Differential equations involving causal operators with nonlinear periodic boundary conditions

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Abstract

The notion of causal operators is extended to periodic boundary value problems with nonlinear boundary conditions in this paper. By utilizing the monotone iterative technique and the method of lower and upper solutions (resp. weakly coupled lower and upper solutions), we establish the existence of the extremal solutions (resp. weakly coupled extremal quasi-solutions) for nonlinear periodic boundary value problems with causal operators.

Keywords: Lower and upper solutions; Causal operators; Weakly coupled lower and upper solutions; Weakly coupled quasi-solutions; Nonlinear periodic boundary conditions

1. Introduction

In this paper, we deal with the following differential equation with a causal operator:

\[ x'(t) = (Qx)(t), \quad t \in J := [0, T], \]
\[ g(x(0), x(T)) = 0, \]  

(1.1)

where \( Q \in C[E, E] \) is a causal operator, namely, a nonanticipative operator. Some authors have focused their interest on differential equations with causal operators recently, because its theory has the powerful quality of unifying ordinary differential equations, integro differential equations, differential equations with finite or infinite delay, Volterra integral equations and neutral functional equations. Readers are referred to the papers [2–4] and references cited therein.

In 2005, Drici et al. [3] extended differential equations with causal operators to the framework of an arbitrary Banach space and proved the basic results such as existence, uniqueness and global existence. Immediately after it, they discussed in [4] the periodic boundary value problems with a causal operator

\[ u'(t) = (Qu)(t), \quad u(0) = u(T), \]  

(1.2)
where \( t \in J \), \( Q \in C[E, E] \) is a causal operator. The authors developed the monotone iterative technique to obtain the existence of extremal solutions for (1.2).

At the same time, periodic boundary value problems with nonlinear boundary conditions have attracted much attention, we may see [1,5,6]. Inspired by the above results, in this paper, we extend the notion of causal operators to the periodic boundary value problems with nonlinear boundary conditions. First, some comparison principles and the existence and uniqueness of the solutions for the first order linear differential equations with linear boundary conditions are presented. Next, by utilizing the monotone iterative technique and the method of lower and upper solutions, we establish the existence of extremal solutions of the problem (1.1). At last, using the notion of weakly coupled lower and upper solutions, we testify the existence of the weakly coupled minimal and maximal quasi-solutions of the problem (1.1). The results obtained in this paper generalized those in [4].

2. Preliminaries

In this section, we present some definitions and lemmas which help to prove our main results.

Let \( E = C[J, \mathbb{R}] \), where \( J = [0, T] \). We define

\[
\|u - v\|(t) = \max_{\tau_0 \leq s \leq t} |u(t) - v(t)|.
\]

Take \( \Omega = E \cap C^1[J, \mathbb{R}] \). A function \( x \in \Omega \) is said to be a solution of problem (1.1) if it satisfies (1.1).

**Definition 2.1.** Suppose that \( Q \in C[E, E] \), then \( Q \) is said to be a causal map or a nonanticipative map if \( u(s) = v(s), t_0 \leq s \leq t \leq T \), where \( u, v \in E \), then

\[
(Qu)(s) = (Qv)(s), \quad t_0 \leq s \leq t.
\]

**Lemma 2.1.** Let \( m \in \Omega \) be such that

\[
m'(t) \leq -Mm(t) - (\xi m)(t), \quad m(0) \leq \lambda m(T),
\]

where \( 0 < \lambda \leq 1, M \geq 0 \) and \( \xi \in C[E, E] \) is a positive linear operator, that is, \( \xi m \geq 0 \) whenever \( m \geq 0 \). Then, \( m(t) \leq 0 \) for \( t \in J \) provided

\[
\int_0^T e^{MT}(\xi e^{-M})t dt \leq \lambda e^{-MT} \quad \text{and} \quad 0 < \lambda e^{-MT} \leq 1;
\]

or

\[
T(M + (\xi 1)(T)) \leq \lambda \quad \text{and} \quad 0 < \lambda \leq 1.
\]

