We show that every c-chordal graph is $(1/2, 1, \lfloor c/2 \rfloor \mathsf{w})$-, $(1/2, 5, \lfloor (c+2)/3 \rfloor \mathsf{w})$- and $(1/2, 4, (\lfloor c/3 \rfloor + 1)\mathsf{w})$-decomposable, every 4-chordal graph is $(1/2, 6, \mathsf{w})$-decomposable and every weakly chordal graph is $(1/2, 4, \mathsf{w})$-decomposable.

In what follows we will need a special ordering of the vertex set of a graph $G = (V, E)$, which refines well known BFS-ordering produced by a *breadth-first search*. *Lexicographic breadth-first search (LexBFS)*, started at a vertex u, orders the vertices of a graph by assigning numbers from n to 1 in the following way. The vertex u gets the number n. Then each next available number k is assigned to a vertex v (as yet unnumbered) which has lexically largest vector $(s_n, s_{n-1}, \ldots, s_{k+1})$, where $s_i = 1$ if v is adjacent to the vertex numbered i, and $s_i = 0$ otherwise. An ordering of the vertex set of a graph G generated by LexBFS we will call a *LexBFS-ordering* of G, and use σ to denote it. The number assigned to a vertex is called *LexBFS-ordering number*. For any vertex v, $\sigma(v)$ is used to denote its LexBFS-ordering number. For convenience, in the sequel, $\sigma(x) > \sigma(y)$ is simplified as $x > y$. The *father* of a vertex v is the vertex in $N[v]$ which has the largest LexBFS-ordering number. $f(v)$ is used to denote the father of v. LexBFS may be seen to generate a rooted tree T with vertex u as the root.

The following properties of a LexBFS-ordering will be used in what follows (see, e.g., [7, 24]).

(P1) If $x \in L_i(u)$, $y \in L_j(u)$ and $i < j$, then $x > y$ in σ.
(P2) If $v \in L_q(u)$ $(q > 0)$ then $f(v) \in L_{q-1}(u)$ and $f(v)$ is the vertex from $N(v) \cap L_{q-1}(u)$ with the largest number in σ.

(P3) If $x > y$, then either $f(x) > f(y)$ or $f(x) = f(y)$.
(P4) If $a < b < c$ and $ac \in E$ and $bc \notin E$ then there exists a vertex d such that $c < d$, $db \in E$ and $da \notin E$.

Note that, properties (P1)–(P3) are guaranteed even by any BFS-ordering. Property (P4), which is the characteristic property of any LexBFS-ordering (see [7]), implies all other three properties and generally is not fulfilled by an arbitrary BFS-ordering. In most cases we will need only properties (P1)–(P3) and hence it would be sufficient to use simply a BFS-ordering of a graph. The full power of LexBFS-orderings (property (P4)) will be used only in the proof of Lemma 7. However, since a LexBFS-ordering of a graph can be easily found in linear time, too (see [24, 27]), we will assume in the sequel that a LexBFS ordering of a graph is given.

Arbitrary c-Chordal Graphs Here, we consider the class of c-chordal graphs, $c \geq 3$. We start with an easy consequence of a result from [16].

Theorem 10 *Every n-vertex c-chordal graph is* $(1/2, 1, \lfloor c/2 \rfloor \mathsf{w})$*-decomposable.*

Proof In [16], we showed that any c-chordal graph has a subset $S \subseteq V$ of vertices computable in $O(n^3)$ time such that any connected component of $G \setminus S$ has at most $n/2$ vertices and any two vertices x and y of S can be connected in G by a path with at most $\lfloor c/2 \rfloor$ edges. Since in our weighted case any edge has weight at most w, we conclude that in G any vertex x of S $(\lfloor c/2 \rfloor \mathsf{w})$-dominates S. Hence, as induced subgraphs of c-chordal graphs are c-chordal, the result follows. □

Corollary 3 *Every n-vertex c-chordal graph admits a system of at most* $\log_2 n$ *collective additive tree* $(2\lfloor c/2 \rfloor \mathsf{w})$*-spanners, and such a system of spanning trees can be found in* $O(n^3 \log n)$ *time.*

In what follows we will show that every c-chordal graph with $c \geq 4$ is also $(1/2, 5, \lfloor (c+2)/3 \rfloor \mathsf{w})$- and $(1/2, 4, (\lfloor c/3 \rfloor + 1)\mathsf{w})$-decomposable. To prove these, we first show that any graph has a special balanced separator S. Let $N(C) := \bigcup_{v \in C} N(v) \setminus C$ and $N[C] = N(C) \cup C$, for any set $C \subseteq V$.

