Positive solutions to a new kind Sturm–Liouville-like four-point
boundary value problem

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\begin{abstract}
In this paper, we considered the following four-point boundary value problem
\begin{equation}
\begin{aligned}
& x''(t) + h(t)f(t,x(t),x'(t)) = 0, \\
& x'(0) - \alpha_1 x'(\xi) = 0, \\
& x'(1) + \alpha_2 x'(\eta) = 0,
\end{aligned}
\end{equation}
where $0 < \alpha_1 < \frac{1}{\xi}$, $0 < \alpha_2 < \frac{1}{1-\xi}$, $0 < \xi < \eta < 1$, $\alpha_1 \alpha_2 \eta - \alpha_1 \alpha_2 \xi + \alpha_1 + \alpha_2 > 0$. By applying fixed-point theorems, we obtain a variety of existence results. In particular, our four-point boundary condition is a new kind Sturm–Liouville-like boundary condition, which has rarely been considered up to now.
\end{abstract}

\section{1. Introduction}
In this paper, we consider the existence and the multiplicity of positive solutions for a kind of four-point boundary value problem
\begin{equation}
\begin{aligned}
& x''(t) + h(t)f(t,x(t),x'(t)) = 0, \\
& x'(0) - \alpha_1 x'(\xi) = 0, \\
& x'(1) + \alpha_2 x'(\eta) = 0,
\end{aligned}
\end{equation}
where $0 < \alpha_1 < \frac{1}{\xi}$, $0 < \alpha_2 < \frac{1}{1-\xi}$, $0 < \xi < \eta < 1$, $\alpha_1 \alpha_2 \eta - \alpha_1 \alpha_2 \xi + \alpha_1 + \alpha_2 > 0$. Throughout, we assume
\begin{enumerate}
\item[(H1)] $h: [0,1] \rightarrow [0,\infty)$ is continuous and $h$ is not identical to zero on $[0,1]$.
\item[(H2)] $f \in C^1([0,1] \times [0,\infty) \times \mathbb{R}, [0, +\infty))$ and $f \not\equiv 0$.
\item[(H3)] $f$ is an $L^1$-Caratheodory function.
\end{enumerate}

The study of multi-point boundary value problem were initiated by Ilin and Moiseev [1]. Motivated by the study of Ilin and Moiseev [1], Gupta [2] studied nonlinear three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the readers to [3–9]. Recently, four-point BVP, as a special multi-point BVP, has been received much attention. We refer the readers to [10–14]. Bai in [10] investigated the following four-point boundary value problem

\begin{enumerate}
\item[(H1)] $h: [0,1] \rightarrow [0,\infty)$ is continuous and $h$ is not identical to zero on $[0,1]$.
\item[(H2)] $f \in C^1([0,1] \times [0,\infty) \times \mathbb{R}, [0, +\infty))$ and $f \not\equiv 0$.
\item[(H3)] $f$ is an $L^1$-Caratheodory function.
\end{enumerate}

\begin{thebibliography}{99}
\bibitem{1} Ilin and Moiseev
\bibitem{2} Gupta
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\bibitem{4} Gupta
\bibitem{5} Gupta
\bibitem{6} Gupta
\bibitem{7} Gupta
\bibitem{8} Gupta
\bibitem{9} Gupta
\bibitem{10} Bai
\end{thebibliography}
\[ \begin{cases} x'(t) + q(t)f(t,x(t),x'(t)) = 0, & 0 < t < 1, \\ x(0) = ax(\xi), \quad x(1) = bx(\eta), \end{cases} \] (1.2)

where \( 0 < \xi < \eta < 1, \quad 0 < \alpha < \frac{1}{\beta}, \quad 0 < \beta < \frac{1}{\gamma}. \) What’s deserving attention is that when \( \xi = 0, \eta = 1 \) and \( a, b \neq 1, \) BVP (1.2) becomes a Dirichlet BVP. So [10] can be seen as a generalization of Dirichlet problem. On the other hand, the generalization of the classical BVPs such as Neuman and the mixed have also been considered extensively, we refer the readers to [15–22].