**Proof.** Let \( v(t) = e^{MT}m(t) \), then \( v \in \Omega \) and

\[
v'(t) \leq -e^{MT}(\xi e^{-M}v)(t), \quad v(0) \leq \lambda e^{-MT} v(T).
\]

Obviously, \( v(t) \leq 0 \) implies \( m(t) \leq 0 \). So, it suffices to show \( v(t) \leq 0 \) for any \( t \in J \). Suppose on the contrary, \( v(t) > 0 \) for some \( t \in J \). Then there are two cases:

(i) there exists \( \bar{t} \in J \) such that \( v(\bar{t}) > 0 \) and \( v(t) \geq 0 \) for all \( t \in J \);

(ii) there exist \( t^* \) and \( t_0 \) such that \( v(t^*) > 0 \) and \( v(t_0) < 0 \).

First, suppose (2.2) holds. In case (i), (2.3) implies \( v'(t) \leq 0 \) on \( J \) and \( v(t) \) is nonincreasing on \( J \). If \( \lambda e^{-MT} = 1 \), \( v(0) \leq v(T) \) shows that \( v(t) \equiv c \), then \( v(t) = 0 \). On the other hand, we have \( 0 = v'(\bar{t}) \leq -e^{M\bar{t}}(\xi e^{-M}v)(\bar{t}) < 0 \) since \( v(\bar{t}) > 0 \), which is a contradiction. If \( 0 < \lambda e^{-MT} < 1 \), then \( v(T) \leq v(0) \leq \lambda e^{-MT} v(T) \), which is also a contradiction.

Now turning to case (ii). Let \( \min_{t \in J} v(t) = -\gamma \), then \( \gamma > 0 \). Without loss of generality, we may suppose \( v(t_0) = -\gamma \).

First suppose \( t_0 < t^* \), integrating the first inequality of (2.3) from \( t_0 \) to \( t^* \) one attains

\[
v(t^*) - v(t_0) = \int_{t_0}^{t^*} v'(t) dt \leq \int_{t_0}^{t^*} -e^{M(t_0 - t)}(\xi e^{-M}v)(t) dt \leq \gamma \int_0^T e^{MT} (\xi e^{-M})(t) dt.
\]
and hence
\[ 0 < v(t^*) \leq -\gamma + \gamma \int_0^T e^{MT}(\xi e^{-M})(t)\,dt \leq -\gamma \left(1 - \int_0^T e^{MT}(\xi e^{-M})(t)\,dt\right), \]
which yields
\[ \int_0^T e^{MT}(\xi e^{-M})(t)\,dt > 1. \]
A contradiction is then elicited due to (2.2).
If \( t_0 > t^* \), we have from (2.3)
\[
v(T) \leq v(t_0) + \gamma \int_{t_0}^T e^{MT}(\xi e^{-M})(t)\,dt
\]
and
\[
v(t^*) \leq v(0) + \gamma \int_0^{t^*} e^{MT}(\xi e^{-M})(t)\,dt,
\]
based on the fact \( v(0) \leq \lambda e^{-MT}v(T) \) and \( 0 < \lambda e^{-MT} \leq 1 \), it then follows from (2.2), (2.4) and (2.5) that
\[
0 < v(t^*) \leq \lambda e^{-MT} \left(-\gamma + \gamma \int_{t_0}^T e^{MT}(\xi e^{-M})(t)\,dt\right) + \gamma \int_0^{t^*} e^{MT}(\xi e^{-M})(t)\,dt \\
\leq -\gamma \lambda e^{-MT} + \gamma \int_{t_0}^T e^{MT}(\xi e^{-M})(t)\,dt + \gamma \int_0^{t^*} e^{MT}(\xi e^{-M})(t)\,dt \\
\leq -\gamma (\lambda e^{-MT} - \int_0^T e^{MT}(\xi e^{-M})(t)\,dt) \leq 0,
\]
which is a contradiction.

If (2.2) holds, by using (2.1) directly instead of (2.3), we may derive contradiction in a similar way. This completes the proof. \( \square \)

Consider the following problem
\[
y'(t) = -My(t) - (\xi y)(t) + \sigma_u(t), \quad t \in J, \\
g(u(0), u(T)) + M_1(y(0) - u(0)) + M_2(y(T) - u(T)) = 0,
\]
where \( \sigma_u(t) = (Qu)(t) + Mu(t) + (\xi a)(t) \).