Lemma 4 *Any graph G has a separator S such that each connected component of* $G \setminus S$ *contains at most* $n/2$ *vertices.*

Proof Let $\sigma = (v_1, v_2, \ldots, v_n)$ be a LexBFS-ordering of G and $B_i := \{v_i, v_{i+1}, \ldots, v_n\}$. Clearly, for any $i = 1, 2, \ldots, n-1$, B_i is connected. Let $C^*(i)$ be a largest (by number of vertices) connected component of $G \setminus B_i$. In what follows, i will be chosen to be the largest index such that $|V(C^*(i))| \leq n/2$. Evidently, $i \geq \lceil n/2 \rceil$ and, by maximality of i, a largest connected component $C^*(i+1)$ of graph $G \setminus B_{i+1}$ must have more than $n/2$ vertices. It is easy to see that if $C_1, C_2, \ldots, C_k, C^*(i+1)$ are the connected components of $G \setminus B_{i+1}$, then the connected components of $G \setminus B_i$ will be $C_1, C_2, \ldots, C_k, C_{k+1}, \ldots, C_{k+p}$, where $C_{k+1}, \ldots, C_{k+p}$ are the connected components of the subgraph of G induced by vertices of $C^*(i+1) \setminus \{v_i\}$.

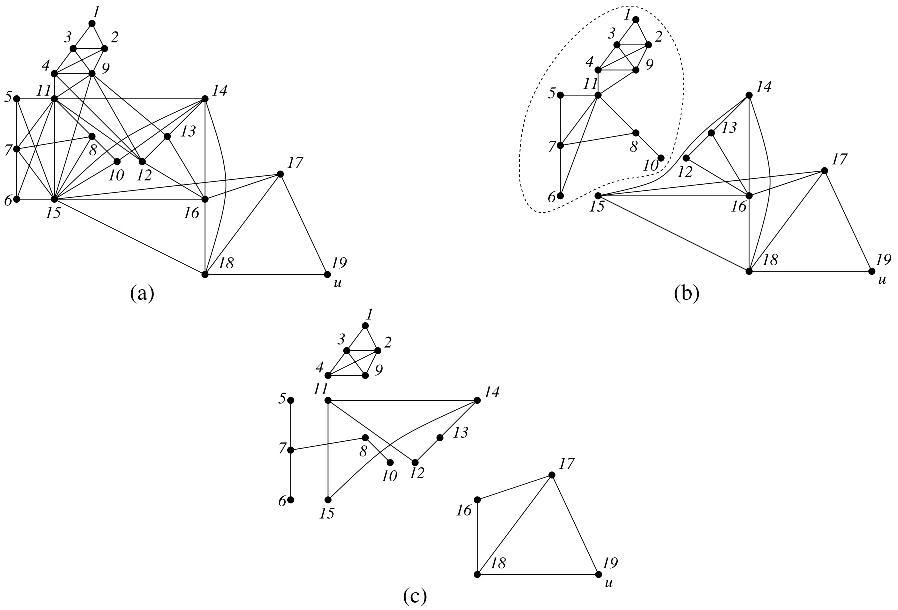

Fig. 2 (**a**) A 4-chordal graph G with a LexBFS-ordering. (**b**) A largest connected component $C^*(12)$ of $G \setminus B_{12}$ (*circled*). A balanced separator $S = \{11, 12, 13, 14, 15\}$ and the connected components of $G \setminus S$

Since $|V(C^*(i+1))| > n/2$, $|B_{i+1}| + \sum_{i=1}^{k} |V(C_i)| < n/2$ holds. Let $C' = V(C^*(i+1))$, $A = B_{i+1} \cap N(C')$ and $S = A \cup \{v_i\}$. Clearly, all connected components of $G \setminus S$ have at most $n/2$ vertices as they coincide with components $C_{k+1}, \ldots, C_{k+p}$ and the connected components of the subgraph of G induced by vertices of $C_1, C_2, \ldots, C_k$ and $B_{i+1} \setminus A$. □

From the proof of Lemma 4, one can easily design a procedure to find such a balanced separator S in at most $O(|V||E|)$ time. Our goal in this section is to show that in c-chordal graphs separator S has a small r_c-dominating set.

Let $G = (V, E)$ be a c-chordal graph with $c \geq 4$ and $\sigma = (v_1, v_2, \ldots, v_n)$ be a LexBFS-ordering of G (the *LexBFS-ordering number* of $v_i = \sigma(i)$ is $i = \sigma^{-1}(v_i)$). For any vertex $x \in V$, define $V_{>x} = \{u \in V : u > x\}$ and $G_{>x}$ to be a subgraph of G induced by $V_{>x}$. Let also $S = A \cup \{v_i\}$ be a separator of G computed as described in the proof of Lemma 4. That is, $C^*(i+1)$ is the largest connected components of $G \setminus B_{i+1}$ and $A = N(C^*(i+1)) \cap B_{i+1}$ (see Fig. 2 for an illustration). By the properties of LexBFS-orderings, the following observation clearly holds.

Proposition 3 *No vertex of* $C^*(i+1)$ *has a neighbor in* $V_{>f(v_i)}$.

We say that a vertex x has the level number $l(x)$ if $x \in L_{l(x)}(v_n)$. Since for any $y \in A$, $v_i < y \leq f(v_i)$ holds, all the vertices of S are either in $L_{l(v_i)}(v_n)$ or in $L_{l(v_i)-1}(v_n)$. Let $S_1 := S \cap L_{l(v_i)}(v_n)$ and $S_2 := S \cap L_{l(v_i)-1}(v_n)$.