Motivated by the above work, in [12], the authors studied the BVPs subject to the following BC:

\[ x(0) - ax'(\xi) = 0, \quad x(1) + bx'(\eta) = 0. \] (1.3)

Obviously, (1.3) can be seen as a generalization of the Sturm–Liouville BC:

\[ ax(0) - bx'(0) = 0, \quad bx(1) + ax'(1) = 0, \] (1.4)

and the authors called it Sturm–Liouville like BC. However, we can notice that

\[ x'(0) - ax'(\xi) = 0, \quad x'(1) + bx'(\eta) = 0 \] (1.5)

can also be seen as a generalization of (1.4) when \( \beta, \gamma \neq 0. \) And to the best knowledge of the authors, few work has been done to deal with BVPs subject to BC (1.5). So, our work intends to fill this gap in the literature. In order to distinguish from the above work, we call (1.1) a new kind Sturm-Liouville like BVP. We not only get the existence of one positive solution, but also give sufficient conditions to guarantee the existence of at least three positive solutions. Part of our approach is similar to that used in [10], i.e., fixed-point theorem in [20] is employed as the main tool of analysis, but the details here are not trivial. For example, in order to apply the ideas in [20] to our problem, we need to obtain the corresponding Green's function and some useful inequalities, which are much more difficult than that of the usual BVP. To establish such special Green's function, very technical arguments are involved due to the fact that we have a multi-point BC.

The outline of this paper is as follows: In Section 2, we present some background material and state the main theorems that will be used in this paper; In Section 3, we give some useful lemmas; In Section 4, the existence of at least one positive solution to BVP (1.1) was considered; In Section 5, growth conditions are imposed on \( f \) to obtain at least three positive solutions to BVP (1.1) and at the end of this paper, we give an example to illustrate our main result.

2. Background material

For convenience, we present some definitions and theorems that will be used in this paper.

**Definition 2.1.** Let \( E \) be a Banach Space, \( P \subset E \) is a nonempty convex closed set, \( P \) is said to be a cone provided that

(i) \( \lambda u \in P \) for all \( \lambda \geq 0, \ u \in P; \)

(ii) \( u \in P, \ -u \in P \) implies \( u = 0. \)

**Definition 2.2.** A map \( \alpha \) is said to be a nonnegative concave (resp. convex) continuous functional on \( P \) provided that \( \alpha: P \to [0, \infty) \) is continuous and

\[ \alpha(\lambda x + (1 - \lambda)y) \geq \lambda \alpha(x) + (1 - \lambda)\alpha(y) \] (resp. \( \alpha(\lambda x + (1 - \lambda)y) \leq \lambda \alpha(x) + (1 - \lambda)\alpha(y) \))

for all \( x, y \in P \) and \( 0 \leq \lambda \leq 1. \)

**Definition 2.3.** We say \( f: [0,1] \times [0,\infty) \times R \to [0,\infty) \) is an \( L^1 \)-Carathéodory function if

(i) \( t \to f(t,x,y) \) is measurable for any \( (x,y) \in [0,\infty) \times R; \)

(ii) \( t \to f(t,x,y) \) is continuous for a.e. \( t \in [0,1]; \)

(iii) For each \( r_1, r_2 > 0 \) there exists \( h_1, h_2 \in L^1[0,1] \) such that \( |x| \leq r_1, |x| \leq r_2 \) implies \( |f(t, x, y)| \leq h_1, h_2(t) \) for a.e. \( t \in [0,1]. \)

**Definition 2.4.** Let \( r > a > 0, \ L > 0 \) be given and \( \psi \) be a nonnegative concave functional and \( \alpha, \beta \) be nonnegative convex functional on the cone \( P. \) Define convex sets

\[ P(\alpha^r, \beta^L) = \{ x \in P | \alpha(x) < r, \beta(x) < L \}, \]

\[ P(\alpha^r, \beta^L, \psi_a) = \{ x \in P | \alpha(x) < r, \beta(x) < L, \psi(x) > a \}, \]

\[ P(\alpha^r, \beta^L, \psi_a) = \{ x \in P | \alpha(x) < r, \beta(x) < L, \psi(x) \geq a \}. \]

Suppose the nonnegative continuous convex functionals \( \alpha, \beta \) on cone \( P \) satisfy
Lemma 3.1. Some Lemmas

