A Novel Approach to Determination of A Transition Function with Repeated Eigenvalues

Constant Matrix

Kao-Shing Hwang  Min-Cheng Tsai  Feng-Cheng Chang
Electrical Engineering Department
National Chung Cheng University
Chia-Yi, Taiwan

Abstract
An analytical function of a matrix with repeated eigenvalues is expressed in terms of constituent matrices. Two approaches to computing the constituent matrices are then presented. For a special case of a companion matrix, the computation can be greatly simplified.

keywords: The generalized Vandermonde matrix; The partial fraction expansion; Constant matrix; Characteristic polynomial; Longhand division;

1.FORMULATION

For a given \( n \times n \) constant matrix \( A \), eigenvalues are determined by solving the characteristic polynomial of degree \( n \),

\[
\det(sI-A) = \sum_{p=0}^{n} a_p s^{n-p} = \prod_{k=1}^{m} (s - \lambda_k)^{r_k}, \quad \sum_{k=1}^{m} r_k = n
\]

That is, the matrix \( A \) has \( m \) eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_m \) with multiplicities \( r_1, r_2, \ldots, r_m \), respectively. Let \( f(s) \) be an analytical function of \( s \), than any analytical matrix function of \( A \), \( f(A) \) may be written as the expression of the \( n \)-term sum [1]:

\[
f(A) = \sum_{k=1}^{m} \sum_{h=0}^{r_k-1} \frac{f^{(h)}(\lambda_k)}{h!} Z_{kh},
\]

where matrices \( Z_{kh}, \ k=1, 2, \ldots, m, \ h=0,1,\ldots, r_k-1, \) are called constituent matrices [2]. The main objective of this article is to derive the most efficient technique to compute \( Z_{kh} \) from a given constant matrix \( A \). In general the computation is very much involved, especially for large value of \( n \) with high multiplicity.

In a usual approach [2], the evaluation of \( Z_{kh} \) may be made by successively
setting in (2) \( f(A) = A^\ell \), \( \ell = 0, 1, \ldots, n-1 \). The results are
\[
A^\ell = \sum_{k=1}^{m} \sum_{h=0}^{\gamma_k - 1} (V_k)_h Z_{kh}, \quad \ell = 0, 1, \ldots, n-1,
\]
(3)

Where coefficients
\[
(V_k)_j = \binom{j}{k} x_k^{-j}, \quad k = 1, 2, \ldots, m, \quad i = 0, 1, \ldots, n-1, \quad j = 0, 1, \ldots, \gamma_k - 1,
\]
(4)
are the elements of the generalized Vandermonde matrix \( V \). The matrices \( Z_{kh} \) are then obtained by solving the \( n \) simultaneous linear equations (3),
\[
Z_{kh} = \sum_{l=0}^{n-1} (V_k^{-1})_{hl} A^l, \quad k = 1, 2, \ldots, m, \quad h = 0, 1, \ldots, \gamma_k - 1,
\]
(5)
where \( (V_k^{-1})_{ij} \), the elements of the inverse Vandermonde matrix \( V^{-1} \), can be found by successively setting \( i = \gamma_k - 1, \ldots, 1 \), into (3):
\[
(V_k^{-1})_{ij} = \frac{1}{(dk)_0}[ (W_k)_{i-1, j} - \sum_{x=i+1}^{\gamma_k} (d_k)_{x-i} (V_k^{-1})_{ij} ],
\]
(6)
with
\[
(W_k)_{ij} = \sum_{p=0}^{n-1-i-j} (n-1-j-p) a_p x_k^{n-1-i-j-p},
\]
(7)
and
\[
(d_k)_{ij} = \sum_{p=0}^{n-1-i-j} (a_p x_k^{n-1-i-j-p},
\]
(8)
and \( (W_k)_{ij} \) are the elements of the modal matrix \( W \).

It is understood that the summation in (6) is zero if the lower limit is greater than the upper limit.

When the order \( n \) is large, the evaluation of the scalar coefficients \( a_p \) in (1) is generally very involved. Because the expansion of an \( nxn \) determinant \( \det(sI - A) \) into a polynomial (of degree \( n \)) is not an easy task. Frame and others[2] have proposed to compute \( a_p \) by the following recurrent formulas:
\[
a_p = -\frac{1}{p} tr(AB_{p-1}), \quad a_0 = 1,
\]
(9)
and
\[
B_p = AB_{p-1} + a_p I, \quad B_0 = I, \quad B_n = 0,
\]
(10)
where \( p = 1, 2, \ldots, n \). It is noted here that \( B_p \) are matrix coefficients of the adjoin
matrix

\[ \text{adj}(sI - A) = \sum_{p=0}^{n-1} B_p s^{n-1-p}. \]

Let \( f(A) = e^{At} \) in (2),

\[ e^{At} = \sum_{k=1}^{m} \sum_{h=0}^{r_k-1} e^{\lambda_k t} \frac{t^h}{h!} Z_{kh}. \]  

(12)