Lemma 5 *There is a set D of at most five vertices in G such that D ($\lfloor(c+2)/3\rfloor\mathsf{w}$)-dominates S. Moreover, if G is a c-chordal graph with $4 \le c \le 6$, then D consists of only four vertices.*

Proof Define vertices $f_0, \dots, f_{a(v_i)}$ as follows: $f_0 := v_i, f_1 := f(v_i), \dots, f_{a(v_i)} := f(f_{a(v_i)-1})$, where $a(v_i) := \min\{\lfloor(c+2)/3\rfloor, l(v_i)\}$. We claim that the set $\{f_1, f_{a(v_i)-1}, f_{a(v_i)}\}$ is a ($\lfloor(c+2)/3\rfloor\mathsf{w}$)-dominating set for S_1.

If $a(v_i) = l(v_i)$ then $f_{a(v_i)} = v_n$ and trivially S_1 is ($\lfloor(c+2)/3\rfloor\mathsf{w}$)-dominated by $f_{a(v_i)} = v_n$ since $S_1 \subseteq L_{a(v_i)}(v_n)$. Therefore, assume that $a(v_i) \neq l(v_i)$, and let x be an arbitrary vertex of $S_1 \setminus \{v_i\}$. Consider vertices $f'_0 := x, f'_1 := f(f'_0), \dots, f'_{a(v_i)} := f(f'_{a(v_i)-1})$. If there is an index i $(0 \le i < a(v_i))$ such that f_i coincides with f'_i or $f'_i f_i \in E(G)$ or $f'_i f_{i+1} \in E(G)$, then the distance between $f_{a(v_i)-1}$ (or $f_{a(v_i)}$) and x is at most $\lfloor(c+2)/3\rfloor\mathsf{w}$ and the claim clearly holds. Hence, we may assume that there is no such index i. Since $f_i < f'_i$ (by property (P3) of LexBFS-orderings), one concludes that $f_i f'_{i+1} \notin E(G)$, too, for any $i = 0, \dots, a(v_i) - 1$.

Let $P_G(f_{a(v_i)}, f'_{a(v_i)})$ be an induced path between $f_{a(v_i)}$ and $f'_{a(v_i)}$ such that, if $(f_{a(v_i)}, f'_{a(v_i)}) \notin E(G)$, then all its inner vertices are from levels L_j, $j \le l(f_{a(v_i)}) - 1$. Let $P_G(f_1, f'_0)$ be an induced path of G obtained by concatenating the two paths $P_G(f_1, f_{a(v_i)}) := (f_1, \dots, f_{a(v_i)})$, $P_G(f'_0, f'_{a(v_i)}) := (f'_0, \dots, f'_{a(v_i)})$ with $P_G(f_{a(v_i)}, f'_{a(v_i)})$. Obviously, $P_G(f_1, f'_0)$ has at least $\lfloor(c+2)/3\rfloor - 1 + \lfloor(c+2)/3\rfloor + 1 = 2\lfloor(c+2)/3\rfloor$ edges. Let also $P'_G(f_1, f'_0)$ be an induced path between f_1 and f'_0 all inner vertices of which are from $C^*(i+1)$. By construction of S, we know that $x > v_i$. This and property (P3) of LexBFS-orderings imply that all inner vertices of $P_G(f_1, f'_0)$ are from $V_{>f(v_i)}$. By Proposition 3, no vertex from $V(P'_G(f_1, f'_0)) \setminus \{f_1, f'_0\}$ is adjacent to a vertex from $V(P_G(f_1, f'_0)) \setminus \{f_1, f'_0\}$. Now, by concatenating the two induced paths $P'_G(f_1, f'_0)$ and $P_G(f_1, f'_0)$, we obtain a chordless cycle in G. Since G is a c-chordal graph, the path $P'_G(f_1, f'_0)$ cannot have more than $\lfloor(c+2)/3\rfloor$ edges (otherwise, G will have an induced cycle with at least $c+1$ edges). Hence $d(f_1, x) \le \lfloor(c+2)/3\rfloor\mathsf{w}$ and our claim that the set $\{f_1, f_{a(v_i)-1}, f_{a(v_i)}\}$ is a ($\lfloor(c+2)/3\rfloor\mathsf{w}$)-dominating set for S_1 is proven.

Clearly, if G is a c-chordal graph with $4 \le c \le 6$, then $a(v_i)$ is at most $2 = \lfloor(c+2)/3\rfloor$. Therefore, in this case, S_1 is w-dominated by $f_{a(v_i)} = v_n$, if $a(v_i) = l(v_i) = 1$, or ($2\mathsf{w}$)-dominated by $f_{a(v_i)-1} = f_1$ and $f_{a(v_i)} = f_2$, if $a(v_i) = \lfloor(c+2)/3\rfloor = 2$.