Lemma 3.1. The Green function for BVP

\[
\begin{cases}
-x''(t) = 0, & 0 < t < 1, \\
x'(0) - \alpha_1 x(\xi) = 0, & x'(1) + \alpha_2 x(\eta) = 0
\end{cases}
\]

is

\[
G(t,s) = \begin{cases}
G_1(t,s), & 0 \leq s \leq \min\{t, \xi\} \leq 1, \\
G_2(t,s), & 0 \leq t \leq s \leq \xi, \\
G_3(t,s), & \xi \leq s \leq \min\{t, \eta\} \leq 1, \\
G_4(t,s), & 0 \leq \max\{\xi, t\} \leq s \leq \eta, \\
G_5(t,s), & \eta \leq s \leq t \leq 1, \\
G_6(t,s), & 0 \leq \max\{\xi, t\} \leq s \leq 1,
\end{cases}
\]

where

\[
\begin{align*}
G_1(t,s) &= \frac{1}{\delta} (\alpha_2 \eta + 1 - \alpha_2 t), \\
G_2(t,s) &= \frac{1}{\delta} (\alpha_2 \eta + 1 - \alpha_2 s) + \frac{1}{\delta} (s - t)(\alpha_1 \alpha_2 \xi - \alpha_1 \alpha_2 \eta - \alpha_1), \\
G_3(t,s) &= \frac{1}{\delta} (\alpha_2 \eta + 1 - \alpha_2 t)(\alpha_1 s + 1 - \alpha_1 \xi), \\
G_4(t,s) &= \frac{1}{\delta} (\alpha_2 \eta + 1 - \alpha_2 s)(\alpha_1 t + 1 - \alpha_1 \xi), \\
G_5(t,s) &= \frac{1}{\delta} (\alpha_1 t + 1 - \alpha_1 \xi) + (s - t), \\
G_6(t,s) &= \frac{1}{\delta} (\alpha_1 t + 1 - \alpha_1 \xi),
\end{align*}
\]

and \( \delta = \alpha_1 \alpha_2 \eta - \alpha_1 \alpha_2 \xi + \alpha_1 + \alpha_2 > 0. \)

Remark 3.1. It is easy to see that when \( 0 < \xi < \eta < 1, \ 0 < \alpha_1 < \frac{1}{\xi}, \ 0 < \alpha_2 < \frac{1}{\eta}, \ \delta = \alpha_1 \alpha_2 \eta - \alpha_1 \alpha_2 \xi + \alpha_1 + \alpha_2 > 0. \)
Lemma 3.2. So, the Green function of BVP (3.1) on \([0,1]\). The proof is completed. □

Lemma 3.2. If \(0 < \alpha_1 < \frac{1}{\nu}, \ 0 < \alpha_2 < \frac{1}{\nu - \eta}\), then

(a) \(\max \left| \frac{\partial G(t,s)}{\partial t} \right| = \max \{1 - \alpha_1/\delta, 1 - \alpha_2/\delta\}\); 

(b) \(G(t,s) \geq 0\) for all \(0 \leq t, s \leq 1\); 

(c) there exists a constant \(K = \min \left\{ \frac{1}{\nu - \eta}, \frac{1}{\nu - \eta - \xi} \right\} \) such that 
\[ G(t,s) \geq K \max G(t,s), \quad \text{for} \ t \in [\xi, \eta], s \in [0,1]. \]