**Lemma 2.2.** \( y \in \Omega \) is a solution of problem (2.6) if and only if \( y \) is a solution of the following integral equation
\[
y(t) = \frac{e^{-MT}Bu}{M_1 - M_2e^{-MT}} + \int_0^T G(t, s)[-(\xi y)(s) + \sigma_u(s)]\,ds, \quad t \in J,
\]
where \( Bu = -g(u(0), u(T)) + M_1u(0) - M_2u(T), M, M_1, M_2 \) are constants satisfying \( M \geq 0, M_1 \neq M_2e^{-MT} \) and
\[
G(t, s) = \begin{cases} 
\frac{M_2}{M_1 - M_2e^{-MT}}e^{-M(t+s)} + e^{-M(t-s)}, & 0 \leq t < s \leq T; \\
\frac{M_1 - M_2e^{-MT}}{M_1 - M_2e^{-MT}}e^{-M(t+s)}, & 0 \leq s < t \leq T.
\end{cases}
\]

The result may be obtained by a simple computation.

Obviously, \( \|G(t, s)\| = \max\{|\frac{M_1}{M_1 - M_2e^{-MT}}|, \frac{M_2e^{-MT}}{M_1 - M_2e^{-MT}}|\} \). In the remainder of the paper, we denote \( \tau = \|G(t, s)\| = \max\{|\frac{M_1}{M_1 - M_2e^{-MT}}|, \frac{M_2e^{-MT}}{M_1 - M_2e^{-MT}}|\} \).
Lemma 2.3. If $M \geq 0$, $M_1 \neq M_2 e^{-MT}$ and
\[ \tau \| \mathcal{L} \| T < 1. \]  
(2.8)

Then problem (2.6) has a unique solution $y \in \Omega$.

Proof. Define an operator $A : \Omega \to \Omega$ by
\[ (Ay)(t) = \frac{e^{-MT}Bu}{M_1 - M_2 e^{-MT}} + \int_0^T G(t, s)(-Qy(s) + \sigma_n(s))ds, \quad t \in J. \]

For $y_1, y_2 \in \Omega$, one derives
\[ |Ay_1 - Ay_2| = \left| \int_0^T G(t, s)(\mathcal{L}(y_2 - y_1))(s)ds \right| \leq \tau \| \mathcal{L} \| T \| y_1 - y_2 \| . \]

Hence, (2.8) shows that $A$ is a contraction. Banach fixed point theorem shows that $A$ has a unique fixed point $y^*$ such that $Ay^* = y^*$. It is clear that this fixed point is the solution of (2.6). The proof is then finished. \qed

3. Main results

In this section, we shall establish the existence of extremal solutions of problem (1.1).

Definition 3.1. Functions $\alpha, \beta \in \Omega$ are said to be lower and upper solutions of problem (1.1), respectively, if
\[ \alpha'(t) \leq (Q\alpha)(t), \quad g(\alpha(0), \alpha(T)) \leq 0 \]
and
\[ \beta'(t) \geq (Q\beta)(t), \quad g(\beta(0), \beta(T)) \geq 0. \]

Now we state our theorem.

Theorem 3.1. Let (2.8) hold and $Q \in C[E, E]$. Suppose the following conditions hold,
(H1) the functions $\alpha, \beta \in \Omega$ are lower and upper solutions of problem (1.1) respectively, such that $\alpha(t) \leq \beta(t)$ for $t \in J$;
(H2) whenever $\alpha(t) \leq x_2(t) \leq x_1(t) \leq \beta(t)$, $Q$ satisfies
\[ (Qx_1)(t) - (Qx_2)(t) \geq -M(x_1(t) - x_2(t)) - (\mathcal{L}(x_1 - x_2))(t), \quad t \in J, \]
where $M > 0$ is a constant and $\mathcal{L} \in C[E, E]$ is a positive linear operator such that
\[ \int_0^T e^{MT}(\mathcal{L}e^{-M})(t)dt \leq \frac{M_2}{M_1} e^{-MT} \]  
(3.1)
or
\[ T(M + (\mathcal{L}1)(T)) \leq \frac{M_2}{M_1} \]  
(3.1')