Let now v' be the vertex of S_2 with the smallest LexBFS-ordering number. Define $a(v') := \min\{\lfloor(c+2)/3\rfloor, l(v_i) - 1\}$. Let $f''_0 := v', f''_1 := f(f''_0), \dots, f''_{a(v')} := f(f''_{a(v')-1})$. Let x be an arbitrary vertex in $S_2 \setminus \{v'\}$. Note that, by the definition of S_2, both v' and x have neighbors in $C^*(i+1)$. Since $C^*(i+1)$ is connected, there is an induced path $P'_G(v', x)$ all inner vertices of which are from $C^*(i+1)$. Using similar arguments as before, one can show that the set $\{v', f''_{a(v')-1}\}$ is a ($\lfloor(c+2)/3\rfloor\mathsf{w}$)-dominating set for S_2.

Set $D := \{v', f''_{a(v')-1}\} \cup \{v_i, f_{a(v_i)-1}, f_{a(v_i)}\}$. Clearly, D is a ($\lfloor(c+2)/3\rfloor\mathsf{w}$)-dominating set for S. This concludes the proof of the lemma. □

In a similar way we can prove

Lemma 6 *There is a set D' of at most four vertices in G such that D' is a* $((\lfloor c/3\rfloor + 1)\mathrm{w})$*-dominating set for S.*

Proof Set $a(v_i) := \min\{\lfloor c/3\rfloor, l(v_i)\}$. Let $f_0 := v_i, f_1 := f(f_0), \dots, f_{a(v_i)} := f(f_{a(v_i)-1})$. We claim that the set $\{f_1, f_{a(v_i)}\}$ is a $((\lfloor c/3\rfloor + 1)\mathrm{w})$-dominating set of S_1.

If $a(v_i) = l(v_i)$, then $f_{a(v_i)} = v_n$ and claim clearly holds. So, assume $a(v_i) \neq l(v_i)$. Let x be an arbitrary vertex in $S_1 \setminus \{v_i\}$. Set $f'_0 := x, f'_1 := f(f'_0), \dots, f'_{a(v_i)} := f(f'_{a(v_i)-1})$. If there is an index i, $0 \le i \le a(v_i)$, such that $f_i = f'_i$ or $f_i f'_i \in E(G)$ or $f_{i+1} f'_i \in E(G)$ (for $i < a(v_i)$), then we are done. Hence, we may assume that no such i exists. We have also $f_i f'_{i+1} \notin E(G)$, for any $i = 0, \dots, a(v_i) - 1$ since $f_i < f'_i$ holds by property (P3) of LexBFS-orderings. Let $P_G(f_{a(v_i)}, f'_{a(v_i)})$ be an induced path (of length at least 2) between $f_{a(v_i)}$ and $f'_{a(v_i)}$ all inner vertices of which are from levels L_j, $j \le l(f_{a(v_i)}) - 1$. By concatenating the paths $P_G(f_1, f_{a(v_i)}) := (f_1, f_2, \dots, f_{a(v_i)})$, $P_G(f'_0, f'_{a(v_i)}) := (f'_0, f'_1, \dots, f'_{a(v_i)})$ with path $P_G(f_{a(v_i)}, f'_{a(v_i)})$, one gets an induced path with at least $(\lfloor c/3\rfloor - 1) + \lfloor c/3\rfloor + 2 = 2\lfloor c/3\rfloor + 1$ edges. Since $f_1 f'_0 \notin E(G)$, there must exist an induced path $P'_G(f_1, f'_0)$ between f_1 and f'_0 all inner vertices of which are from $C^*(i+1)$. By Proposition 3 and the fact that $x > v_i$, we have also that no inner vertex of $P'_G(f_1, f'_0)$ is adjacent to inner vertices of $P_G(f_1, f'_0)$. Therefore, these two paths form an induced cycle. Since G is a c-chordal graph, $P'_G(f_1, f'_0)$ must have at most $\lfloor c/3\rfloor + 1$ edges. This proves the claim.

Let now v' be the vertex with the smallest LexBFS-ordering number in S_2. Define $a(v') := \min\{\lfloor c/3\rfloor, l(v_i) - 1\}$. Let $f''_0 := v', f''_1 := f(f''_0), \dots, f''_{a(v')} := f(f''_{a(v')-1})$. Using similar arguments as before, one can show that the set $\{f''_0, f''_{a(v')}\}$ is a $((\lfloor c/3\rfloor + 1)\mathrm{w})$-dominating set for S_2.

Set $D' := \{f_1, f_{a(v_i)}, f''_0, f''_{a(v')}\}$. Clearly, D' is a $((\lfloor c/3\rfloor + 1)\mathrm{w})$-dominating set for S. This completes the proof. □

Clearly, for a given S, both sets D and D' can be found in linear time. Thus, we have proven the following results.

Theorem 11 *Let G be a c-chordal graph. Then, G is* $(1/2, 4, \lfloor (c+2)/3\rfloor \mathrm{w})$*-decomposable, if* $4 \le c \le 6$, *and is* $(1/2, 5, \lfloor (c+2)/3\rfloor \mathrm{w})$- *and* $(1/2, 4, (\lfloor c/3\rfloor + 1)\mathrm{w})$*-decomposable, if* $c > 6$.