Proof. From Lemma 3.1, we have 

\[ \frac{\partial G(t,s)}{\partial t} = \begin{cases} 
\frac{\partial G(t,s)}{\partial t} = -\frac{\alpha_2}{2}, & 0 \leq t \leq 1, \\
\frac{\partial G(t,s)}{\partial t} = \frac{\alpha_2 s_1 + 2 \alpha_2 s_2}{3}, & 0 \leq s \leq \xi, \\
\frac{\partial G(t,s)}{\partial t} = -\frac{\alpha_2 s_1 + 2 \alpha_2 s_2}{3}, & \xi \leq s \leq \min\{t, \eta\}, \\
\frac{\partial G(t,s)}{\partial t} = -\frac{\alpha_2 s_1 + 2 \alpha_2 s_2}{3}, & 0 \leq t \leq \xi, \\
\frac{\partial G(t,s)}{\partial t} = \frac{\alpha_2 s_1 + 2 \alpha_2 s_2}{3}, & \xi \leq s \leq \min\{t, \eta\}, \\
\frac{\partial G(t,s)}{\partial t} = -\frac{\alpha_2}{2}, & 0 \leq \min\{t, \eta\}, \ s \leq \xi, \\
\frac{\partial G(t,s)}{\partial t} = \frac{\alpha_2}{3}, & \xi \leq s \leq \min\{t, \eta\}, \\
\frac{\partial G(t,s)}{\partial t} = 0, & 0 \leq \min\{t, \eta\}, \ s \leq \xi, \\
\frac{\partial G(t,s)}{\partial t} = 0, & 0 \leq \min\{t, \eta\}, \ s \leq \xi. 
\end{cases} \tag{3.5} \]

Clearly, \(\max \left| \frac{\partial G(t,s)}{\partial t} \right| = \max \{1 - \alpha_1/\delta, 1 - \alpha_2/\delta\}\), which is the desired result in (a).

Next, we show that (b) holds. It is clear that \(G_1(t,s), G_3(t,s), G_4(t,s), G_5(t,s) \geq 0\), we only need to prove \(G_2(t,s), G_3(t,s) \geq 0\).
When $0 \leq t \leq s \leq \xi$, we have
\[ \delta G_2(t, s) = (\alpha_1 \alpha_2 (\eta - \xi) + \alpha_1) t + (\alpha_1 \alpha_2 (\xi - \eta) - \alpha_1 - \alpha_2) s + 1 - \alpha_2 \eta. \]
Since $\delta G_2(t, s)$ is increasing on $t \in [0, 1]$ and decreasing on $s \in [0, 1]$, we have
\[ \min_{t \in [0, 1]} (\delta G_2(t, s)) = 1 + \alpha_2 (\eta - \xi) \quad \text{when} \quad t = s = \xi \]
and we can see that $1 + \alpha_2 (\eta - \xi) > 0$, then $G_2(t, s) > 0$.

When $\eta \leq s \leq t \leq 1$, we have
\[ G_5(t, s) = \frac{1}{\delta} (\alpha_1 t + 1 - \alpha_2 s) + (s - t) = \frac{1}{\delta} (\alpha_1 + \delta) t + \frac{1}{\delta} (1 - \alpha_2 s) + s = \frac{1}{\delta} (\alpha_1 \alpha_2 s - \alpha_1 \alpha_2 \eta - \alpha_2) + \frac{1}{\delta} (1 - \alpha_2 s) + s, \]
since $\alpha_1 \alpha_2 \eta - \alpha_2 < 0$, we know that $G_5(t, s)$ is decreasing on $t$ and increasing on $s$, so $G_5(t, s)$ arrives its minimum at $t = 1, s = \eta$, then,
\[ \min_{t \in [0, 1]} (\delta G_5(t, s)) = (1 + \alpha_2 \eta - \alpha_2)(1 + \alpha_1 \eta - \alpha_1 \xi) \geq 0, \]
so we have $G_5(t, s) \geq 0$.

Finally, we show (c) also holds. According to Lemma 3.1, we have
\[ \min_{t \in [0, 1]} G(t, s) = \min_{t \in [0, 1]} \{ \min_{t \in [0, 1]} G_1(t, s), \min_{t \in [0, 1]} G_2(t, s), \min_{t \in [0, 1]} G_3(t, s), \min_{t \in [0, 1]} G_4(t, s), \min_{t \in [0, 1]} G_5(t, s) \}. \]
\[ \max_{t \in [0, 1]} G(t, s) = \max \left\{ \begin{array}{ll}
\max_{t \in [0, 1]} G_1(t, s), & \max_{t \in [0, 1]} G_2(t, s), \max_{t \in [0, 1]} G_3(t, s), \\
\max_{t \in [0, 1]} G_4(t, s), & \max_{t \in [0, 1]} G_5(t, s) \end{array} \right\}. \]
From (3.5), we can easily get
\[ \min_{t \in [0, 1]} G(t, s) = \frac{1}{\delta} \max_{t \in [0, 1]} \max_{t \in [0, 1]} \left\{ \frac{1}{\delta} (1 + \alpha_2 \eta), \frac{1}{\delta} (\alpha_1 + 1 - \alpha_1 \xi) \right\}. \]
Set
\[ K = \min_{t \in [0, 1]} \left\{ \frac{1}{1 + \alpha_2 \eta}, \frac{1}{\alpha_1 + 1 - \alpha_1 \xi} \right\}, \]
it holds
\[ \min_{t \in [0, 1]} G(t, s) \geq K \max_{t \in [0, 1]} G(t, s), \quad \text{for} \quad s \in [0, 1]. \]
The proof is completed. \( \square \)