hold and $M_1 \geq M_2 > 0$;
(H3) the function $g(x, y) \in C(\mathbb{R}^2, \mathbb{R})$ satisfies
\[ g(x_1, \tilde{x}_1) - g(x_2, \tilde{x}_2) \leq M_1(x_1 - x_2) - M_2(\tilde{x}_1 - \tilde{x}_2) \]
whenever $\alpha(0) \leq x_2 \leq x_1 \leq \beta(0)$, $\alpha(T) \leq \tilde{x}_2 \leq \tilde{x}_1 \leq \beta(T)$, where $M_1 \geq M_2 > 0$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, such that
\[ \lim_{n \to \infty} \alpha_n(t) = x_*(t), \quad \lim_{n \to \infty} \beta_n(t) = x^*(t) \]
uniformly on $J$, and $x_*(t), x^*(t)$ are the minimal and the maximal solutions of problem (1.1), respectively, such that $\alpha(t) \leq x_* \leq x^* \leq \beta(t)$ on $J$.
Proof. For any \( u \in [\alpha, \beta] \), where \([\alpha, \beta] = \{ u \in \Omega \mid \alpha(t) \leq u(t) \leq \beta(t), \ t \in J \}\). Consider the problem (2.6) with \( \sigma_u(t) = (Qu)(t) + Mu(t) + (Eu)(t) \). Noticing that (2.8) holds, then by Lemma 2.3 one may see that problem (2.6) has exactly one solution \( y \in \Omega \). Denote \( y(t) = Au(t) \), then the operator \( A \) has the following properties:

(i) \( \alpha \leq A\alpha, \ A\beta \leq \beta \);

(ii) \( A \) is monotонically nondecreasing in \([\alpha, \beta]\), i.e. for any \( u_1, u_2 \in [\alpha, \beta] \), \( u_1 \leq u_2 \) implies \( Au_1 \leq Au_2 \).

To prove (i), set \( m = \alpha - \alpha_1 \), where \( \alpha_1 = A\alpha \). Owing to (H1), we acquire

\[
m'(t) = \alpha'(t) - \alpha_1'(t)
\]

\[
\leq (Q\alpha)(t) - [-M\alpha_1(t) - (E\alpha_1)(t) + (Q\alpha)(t) + M\alpha(t) + (E\alpha)(t)]
\]

\[
= -Mm(t) - (Em)(t), \quad t \in J,
\]

and

\[
m(0) = \alpha(0) - \left[ -\frac{1}{M_1} g(\alpha(0), \alpha(T)) + \frac{M_2}{M_1} (\alpha_1(T) - \alpha(T)) + \alpha(0) \right]
\]

\[
= -\frac{1}{M_1} g(\alpha(0), \alpha(T)) + \frac{M_2}{M_1} \alpha(T) - \alpha_1(T)
\]

\[
\leq \frac{M_2}{M_1} m(T).
\]

Based on the fact \( M_1 \geq M_2 > 0 \), (3.1) (or (3.1)') and Lemma 2.1, we get \( m(t) \leq 0 \) on \( J \), i.e. \( \alpha \leq A\alpha \). Similar arguments may show that \( A\beta \leq \beta \).

To prove (ii), take \( y_1 = Au_1, \ y_2 = Au_2 \), where \( u_1 \leq u_2 \) on \( J \) and \( u_1, u_2 \in [\alpha, \beta] \). Set \( m = y_1 - y_2 \), employing (H2) and (H3), we achieve

\[
m'(t) = y'_1(t) - y'_2(t)
\]

\[
= [-M_y_1(t) - (E\gamma_1)(t) + \sigma_{u_1}(t)] - [-M_y_2(t) - (E\gamma_2)(t) + \sigma_{u_2}(t)]
\]

\[
= -Mm(t) - (Em)(t) + [(Qu_1)(t) - (Qu_2)(t) + M(u_1(t) - u_2(t)) + (Eu_1 - Eu_2)(t)]
\]