Corollary 4 *Let G be an n-vertex and m-edge c-chordal graph. If $c > 6$, then G admits a system of at most $5\log_2 n$ collective additive tree* $(2\lfloor (c+2)/3\rfloor \mathrm{w})$*-spanners and a system of at most $4\log_2 n$ collective additive tree* $(2(\lfloor c/3\rfloor + 1)\mathrm{w})$*-spanners. If $4 \le c \le 6$, then G admits a system of at most $4\log_2 n$ collective additive tree* $(2\lfloor (c+2)/3\rfloor \mathrm{w})$*-spanners. Moreover, such systems of spanning trees can be found in $O(nm\log n)$ time.*

From Theorem 10 and Theorem 11 we conclude that any 3-chordal graph is $(1/2, 1, \mathrm{w})$-decomposable, any 4-chordal graph or 5-chordal graph is $(1/2, 1, 2\mathrm{w})$-decomposable, any 6-chordal graph is $(1/2, 1, 3\mathrm{w})$- and $(1/2, 4, 2\mathrm{w})$-decomposable, any 7-chordal graph is $(1/2, 1, 3\mathsf{w})$-decomposable, and any 8-chordal graph is $(1/2, 1, 4\mathsf{w})$- and $(1/2, 4, 3\mathsf{w})$-decomposable. In the next section we will show that the result for 4-chordal graphs can be refined. In Table 2 we present our decomposition results for all c-chordal graphs.

4-Chordal Graphs Here, we show that every 4-chordal graph is $(1/2, 6, \mathsf{w})$-decomposable and every weakly chordal graph is $(1/2, 4, \mathsf{w})$-decomposable.

Let $G = (V, E)$ be a 4-chordal graph and $\sigma = (v_1, v_2, \ldots, v_n)$ be a LexBFS-ordering of G. Let also $S = A \cup \{v_i\}$ be a separator of G computed as described in the proof of Lemma 4. That is, $C^*(i+1)$ is the largest connected components of $G \setminus B_{i+1}$ and $A = N(C^*(i+1)) \cap B_{i+1}$.

Denote by $\overline{C}_6$ the complement of an induced cycle C_6 on 6 vertices. First we will show that any 4-chordal graph not containing $\overline{C}_6$ as an induced subgraph is $(1/2, 4, \mathsf{w})$-decomposable. Clearly, these graphs contain all weakly chordal graphs.

Lemma 7 *If G is a 4-chordal graph not containing $\overline{C}_6$ as an induced subgraph, then there exists a set D of at most four vertices in G such that $S \subseteq N[D]$.*

Proof Let $A^- := \{w \in A : wv_i, wf(v_i) \notin E(G)\}$. We will show that there are at most two vertices a, b in G such that $A^- \subseteq N[\{a, b\}]$. Consider a vertex $w \in A^-$. Obviously, $w > v_i$. By properties (P2) and (P3) of LexBFS-orderings, $f(v_i) < f(w)$ must hold. By Proposition 3, one concludes that $w < f(v_i)$ holds. Let $x \in C^*(i+1)$ be a vertex from $N(w)$ which can be connected to v_i in $C^*(i+1)$ with minimum number of edges.

Claim 1 $f(v_i)x \in E(G)$.

Proof The claim can be proved by contradiction. Assume $f(v_i)x \notin E(G)$. Let $P = (v_i = u_0, u_1, \ldots, u_l = x)$ be a path between v_i and x in $C^*(i+1)$ with minimum number of edges. Clearly, $N(w) \cap P = \{x\}$. Let $u_{l'}$ be the vertex of P with largest index which is adjacent to $f(v_i)$. Then path $P_1 = (f(v_i), u_{l'}, u_{l'+1}, \ldots, u_l, w)$ is an induced path connecting $f(v_i)$ and w, and it consists of at least 3 edges. Since $f(v_i) < f(w)$, there must be an induced path P_2 between $f(v_i)$ and w all inner vertices of which are from $V_{>f(v_i)}$. Moreover, no vertex from P_2 can be adjacent to any vertex from $P_1 \setminus \{f(v_i), w\}$. Since P_2 consists of at least 2 edges, by combining P_1 and P_2, one gets an induced cycle in G with at least 5 edges. As G is a 4-chordal graph, that is impossible. □

Consider a layering $\{v_n\}, L_1(v_n), L_2(v_n), L_3(v_n), \ldots$ of graph G, where $L_i(v_n) = \{x \in V : x$ can be connected to v_n by a path with i edges but not by a path with $i-1$ edges$\}$. Since all the vertices in B_{i+1} have larger LexBFS-ordering numbers than v_i, by property (P1) of LexBFS-orderings, each vertex in A^- is either in level $L_{l(v_i)}(u_n)$ or in level $L_{l(v_i)-1}(u_n)$ (recall that $w < f(v_i)$ for any $w \in A^-$). Define $A_u = \{u : u \in A^- \cap L_{l(v_i)}(v_n)\}$ and $A_d = \{u : u \in A^- \cap L_{l(v_i)-1}(v_n)\}$. Set also $N_\downarrow(x) := N(x) \cap (L_{l(x)-1}(v_n) \cap V_{>f(v_i)})$ for any $x \in V$. Since for every vertex $w \in A^-$, $f(w) > f(v_i)$ holds, $N_\downarrow(w)$ is not empty for any $w \in A^-$. We have $l(v_i) > 1$, since otherwise, $f(v_i) = v_n$ and therefore w must be adjacent to or coincide with $f(v_i)$.