Let $X = C^1([0, 1])$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0, 1]$, and the maximum norm $\|x\| = \max \{|x|_1, |x'|_1\}$, where $|x|_1 = \max_{0 \leq t \leq 1} |x(t)|$. Cone $P \subset X$ is defined by
\[ P = \left\{ x \in X : x(t) \geq 0, \min_{t \in [0, 1]} x(t) \geq K \max_{t \in [0, 1]} x(t), x \text{ is concave on } [0, 1] \right\}. \]
Set $T : P \rightarrow X$ by
\[ (Tx)(t) = \int_0^1 G(t, s) h(s) f(s, x(s), x'(s)) ds, \]
$0 \leq t \leq 1$, where $G(t, s)$ is as defined in Lemma 3.1.

**Lemma 3.3.** Let $(H_1)$ holds, $(H_2)$ or $(H_3)$ holds, and $G(t, s) \geq 0$. Then $T : P \rightarrow P$ is completely continuous.

**Proof.** To justify this, we first show that $T : P \rightarrow P$ is well defined. Let $x \in P$, by Lemma 3.1,
\[ Tx(t) := \int_0^1 G(t, s) h(s) f(s, x(s), x'(s)) ds \geq 0. \tag{3.6} \]
Next, by Lemma 3.2, we get
\[ \min_{t \in [0, 1]} T x(t) = \min_{t \in [0, 1]} \int_0^1 G(t, s) h(s) f(s, x(s), x'(s)) ds \geq K \int_0^1 \max_{t \in [0, 1]} G(t, s) h(s) f(s, x(s), x'(s)) ds \geq K \max_{t \in [0, 1]} T x(t). \]
Moreover, $(Tx)'(t) \leq 0$ on $t \in [0, 1]$ can be verified easily.
So $T : P \rightarrow P$ is well defined.

$T$ is completely continuous if and only if $T$ is continuous with respect to $x$ and maps a bounded subset of $P$ into a relatively compact set.
Let $x_n \to x$ as $n \to +\infty$ in $P$. Then there exists $r > 0$ such that $\|x_n\| < r$, hence

\[ 0 < Tx(t) := \int_0^1 G(t, s)h(s)f(s, x(s), x'(s))ds < +\infty, \]

\[ |(Tx)'(t)| = \left| \int_0^1 \frac{\partial G(t, s)}{\partial t} h(s)f(s, x(s), x'(s))ds \right| < +\infty, \]

we have

\[ |(Tx_n)(t) - (Tx)(t)| \leq \int_0^1 |G(t, s)||h(s)||f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))|ds \]

\[ \to 0 \quad \text{as} \quad n \to +\infty, \quad (3.7) \]

and

\[ |(Tx_n)'(t) - (Tx)'(t)| \leq \max_{0 \leq t \leq 1} \left| \frac{\partial G(t, s)}{\partial t} \right| \int_0^1 |h(s)||f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))|ds \]

\[ \to 0 \quad \text{as} \quad n \to +\infty. \quad (3.8) \]

Let $D \subset P$ be a bounded subset, so there exists $\rho$, s.t. $D \subset \{x \in P: \|x\| \leq \rho\}$, $\forall x \in D$, we have

\[ 0 \leq \int_0^1 f(s, x(s), x'(s))ds \leq \max_{(s, u, u') \in [0,1] \times [-\rho, \rho] \times [-\rho, \rho]} f(s, u, u') < \infty, \]

so $TD$ is uniformly bounded according to the properties of $f, h$.