\[
\leq -Mm(t) - (Em)(t), \quad t \in J,
\]

\[
m(0) = y_1(0) - y_2(0)
\]

\[
= -\frac{1}{M_1} g(u_1(0), u_1(T)) + \frac{M_2}{M_1} (y_1(T) - u_1(T)) + u_1(0)
\]

\[
- \left[ -\frac{1}{M_1} g(u_2(0), u_2(T)) + \frac{M_2}{M_1} (y_2(T) - u_2(T)) + u_2(0) \right]
\]

\[
= \frac{M_2}{M_1} [y_1(T) - y_2(T)] + \frac{1}{M_1} [g(u_2(0), u_2(T)) - g(u_1(0), u_1(T))]
\]

\[
- (u_2(0) - u_1(0)) + \frac{M_2}{M_1} [u_2(T) - u_1(T)]
\]

\[
\leq \frac{M_2}{M_1} m(T).
\]

By virtue of Lemma 2.1, one has \( m(t) \leq 0 \) on \( J \), i.e. \( Au_1 \leq Au_2 \). From (i) and (ii), one knows that \( \alpha \leq A\alpha \leq \alpha_1 \leq A\beta \leq \beta \).

Now, define the sequences \( \{\alpha_n(t)\}, \ {\beta_n(t)\} \) with \( \alpha_0 = \alpha, \ \beta_0 = \beta \) such that \( \alpha_{n+1} = A\alpha_n, \ \beta_{n+1} = A\beta_n, \ n = 1, 2, \ldots \). Due to (i) and (ii), one achieves\n
\[
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0 \quad \text{on} \ J.
\]

Apparently, \( \alpha_n, \ \beta_n (n = 1, 2, \ldots) \) satisfy

\[
\begin{cases}
\alpha'_n(t) = -M\alpha_n - (E\alpha_n)(t) + (Q\alpha_n)(t) + M\alpha_n(t) + (E\alpha_n)(t), \\
g(\alpha_{n-1}(0), \alpha_{n-1}(T)) + M_1(\alpha_n(0) - \alpha_n(0)) - M_2(\alpha_n(T) - \alpha_{n-1}(T)) = 0
\end{cases}
\]

and

\[
\begin{cases}
\beta'_n(t) = -M\beta_n - (E\beta_n)(t) + (Q\beta_n)(t) + M\beta_n(t) + (E\beta_n)(t), \\
g(\beta_{n-1}(0), \beta_{n-1}(T)) + M_1(\beta_n(0) - \beta_n(0)) - M_2(\beta_n(T) - \beta_{n-1}(T)) = 0
\end{cases}
\]
Consequently, there exist $x_*$ and $x^*$ such that $\lim_{n \to \infty} \alpha_n(t) = x_*(t)$, $\lim_{n \to \infty} \beta_n(t) = x^*(t)$ uniformly on $J$. Apparently, $x_*$, $x^*$ are the solutions of problem (1.1).

To show $x_*$, $x^*$ are extremal solutions of problem (1.1), let $x(t)$ be any solution of (1.1) such that $\alpha(t) \leq x(t) \leq \beta(t)$. Assume that there exists a positive integer $n$ such that $\alpha_n(t) \leq x(t) \leq \beta_n(t)$ on $J$. Based on the monotonically nondecreasing property of $A$, it then follows that $\alpha_{n+1} = A\alpha_n \leq Ax = x$, i.e. $\alpha_{n+1}(t) \leq x(t)$ on $J$. Similarly, one derives $x(t) \leq \beta_{n+1}(t)$ on $J$. Since $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ on $J$, by induction we see that $\alpha_n(t) \leq x(t) \leq \beta_n(t)$ on $J$ for every $n$. Therefore $x_*(t) \leq x(t) \leq x^*(t)$ on $J$ as $n \to \infty$. The proof is then finished. □

4. Weakly coupled lower and upper solutions

**Definition 4.1.** Functions $\alpha, \beta \in \Omega$ are called weakly coupled lower and upper solutions of problem (1.1) if

\[ \alpha'(t) \leq (Q\alpha)(t), \quad g(\beta(0), \alpha(T)) \leq 0 \]

and

\[ \beta'(t) \geq (Q\beta)(t), \quad g(\alpha(0), \beta(T)) \geq 0. \]