Claim 2 *For any vertex $w \in A^-$, $N_\downarrow(w) \subseteq N_\downarrow(f(v_i))$ holds.*

Proof Assume that the statement is not true. Then, one can find a vertex $w' \in N_\downarrow(w)$ such that $w' > f(v_i)$ and $w'f(v_i) \notin E(G)$. By Claim 1, there is a vertex x in $C^*(i+1)$ which is adjacent to both $f(v_i)$ and w. We know also that x is not adjacent to any vertex of $V_{>f(v_i)}$. We distinguish between two cases. First assume $w \in A_u$. There must exist an induced path $P_{f(v_i),w'}$ between $f(v_i)$ and w' such that its inner vertices are all from layers $L_s(v_n)$, $s \leq l(f(v_i)) - 1$. This path has at least 2 edges. Moreover, no inner vertex of $P_{f(v_i),w'}$ is adjacent to w or x. Therefore, paths $(f(v_i), x, w, w')$ and $P_{f(v_i),w'}$ will form a chordless cycle with at least 5 edges in G, which is impossible.

Assume now that $w \in A_d$. Since $w < f(v_i) < w'$ and $w'f(v_i) \notin E$ but $w'w \in E$, by property (P4) of LexBFS-orderings, there is a vertex $t > w'$ such that $tf(v_i) \in E(G)$ and $tw \notin E(G)$. Let $P_{t,w'}$ be an induced path connecting t and w' all inner vertices of which are from $\bigcup_{i \leq l(w')-1} L_i(v_n)$. $P_{t,w'}$ has at least one edge. Hence, the path $P_{t,w'}$ together with $(t, f(v_i), x, w, w')$ will form an induced cycle with at least 5 edges in G, which is impossible. □

Claim 3 *For any two vertices $w, z \in A_u$ or $w, z \in A_d$, sets $N_\downarrow(w)$ and $N_\downarrow(z)$ are comparable.*

Proof The claim can be proved by contradiction. Assume $w, z \in A_u$ and $N_\downarrow(w)$ and $N_\downarrow(z)$ are not comparable. Then, there exist two vertices $w' \in N_\downarrow(w)$ and $z' \in N_\downarrow(z)$ such that $w'z, z'w \notin E(G)$. By Claim 2, we know $f(v_i)w', f(v_i)z' \in E(G)$. Let $x, y \in C^*(i+1)$ be two vertices such that $xw, xf(v_i), yz, yf(v_i) \in E$, the existence of which follows from Claim 1. As w', z' are from $V_{>f(v_i)}$ and x, y are from $C^*(i+1)$, there cannon be an edge between sets $\{x, y\}$ and $\{z', w'\}$.

First, we show that both wz and $w'z'$ must be in $E(G)$. Assume $w'z' \notin E(G)$. Let $P_{w,z}$ be an induced path between w and z such that all its inner vertices are from $G^*(i+1)$. $P_{w',z'}$ is used to denote an induced path between w' and z' such that its inner vertices are from $\bigcup_{i \leq l(w')-1} L_i(v_n)$. Clearly, the inner vertices of $P_{w',z'}$ are not adjacent to any vertex from $P_{w,z}$. Since $P_{w,z}$ has at least one edge and $P_{w',z'}$ has at least 2 edges, $P_{w,z}$, ww', zz' and $P_{w',z'}$ will form a hole in G, which is impossible. This proves that $w'z'$ must be in $E(G)$. Similarly, if $wz \notin E(G)$, then $P_{w,z}$ has at least 2 edges. Moreover, any inner vertex of $P_{w,z}$ is adjacent neither to w' nor to z'. Hence, $P_{w,z}$, ww', $w'z'$, $z'z$ form an induced cycle with at least 5 edges in G, which is impossible. Thus, both wz and $w'z'$ are in $E(G)$.

Second, we claim that neither wy nor zx is in $E(G)$. If $wy \in E(G)$, then since $wz, w'z' \in E(G)$, vertices w, y, z, w', z' and $f(v_i)$ would give an induced $\overline{C}_6$ which is also forbidden in G. In a similar way, one can show that $zx \in E(G)$ is impossible.