\[ |(Tx)(t_1) - (Tx)(t_2)| \leq \int_0^1 |G(t_1, s) - G(t_2, s)||h(s)||f(s, x(s), x'(s))|ds \]

\[ \to 0 \quad \text{as} \quad t_1 \to t_2, \quad (3.9) \]

and

\[ |(Tx)'(t_1) - (Tx)'(t_2)| \leq \left| \frac{\partial G(t, s)}{\partial t} \right| \int_{t_1}^{t_2} |h(s)||f(s, x(s), x'(s))|ds \]

\[ \to 0 \quad \text{as} \quad t_1 \to t_2. \quad (3.10) \]

Therefore, combining (3.7)–(3.10) and by Arzela-Ascoli Theorem we can get $T: P \to P$ is completely continuous. \qed

4. The existence of one positive solution

Denote

\[ \|h\| = \int_0^1 |h(s)|ds, \quad \Gamma_1 = \max_{0 \leq t \leq 1} \{G(t, s)\} \|h\|, \quad \Gamma_2 = \max_{0 \leq t \leq 1} \left\{ \left| \frac{\partial G(t, s)}{\partial t} \right| \right\} \|h\|. \]

Theorem 4.1. Suppose that (H2) holds and $f(t, 0, 0) \neq 0$ for a.e. $t \in [0, 1]$, there exist functions $a, b, c \in L^1([0,1], [0, \infty))$ satisfying

\[ \|b\|_{L^1} + \|c\|_{L^1} < \min \left\{ \frac{1}{\Gamma_1}, \frac{1}{\Gamma_2} \right\} \]

such that

\[ f(t, x, y) \leq a(t) + b(t)x + c(t)y. \]

Then problem (1.1) has at least one nontrivial positive solution.

Proof. From Lemma 3.3 we have known that $T: P \to P$ is completely continuous. Let

\[ R > \max \left\{ \frac{\Gamma_1 \|a\|_{L^1}}{1 - \Gamma_1 \|b\|_{L^1} + \|c\|_{L^1}}, \frac{\Gamma_2 \|a\|_{L^1}}{1 - \Gamma_2 \|b\|_{L^1} + \|c\|_{L^1}} \right\}. \]

Now define $\Omega = \{x \in P: \|x\| < R\}$. Then, for any $x \in \partial \Omega$, there is $\|x\| = R$, $\|x'\| = R$. Thus,

\[ |(Tx)(t)| = \left| \int_0^1 G(t, s)h(s)f(s, x(s), x'(s))ds \right| \leq \max_{0 \leq t \leq 1} G(t, s) \int_0^1 |h(s)||f(s, x(s), x'(s))|ds \]

\[ \leq \Gamma_1 \|a\|_{L^1} + (\|b\|_{L^1} + \|c\|_{L^1})R \leq \Gamma_1 \int_0^1 |f(s, x(s), x'(s))|ds \leq \Gamma_1 \int_0^1 |f(s, x(s), x'(s))|ds \]

\[ \leq \Gamma_1 \|a\|_{L^1} + (\|b\|_{L^1} + \|c\|_{L^1})R < R = \|x\|, \]

\[ (Tx)(t) \neq \lambda x \quad \text{for all} \quad \lambda \in (0, \frac{1}{\Gamma_1 \|a\|_{L^1}}). \]

\[ \frac{\|Tx\|}{\|x\|} \leq \frac{\max_{t \in [0,1]} \|G(t,s)\|}{\|x\|} \leq R = N. \]

So \( \|Tx\| \leq \|x\| \), i.e., taking \( p = 0 \) in Theorem 2.2, for any \( x \in \mathcal{P} \), \( x = \lambda Tx (0 < \lambda < 1) \) does not hold. Thus Theorem 2.2 implies that the operator \( T \) has at least one fixed point. In view of

\[ \int_0^1 f(s,x(s),x'(s)) ds \leq \frac{1}{2} \int_0^1 h(s)f(s,x(s),x'(s)) ds \leq \frac{1}{2} \int_0^1 h(s)f(s,x(s),x'(s)) ds \leq \frac{1}{2} \int_0^1 f(s,x(s),x'(s)) ds \]

\[ \leq \Gamma R < R = \|x\|. \]

Therefore, \( \|Tx\| \leq \|x\| \) for all \( x \in X \).