**Definition 4.2.** A pair $(U, V)$, $U, V \in \Omega$, is called a weakly coupled quasi-solution of problem (1.1) if for $t \in J$, there are

\[ U(t) = (QU)(t), \quad g(V(0), U(T)) = 0 \]

and

\[ V(t) = (QV)(t), \quad g(U(0), V(T)) = 0. \]

**Definition 4.3.** A weakly coupled quasi-solution $(\rho, \gamma)$, $\rho, \gamma \in \Omega$, is called weakly coupled minimal and maximal quasi-solution of problem (1.1), if for any weakly coupled quasi-solution $(U, V)$ of (1.1), we have $\rho(t) \leq U(t)$, $V(t) \leq \gamma(t)$ on $J$.

**Theorem 4.1.** Assume condition (H2) and (2.8) hold, let $Q \in C([E, E])$. In addition, suppose that

(H4) $\alpha, \beta \in \Omega$ are weakly coupled lower and upper solutions such that $\alpha(t) \leq \beta(t)$;

(H5) the function $g(x, y) \in C([\Omega^2, \mathbb{R}]$ is nondecreasing in the first variable and

\[ g(x, y_1) - g(x, y_2) \leq -M_2(y_1 - y_2) \quad \text{if} \quad \alpha(T) \leq y_2 \leq y_1 \leq \beta(T), \]

where $M_1 \geq M_2 > 0$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, such that

\[ \lim_{n \to \infty} \alpha_n(t) = x_*(t), \quad \lim_{n \to \infty} \beta_n(t) = x^*(t) \]

uniformly on $J$, and $(x_*(t), x^*(t))$ is the weakly coupled minimal and maximal quasi-solution of problem (1.1), respectively, such that $\alpha(t) \leq x_* \leq x^* \leq \beta(t)$ on $J$.

**Proof.** Let

\[
\begin{align*}
\alpha_n'(t) &= -M\alpha_n - (\xi\alpha_n)(t) + (Q\alpha_{n-1})(t) + M\alpha_{n-1}(t) + (\xi\alpha_{n-1})(t), \\
g(\beta_{n-1}(0), \alpha_{n-1}(T)) + M_1(\alpha_n(0) - \alpha_{n-1}(0)) - M_2(\alpha_n(T) - \alpha_{n-1}(T)) &= 0
\end{align*}
\]

and

\[
\begin{align*}
\beta_n'(t) &= -M\beta_n - (\xi\beta_n)(t) + (Q\beta_{n-1})(t) + M\beta_{n-1}(t) + (\xi\beta_{n-1})(t), \\
g(\alpha_{n-1}(0), \beta_{n-1}(T)) + M_1(\beta_n(0) - \beta_{n-1}(0)) - M_2(\beta_n(T) - \beta_{n-1}(T)) &= 0
\end{align*}
\]

for $n = 1, 2, \ldots$, where $\alpha_0 = \alpha$, $\beta_0 = \beta$.

First, we show that $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$. 
Set $m = \alpha_0 - \alpha_1$, then owe to (H4), one arrives at
\[
m'(t) = \alpha'_0(t) - \alpha'_1(t)
\leq Q\alpha_0(t) - \left[ -M\alpha_1(t) - (\varepsilon \alpha_1)(t) + (Q\alpha_0)(t) + M\alpha_0(t) + (\varepsilon \alpha_0)(t) \right]
\leq -Mm(t) - (\varepsilon m)(t), \quad t \in J,
\]
\[
m(0) = \alpha_0(0) - \alpha_1(0)
\]
\[
= \alpha_0(0) - \left[ -\frac{1}{M_1}g(\beta_0(0), \alpha_0(T)) + \frac{M_2}{M_1}(\alpha_1(T) - \alpha_0(T)) + \alpha_0(0) \right]
\]
\[
= \frac{1}{M_1}g(\beta_0(0), \alpha_0(T)) + \frac{M_2}{M_1}(\alpha_0(T) - \alpha_1(T))
\]
\[
\leq \frac{M_2}{M_1}m(T).
\]

It then follows from (3.1) (or (3.1)') and Lemma 2.1 that $m(t) \leq 0$, that is, $\alpha_0(t) \leq \alpha_1(t)$ for $t \in J$. Similarly, we may obtain $\beta_1(t) \leq \beta_0(t)$ on $J$.