It is easy to see now that vertices $w, z, y, f(v_i), w'$ form an induced cycle with 5 edges in G. A contradiction obtained proves that $N_\downarrow(w)$ and $N_\downarrow(z)$ are comparable for any $w, z \in A_u$. When $w, z \in A_d$, one can show that $N_\downarrow(w)$ and $N_\downarrow(z)$ are comparable in a similar way. □

Claim 3 ensures that there can be found two vertices a and b in G such that $a \in \bigcap_{w \in A_u} N_{\downarrow}(w)$ and $b \in \bigcap_{w \in A_d} N_{\downarrow}(w)$. Hence, $A^- = A_d \cup A_u$ is completely contained in $N[\{a, b\}]$, implying $S \subseteq N[\{v_i, f(v_i), a, b\}]$. □

Hence, we have the following results.

Theorem 12 *Let G be a 4-chordal graph not containing $\overline{C}_6$ as an induced subgraph. Then G is $(1/2, 4, \mathsf{w})$-decomposable.*

Corollary 5 *Any n-vertex m-edge 4-chordal graph G not containing $\overline{C}_6$ as an induced subgraph (in particular, any weakly chordal graph) admits a system of at most $4 \log_2 n$ collective additive tree $(2\mathsf{w})$-spanners. Moreover, such a system of spanning trees can be constructed in $O(nm \log n)$ time.*

Note that the class of weakly chordal graphs properly contains such known classes of graphs as interval graphs, chordal graphs, chordal bipartite graphs, permutation graphs, trapezoid graphs, House-Hole-Domino-free graphs, distance-hereditary graphs and many others. Hence, the results of this subsection generalize some known results from [10, 16]. We recall also that, as it was shown in [10], no system of constant number of collective additive tree r-spanners can exist for unweighted weakly chordal graphs for any constant $r \geq 0$.

The above results can easily be extended to all 4-chordal graphs (note that in the proof of Lemma 7 the absence of $\overline{C}_6$ in G was important only for Claim 3). We can show that every 4-chordal graph is $(1/2, 6, \mathsf{w})$-decomposable.

Lemma 8 *If G is a 4-chordal graph, then there exists a set D of at most six vertices in G such that $S \subseteq N[D]$.*

Proof Let A_u, A_d be the same vertex sets as defined in the proof of Lemma 7. Let x be a vertex of A_u with minimum $|N_{\downarrow}(x)|$ among all vertices of A_u. Similarly, let y be a vertex of A_d with minimum $|N_{\downarrow}(y)|$ among all vertices of A_d. We claim that for any vertex $z \in A_u$, if $xz \notin E(G)$, then $zf(x) \in E(G)$ must hold.

Assume $zf(x) \notin E(G)$ for some $z \in A_u$. By the choice of x, there must exist a vertex $z' \in N_{\downarrow}(z)$ such that $xz' \notin E(G)$. Since x, z are in A_u, there must exist an induced path $P_G(x, z)$ all inner vertices of which are from $C^*(i+1)$. This path has at least 2 edges. On the other hand, there is a path $P_G(f(x), z')$ in G such that, if $f(x)z' \notin E(G)$, then all its inner vertices are from levels $L_s(v_n), s < l(v_i) - 1$. Path $P_G(f(x), z')$ has at least 1 edge. Furthermore, by Proposition 3, no vertex on $P_G(f(x), z')$ can be adjacent to inner vertices of $P_G(x, z)$. Therefore, $P_G(f(x), z')$, $P_G(x, z)$ and two edges $xf(x), zz'$ form an induced cycle with at least 5 edges, which is impossible. This contradiction proves our claim.

Analogously, one can show that for any vertex $z \in A_d$, if $yz \notin E(G)$, then $zf(y) \in E(G)$ must hold. Now, since $S \subseteq N[v_i] \cup N[f(v_i)] \cup A_u \cup A_d$ and $A_u \subseteq N[x] \cup N[f(x)]$, $A_d \subseteq N[y] \cup N[f(y)]$, we conclude that $S \subseteq N[D]$, where $D := \{v_i, f(v_i), x, f(x), y, f(y)\}$. □

Thus, the following results true.

Table 2 Summary of the decomposition results obtained for c-chordal graphs

Chordality of the graph	Decomposition results
3	(1/2, 1, w)
4	(1/2, 1, 2w), (1/2, 6, w)
5	(1/2, 1, 2w)
6	(1/2, 1, 3w), (1/2, 4, 2w)
7	(1/2, 1, 3w)
8	(1/2, 1, 4w), (1/2, 4, 3w)
9	(1/2, 1, 4w), (1/2, 5, 3w)
$c \geq 10$	(1/2, 1, $\lfloor c/2 \rfloor$w), (1/2, 4, ($\lfloor c/3 \rfloor + 1$)w)
$c = 3k$, $k \geq 4$	(1/2, 1, $\lfloor 3k/2 \rfloor$w), (1/2, 4, ($k+1$)w), (1/2, 5, kw)

Theorem 13 *Every* 4*-chordal graph is* $(1/2, 6, \mathsf{w})$*-decomposable.*

Corollary 6 *Any* n*-vertex* m*-edge* 4*-chordal graph* G *admits a system of at most* $6 \log_2 n$ *collective additive tree* $(2\mathsf{w})$*-spanners. Moreover, such a system of spanning trees can be constructed in* $O(nm \log n)$ *time.*

Corollary 7 *Any* n*-vertex* m*-edge* 4*-chordal graph* G *admits an additive* $(2\mathsf{w})$*-spanner with at most* $O(n \log n)$ *edges. Moreover, such a sparse spanner can be constructed in* $O(nm \log n)$ *time.*

The last result improves and generalizes the known from [9, 16, 31] results on sparse spanners of unweighted chordal graphs.