**5. The existence of triple positive solutions**

In this section, we will use Lemmas 3.1–3.3 to acquire some new results of positive solutions for problem (1.1).

Denote

\[ \alpha(x) = \max_{t \in [0,1]} |x(t)|, \quad \beta(x) = \max_{t \in [0,1]} |x'(t)|, \quad \psi(x) = \min_{t \in [0,1]} |x(t)|. \]

Obviously, \( \alpha, \beta, \psi : P \to [0,\infty) \) are nonnegative continuous convex functionals satisfying (A1) and (A2); \( \psi \) is nonnegative continuous concave functional with \( \psi(x) \leq \alpha(x) \) for all \( x \in X \). Then \( \alpha(x), \beta(x), \psi(x) \) satisfy the conditions in Theorem 2.1.

Let

\[ L = \max_{t \in [0,1]} \int_0^1 G(t,s) h(s) ds, \]

\[ N = \max_{0 < t < 1} \left\{ \int_0^1 \frac{\partial G(t,s)}{\partial s} h(s) ds, \int_0^1 \frac{\partial G(t,s)}{\partial s} h(s) ds \right\}, \]

\[ C = \min \left\{ \int_0^\eta G(\xi,\eta) h(\eta) d\eta, \int_0^\eta G(\eta,\eta) h(\eta) d\eta \right\}. \]

**Theorem 5.1.** Suppose that \( f \in C^1([0,1] \times [0,1] \times R, [0,\infty)) \), and there exist numbers \( r_2 \geq d = b/K > b > r_1 > 0, L_2 > L_1 \) such that \( b/C \leq \min(r_2/L, L_2/N) \), further suppose that

\[ (S_1) \ f(t,u,v) < \min_{r \in [0,1]} \max r(L_2/N), \text{ for } (t,u,v) \in [0,1] \times [0,r_1] \times [-L_1,L_1]; \]

\[ (S_2) \ f(t,u,v) > b/C, \text{ for } (t,u,v) \in [z,\eta] \times [b,b/K] \times [-L_2,L_2]; \]

\[ (S_3) \ f(t,u,v) \leq \min_{r \in [0,1]} \max r(L_2/N), \text{ for } (t,u,v) \in [0,1] \times [0,r_2] \times [-L_2,L_2], \]

then BVP (1.1) has at least three positive solutions \( x_1, x_2, x_3 \) such that

\[ \max_{0 < t < 1} |x_1(t)| < r_1, \quad \max_{0 < t < 1} |x'_1(t)| < L_1, \]

\[ b < \min_{0 < t < 1} |x_2(t)| \leq \max_{0 < t < 1} |x_2(t)| < r_2, \quad \max_{0 < t < 1} |x'_2(t)| < L_2, \]

\[ \max_{0 < t < 1} |x_3(t)| < b/K, \quad \min_{0 < t < 1} |x_3(t)| \leq b, \quad \max_{0 < t < 1} |x'_3(t)| < L_2. \]

**Proof.** Let \( X, P, \alpha, \beta, \psi \), and \( T \) be defined as above respectively. Problem (1.1) has a solution \( x = \alpha(x) \) if and only if \( x \) solves the operator equation \( x = Tx \). Thus, we set out to verify that the operator \( T \) satisfies the fixed-point Theorem 2.1 which will prove the existence of three fixed points of \( T \).

If \( x \in \mathcal{P}(x^*, \beta^*) \), then \( \alpha(x) \leq r_2, \beta(x) \leq L_2, \) and assumption (S3) implies that \( f(t,x(t),x'(t)) \leq \min_{r \in [0,1]} \max r(L_2/N) \) for all \( t \in [0,1] \). Therefore,

\[ \alpha(Tx) = \max_{0 < t < 1} |(Tx)(t)| = \max_{0 < t < 1} \int_0^1 G(t,s) h(s) f(s,x(s),x'(s)) ds \leq r_2/L \int_0^1 G(t,s) h(s) ds = r_2. \]