Next, take $m = \alpha_1 - \beta_1$, by way of (H2), one attains
\[
m'(t) = \alpha'_1(t) - \beta'_1(t)
\]
\[
= -M\alpha_1(t) - (\varepsilon \alpha_1)(t) + (Q\alpha_0)(t) + M\alpha_0(t) + (\varepsilon \alpha_0)(t)
\]
\[
- \left[ -M\beta_1(t) - (\varepsilon \beta_1)(t) + (Q\beta_0)(t) + M\beta_0(t) + (\varepsilon \beta_0)(t) \right]
\leq -Mm(t) - (\varepsilon m)(t), \quad t \in J,
\]
noticing $\alpha_0 \leq \beta_0$ and (H5), we derive
\[
m(0) = \alpha_1(0) - \beta_1(0)
\]
\[
= -\frac{1}{M_1}g(\beta_0(0), \alpha_0(T)) + \frac{M_2}{M_1}(\alpha_1(T) - \alpha_0(T)) + \alpha_0(0)
\]
\[
- \left[ -\frac{1}{M_1}g(\alpha_0(0), \beta_0(T)) + \frac{M_2}{M_1}(\beta_1(T) - \beta_0(T)) + \beta_0(0) \right]
\]
\[
\leq \frac{1}{M_1}[g(\alpha_0(0), \beta_0(T)) - g(\beta_0(0), \alpha_0(T))] + \frac{M_2}{M_1}[\alpha_1(T) - \beta_1(T) - (\alpha_0(T) - \beta_0(T))]
\]
\[
= \frac{1}{M_1}[g(\alpha_0(0), \beta_0(T)) - g(\alpha_0(0), \alpha_0(T))] + \frac{M_2}{M_1}[\beta_0(T) - \alpha_0(T)]
\]
\[
+ \frac{1}{M_1}[g(\alpha_0(0), \alpha_0(T)) - g(\beta_0(0), \alpha_0(T))] + \frac{M_2}{M_1}[\alpha_1(T) - \beta_1(T)]
\]
\[
\leq \frac{M_2}{M_1}m(T).
\]

(3.1) (or (3.1)') and Lemma 2.1 then yield $m(t) \leq 0$ on $J$, i.e., $\alpha_1 \leq \beta_1$.

In the following, we shall show that $\alpha_1$, $\beta_1$ are the weakly coupled lower and upper solutions of (1.1). Following (H2) and $\alpha_0 \leq \alpha_1$, we achieve
\[
\alpha'_1(t) = (Q\alpha_1)(t) - (Q\alpha_1)(t) - M\alpha_1(t) - (\varepsilon \alpha_1)(t) + (Q\alpha_0)(t) + M\alpha_0(t) + (\varepsilon \alpha_0)(t)
\]
\[
= (Q\alpha_1)(t) + ((Q\alpha_0)(t) - (Q\alpha_1)(t) + M(\alpha_0(t) - \alpha_1(t)) + (\varepsilon (\alpha_0 - \alpha_1))(t))
\]
\[
\leq (Q\alpha_1)(t), \quad (4.3)
\]
and by means of the fact $\alpha_0 \leq \alpha_1$, $\beta_1 \leq \beta_0$, (H4) and (H5), one reaches
\[
g(\beta_1(0), \alpha_1(T)) = g(\beta_1(0), \alpha_1(T)) - g(\beta_0(0), \alpha_0(T)) - M_1(\alpha_1(0) - \alpha_0(0)) + M_2(\alpha_1(T) - \alpha_0(T))
\]
\[
\leq g(\beta_1(0), \alpha_1(T)) - g(\beta_1(0), \alpha_0(T)) + M_2(\alpha_1(T) - \alpha_0(T))
\]
\[
+ g(\beta_1(0), \alpha_0(T)) - g(\beta_0(0), \alpha_0(T))
\]
\[
\leq 0. \quad (4.4)
\]
Similarly, we may get
\[
\beta_1'(t) \geq (Q\beta_1)(t), \quad g(\alpha_1(0), \beta_1(T)) \geq 0. \tag{4.5}
\]
(4.3)–(4.5) imply that \(\alpha_1, \beta_1\) are weakly coupled lower and upper solutions of (1.1). Employing the mathematical induction, one may show that there exist \(x_*, x^*\) such that \(\lim_{t \to \infty} \alpha_n(t) = x_*(t), \lim_{t \to \infty} \beta_n(t) = x^*(t)\) uniformly on \(J\). Obviously, \(x_*, x^*\) are fulfilling that
\[
x_1'(t) = (Qx_1(t), \quad g(x^*, x_0) = 0,
\]
\[
x_n'(t) = (Qx_n(t), \quad g(x^*, x^*) = 0.
\]
This means that the pair \((x_*, x^*)\) is a weakly coupled quasi-solution of problem (1.1).