In Table 2, we summaries all our decomposition results obtained for c-chordal graphs.

4 Conclusion

In this paper, we continued the approach taken in [10, 15, 16, 26] of studying *collective tree spanners* of graphs. The method of [16] for constructing a "small" system of collective additive tree r-spanners with small values of r was refined and generalized to weighted and larger families of "well" decomposable graphs.

We defined a large class of graphs, called (α, γ, r)*-decomposable*, and showed that any weighted (α, γ, r)-decomposable graph G with n vertices admits a system of at most $\gamma \log_{1/\alpha} n$ collective additive tree $2r$-spanners. Using this, we showed that any weighted planar graph admits a system of $O(\sqrt{n})$ collective additive tree 0-spanners, any weighted graph with genus at most g admits a system of $O(\sqrt{gn})$ collective additive tree 0-spanners, any weighted graph with tree-width at most $k-1$ admits a system of $k \log_2 n$ collective additive tree 0-spanners, any weighted graph with clique-width at most k admits a system of $k \log_{3/2} n$ collective additive tree $(2\mathsf{w})$-spanners, any weighted c-chordal graph admits a system of $\log_2 n$ ($5 \log_2 n$ and $4 \log_2 n$) collective additive tree $(2\lfloor c/2 \rfloor \mathsf{w})$-spanners (respectively, $(2\lfloor (c+2)/3 \rfloor \mathsf{w})$-spanners and

Table 3 Routing labeling schemes obtained via collective additive tree spanners. The scheme construction time is equal to the time needed to construct an appropriate system of collective additive tree spanners plus $O(n\log n)$ times the number of spanning trees in the system. Thus, the construction time is $O(n^{3/2}\log^2 n)$ for planar graphs, $O(n^{3/2}g^{1/2}\log^2 n)$ for graphs with genus g, $O(h^{3/2}n^{1/2}(m\log n + n\log^2 n))$ for graphs without an h-vertex minor, $O((n^2 + km + kn\log n)\log n)$ for graphs with tree-width $k-1$, polynomial for graphs with clique-width k, and $O(nm\log n)$ for c-chordal graphs ($c \geq 4$)

Graph class	Addresses and routing tables (bits per vertex)	Message initiation time	Routing decision time	Deviation
Planar	$O(\sqrt{n}\log^2 n/\log\log n)$	$O(\sqrt{n})$	$O(1)$	0
Of genus g	$O(\sqrt{gn}\log^2 n/\log\log n)$	$O(\sqrt{gn})$	$O(1)$	0
w/o an h-vertex minor	$O(\sqrt{h^3 n}\log^2 n/\log\log n)$	$O(\sqrt{h^3 n})$	$O(1)$	0
Of tree-width $k-1$	$O(k\log^3 n/\log\log n)$	$k\log_2 n$	$O(1)$	0
Of clique-width k	$O(k\log^3 n/\log\log n)$	$k\log_{3/2} n$	$O(1)$	2w
c-chordal ($c \geq 5$)	$O(\log^3 n/\log\log n)$	$\log_2 n$	$O(1)$	$2\lfloor c/2\rfloor$w
		$4\log_2 n$		$2(\lfloor c/3\rfloor + 1)$w
		$5\log_2 n$		$2\lfloor (c+2)/3\rfloor$w
4-chordal	$O(\log^3 n/\log\log n)$	$6log_2 n$	$O(1)$	2w
Weakly chordal	$O(\log^3 n/\log\log n)$	$4\log_2 n$	$O(1)$	2w

$(2(\lfloor c/3\rfloor + 1)$w)-spanners), any weighted 4-chordal graph admits a system of $6\log_2 n$ collective additive tree (2w)-spanners, and any weighted weakly chordal graph admits a system of $4\log_2 n$ collective additive tree (2w)-spanners.

Combining our decomposition results also with Corollary 1, we obtain the following routing labeling schemes presented in Table 3.

We conclude this paper with few open problems:

(1) Find the complexity of the problem "Given a graph G, an integers μ, and a real number r, decide whether G has a system of at most μ collective additive tree r-spanners" for different $\mu \geq 1$, $r \geq 0$ on general graphs and on different restricted families of graphs.

(2) Find better trade-offs between the number of trees μ and the additive stretch factor r on planar graphs, graphs with genus g and graphs without an h-vertex minor.

(3) Find some more applications where collective tree spanners could be useful. The fact that collective tree spanners give a collection of (good) trees might make it easy to adapt many tree algorithms for the graphs that have collective tree r-spanners.

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