For \( x \in P \), we have \( Tx \in P \). Then \( Tx(t) \) is concave on \([0,1]\), it follows that

\[ \beta(Tx) = \max_{0 < t < 1} |(Tx)'(t)| = \max_{0 < t < 1} \{(Tx)'(0)|, (Tx)'(1)\} \]

\[ = \max \left\{ \int_0^1 \frac{\partial G(t,s)}{\partial t} h(s)f(s,x(s),x'(s)) ds, \int_0^1 \frac{\partial G(t,s)}{\partial t} h(s)f(s,x(s),x'(s)) ds \right\} \leq \frac{L_2}{N} N = L_2. \]
So, \( T : \overline{P}(x^i, \beta^i) \rightarrow \overline{P}(x^i, \beta^i) \). And by Lemma 3.3 we know that \( T \) is completely continuous. With assumption (S1) by the similar argument, we can get that \( T : \overline{P}(x^i, \beta^i) \rightarrow \overline{P}(x^i, \beta^i) \), hence, condition (B2) in Theorem 2.1 is satisfied.

Next, to check condition (B1) of Theorem 2.1, we choose \( x(t) = b/K \in \overline{P}(x^i, \beta^i) \), then all the conditions of Theorem 5.1 are satisfied, thus problem (5.2) has at least three positive solutions satisfied.

In addition, condition (H1), (H2) and Lemma 3.1 guarantees that those fixed points are positive. So (1.1) has at least three positive solutions \( x_1, x_2, x_3 \) satisfying (5.1) and we complete our proof. \( \square \)

At the end of this section, we give an example to illustrate our main results.

**Example 5.2.**

\[
\begin{align*}
\begin{cases}
\dot{x}(t) + f(t, x(t), \dot{x}(t)) = 0, \\
\dot{x}(0) - \frac{1}{4}x(\frac{1}{4}) = 0, \\
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
& f(t, u, v) = \begin{cases}
2 \sin t + \left(\frac{5}{3}\right)^7 + \frac{|v|}{1000}, & u \leq 11, \\
2 \sin t + \left(\frac{11}{3}\right)^7 + \frac{|v|}{1000}, & u \geq 11,
\end{cases} \\
& h(t) = 1, \quad \xi = \frac{1}{2}, \quad \eta = \frac{2}{3}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = 2.
\end{align*}
\]

By direct calculation, we have \( K = \frac{3}{2}, \quad \delta = \frac{1}{2}, \quad M = 3, \quad L = \frac{7}{2}, \quad C = \frac{400}{384}, \quad N \leq \frac{14}{16} \). Let \( r_1 = 2, r_2 = 10000, b = 6, b/K = 14, L_1 = 10, L_2 = 10000 \).

We can verify that \( f \) satisfies (H2), furthermore we have

\[
\begin{align*}
& f(t, u, v) < r_1/L = 16/7, \quad \text{for} (t, u, v) \in [0, 1] \times [0, 2] \times [-10, 10]; \\
& f(t, u, v) > b/C = 2304/25, \quad \text{for} (t, u, v) \in \left[1/2, 2/3\right] \times [6, 14] \times [-10000, 10000]; \\
& f(t, u, v) \leq \min \left(r_2/L, L_2/N\right) = 80000/7, \quad \text{for} (t, u, v) \in [0, 1] \times [0, 10000] \times [-10000, 10000].
\end{align*}
\]

Then all the conditions of Theorem 5.1 are satisfied, thus problem (5.2) has at least three positive solutions satisfied

\[
\begin{align*}
& \max_{0 \leq t \leq 1} x_1(t) < 2, \quad \max_{0 \leq t \leq 1} |x'_1(t)| < 10, \\
& 6 < \min_{0 \leq t \leq 1} x_2(t) \leq \max_{0 \leq t \leq 1} x_2(t) < 10000, \quad \max_{0 \leq t \leq 1} |x'_2(t)| < 10000, \\
& \max_{0 \leq t \leq 1} x_3(t) < 14, \quad \min_{0 \leq t \leq 1} x_3(t) \leq 6, \quad \max_{0 \leq t \leq 1} |x'_3(t)| < 10000.
\end{align*}
\]

**References**