It remains to demonstrate \(x_*, x^*\) are the weakly coupled minimal and maximal quasi-solutions of problem (1.1). Let \((u, v)\) be any weakly coupled quasi-solutions of (1.1) such that \(u, v \in [\alpha, \beta]\). Suppose there exist a positive integer \(n\) such that \(\alpha_n(t) \leq u, v \leq \beta_n(t)\), for \(t \in J\).

Putting \(m(t) = \alpha_{n+1}(t) - u(t)\), then for \(t \in J\) and utilizing \(\alpha_n \leq u\) and \((H_2)\), one arrives at
\[
m'(t) = \alpha_{n+1}'(t) - u'(t) = -M\alpha_{n+1}(t) - (Q\alpha_{n+1})(t) + M\alpha_n(t) + (Q\alpha_n)(t) - Q(t)\]
\[
= (Q\alpha_n(t) - (Qu)(t) + M(\alpha_n(t) - u(t)) + (Q(\alpha_n - u)(t))\]
\[
= -M\alpha_{n+1}(t) - (Q\alpha_{n+1})(t) + M\alpha_n(t) + (Q(\alpha_n - u)(t))\]
\[
\leq -Mm(t) - (Qu)(t), \quad t \in J,
\]
and using the fact \(g(u(0), u(T)) = 0\) and \((H_3)\), we have
\[
m(0) = \alpha_{n+1}(0) - u(0) \leq \frac{1}{M_1} g(\beta_n(0), \alpha_n(T)) + \frac{M_2}{M_1} (\alpha_{n+1}(T) - \alpha_n(T)) + M_2\alpha_n(0) - u(0)
\]
\[
\leq \frac{1}{M_1} g(\beta_n(0), \alpha_n(T)) + \frac{1}{M_1} g(u(0), u(T)) + \frac{M_2}{M_1} (\alpha_{n+1}(T) - \alpha_n(T))
\]
\[
= \frac{1}{M_1} [g(u(0), \alpha_n(T)) - g(\beta_n(0), \alpha_n(T))] - \frac{1}{M_1} [g(u(0), \alpha_n(T)) - g(u(0), u(T))]
\]
\[
+ \frac{M_2}{M_1} (\alpha_{n+1}(T) - \alpha_n(T))
\]
\[
\leq -\frac{M_2}{M_1} (u(T) - \alpha_n(T)) + \frac{M_2}{M_1} (\alpha_{n+1}(T) - \alpha_n(T))
\]
\[
\leq \frac{M_2}{M_1} m(T).
\]

(3.1) (or \(3.1')\) and Lemma 2.1 then leads to \(m(t) \leq 0\) for \(t \in J\), i.e., \(\alpha_{n+1} \leq u\). Similarly, we obtain \(v \leq \beta_{n+1}\). Following the induction, one has \(\alpha_n(t) \leq u(t), v(t) \leq \beta_n(t)\) for all \(t \in J\) and any \(n\), which implies \(x_* \leq u, v \leq x^*\). This completes the proof. \(\Box\)

References