Applying Laplace transform to both sides of (12) and noting

\[ L[e^{At}] = [sI - A]^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} \]

yields

\[ \sum_{p=0}^{n-1} B_p s^{n-1-p} = \sum_{k=1}^{m} \sum_{h=0}^{r_k-1} Z_{kh} (s - \lambda_k)^{h+1}, \]

(13)

Now letting

\[ \frac{s^1}{\sum_{j=0}^{n} a_j s^{n-p}} = \sum_{k=1}^{m} \sum_{h=0}^{r_k-1} (W_k^{-1})_{hl} (s - \lambda_k)^{h+1}, \]

(14)

and comparing it with (13) leads to

\[ Z_{kh} = \sum_{j=0}^{n} (W_k^{-1})_{hl} B_{n-1-I}, \quad k=1,2,\ldots,m, \quad h=0,1,\ldots,r_k-1 \]

(15)

where \((W_k^{-1})_{ij}\), the elements of the inverse modal matrix \(W^{-1}\), relating the coefficients of the numerator of a proper rational function to the coefficients of the partial fraction expansion of the function, can be found by successively setting \(i = r_k - 1, \ldots, 1, 0\), into [4]:

\[ (W_k^{-1})_{ij} = \frac{1}{(d_k^i)_{j0}} [(V_k)_{j0} - \sum_{p=i+1}^{r_k-1} (d_k^i)_{j0}(W_k^{-1})_{pj}], \]

(16)

where the value of \((V_k)_{ij}\) and \((d_k^i)_{j0}\) are given by (4) and (8).

Obviously both matrices \(V^{-1}\) and \(W^{-1}\) are uniquely determined for a given characteristic polynomial of matrix \(A\), and are independent of any analytical function \(f(s)\). As comparable equal effort is needed to calculate either \((V_k)_{ij}\) or \((W_k^{-1})_{ij}\), the computation of \(Z_{kh}\) is certainly more efficient with the use of (15) than with the use of (5) for large \(n\), since with the use of (15), \(B_0, B_1, \ldots, B_{n-1}\) are obtained at the same
time when \(a_0, a_1, \ldots, a_n\) are computed, while with the use of (5), additional effort is needed to compute \(A^2, A^3, \ldots, A^{n-1}\). Furthermore, the entire elements of \(V, W, V^{-1}, W^{-1}\) and \(d\) may be easily computed by employing repeated synthetic divisions and longhand divisions[3]. The technique is especially useful for hand calculation.

For a special case, if the given matrix \(A\) is a companion matrix

\[
A = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
& & \cdots & \cdots \\
& & & -a_n - a_{n-1} \cdots - a_1
\end{bmatrix}
\]  \tag{17}

then the characteristic polynomial may be simply given without any calculation

\[
det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \tag{18}
\]

The constituent matrices can then be determined by either

\[
Z_{kh} = \begin{bmatrix}
(V_k)_0 & 0 \cdots & (V_k)_{n-1-k} \\
\vdots & \ddots & \vdots \\
(V_k)_{n-10} \cdots & (V_k)_{n-1-r_k-1-k}
\end{bmatrix}
\begin{bmatrix}
(V_k^{-1})_0 & 0 \cdots & (V_k^{-1})_{n-1} \\
\vdots & \ddots & \vdots \\
(V_k^{-1})_{n-1-1} \cdots & (V_k^{-1})_{n-1-r_k-1-k}
\end{bmatrix}
\]  \tag{19}

or

\[
Z_{kh} = \begin{bmatrix}
(W_k)_0 & 0 \cdots & (W_k)_{n-1-k} \\
\vdots & \ddots & \vdots \\
(W_k)_{n-10} \cdots & (W_k)_{n-1-r_k-1-k}
\end{bmatrix}
\begin{bmatrix}
(W_k^{-1})_0 & 0 \cdots & (W_k^{-1})_{n-1} \\
\vdots & \ddots & \vdots \\
(W_k^{-1})_{n-1-1} \cdots & (W_k^{-1})_{n-1-r_k-1-k}
\end{bmatrix}
\]  \tag{20}

\[k=1,2,\ldots,m, \quad h=0,1,\ldots,r_k-1,
\]

where the matrix elements are from either Vandermonde matrix or modal matrix with its respective inverse.

2.EXAMPLE

Evaluate a transition matrix \(\exp(At)\), if a given constant matrix is given by

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
24 & -44 & -30 & -9
\end{bmatrix}
\]

The given matrix \(A\) is a companion matrix, however, we would assume it is an
arbitrary constant matrix. Therefore applying the recurrent formulas (9) and (10) the scalars \( a_0, a_1, \ldots \) and the matrices \( B_0, B_1, \ldots \) can be successively determined:

Also the integer power of \( A \) are computed for later use:

\[
A^2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-24 & -44 & 30 & -9 \\
216 & 372 & 226 & 51
\end{bmatrix},
\]

\[
a_1 = 9, \quad 0B_1 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
-24 & -44 & 30 & -9 \\
216 & 372 & 226 & 51
\end{bmatrix}
\]

\[
A^3 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-24 & -44 & 30 & -9 \\
216 & 372 & 226 & 51
\end{bmatrix},
\]

\[
a_2 = 9, \quad B_2 = \begin{bmatrix}
0 & 0 & 0 & 9 \\
-24 & -44 & 30 & 0 \\
0 & 0 & 0 & 0 \\
44 & 30 & 9 & 1
\end{bmatrix}
\]

\[
a_3 = 44, \quad B_3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-24 & 0 & 0 & 0 \\
0 & -24 & 0 & 0 \\
0 & 0 & -24 & 0
\end{bmatrix}
\]

\[
a_4 = 24, \quad B_4 = 0
\]

The characteristic polynomial of \( A \) is thus

\[
det(sI - A) = s^4 + 9s^3 + 30s^2 + 44s + 24 = (s + 2)^3 (s + 3)
\]

from which eigenvalues and multiplicity are found:

\[
\lambda_1 = -2, \quad \gamma_1 = 3
\]

\[
\lambda_2 = -3, \quad \gamma_2 = 1
\]

From the fundamental formula (12) we may express the transition matrix as

\[
\exp(At) = e^{-2t}(Z_{00} + tZ_{01} + \frac{1}{2} t^2 Z_{02}) + e^{-3t}Z_{10}
\]

where the constituent matrix \( Z_{00}, Z_{01}, Z_{02} \) and \( Z_{10} \) are to be determined by either (5) or (15), whenever the inverse matrices \( V^{-1} \) or \( W^{-1} \) is evaluated.

The matrices \( V, W \) and \( d \) can be obtained simply by the procedural of the
repeated synthetic divisions depicted in Appendix (a).

So, by means of the procedural, the $V$, $W$ and $d$ can be evaluated as:

$$
V = \begin{bmatrix}
1 & 0 & 0 & | & 1 \\
-2 & 1 & 0 & | & -3 \\
4 & -4 & 1 & | & 9 \\
-8 & 12 & -6 & | & -27
\end{bmatrix}, \quad W = \begin{bmatrix}
12 & 6 & 3 & | & 8 \\
16 & 5 & 1 & | & 12 \\
7 & 1 & 0 & | & 6 \\
1 & 0 & 0 & | & 1
\end{bmatrix}, \quad d = \begin{bmatrix}
1 & 1 & 0 & | & -1
\end{bmatrix}
$$

The inverse matrices $V^{-1}$ and $W^{-1}$ are then obtained from the longhand division procedural, where is also described in Appendix (b).

Once the $V^{-1}$ and $W^{-1}$ are obtain as:

$$
V^{-1} = \begin{bmatrix}
9 & 12 & 6 & 1 \\
-6 & -11 & -6 & -1 \\
12 & 16 & 7 & 1 \\
-8 & -12 & -6 & -1
\end{bmatrix}, \quad W^{-1} = \begin{bmatrix}
1 & -3 & 9 & -26 \\
-1 & 3 & -8 & 20 \\
1 & -2 & 4 & -8 \\
-1 & 3 & -9 & 27
\end{bmatrix}
$$

The desired constituent matrices are therefore computed either by (5):

\[
\begin{align*}
Z_{00} &= 9I + 12A + 6A^2 + A^3 \\
Z_{01} &= -6I - 11A - 6A^2 - A^3 \\
Z_{02} &= 12I + 16A + 7A^2 + A^3 \\
Z_{10} &= -8I - 12A - 6A^2 - A^3
\end{align*}
\]

or alternatively by (15),

\[
\begin{align*}
Z_{00} &= B_3 - 3B_2 + 9B_1 - 26B_0 \\
Z_{01} &= -B_3 + 3B_2 - 8B_1 + 20B_0 \\
Z_{02} &= B_3 - 2B_2 + 4B_1 - 8B_0 \\
Z_{10} &= -B_3 + 3B_2 - 9B_1 + 27B_0
\end{align*}
\]

Since the given matrix $A$ is a form of companion matrix, the constituent matrices may also be easily obtained either by (19),
No matter what approaches we use, the calculated results are the

\[
Z_{00} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -4 & 1 & 0 \\ -8 & 12 & -6 & 0 \end{bmatrix} \quad Z_{01} = \begin{bmatrix} -6 & -11 & -6 & -1 \\ 9 & 12 & 6 & 1 \\ -6 & -11 & -6 & -1 \\ 12 & 16 & 7 & 1 \end{bmatrix} \quad Z_{02} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}
\]

\[
Z_{10} = \begin{bmatrix} -8 & -12 & -6 & -1 \\ -2 & 1 & 0 & 0 \\ 4 & -4 & 1 & 0 \\ -8 & 12 & -6 & 0 \end{bmatrix} \quad Z_{10} = \begin{bmatrix} 1 \\ -3 \\ 9 \\ -27 \end{bmatrix}
\]

or by (20),

\[
Z_{00}^\tau = \begin{bmatrix} 12 & 6 & 3 \\ 16 & 5 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad Z_{02}^\tau = \begin{bmatrix} 12 \\ 16 \\ 7 \\ 1 \end{bmatrix}
\]

\[
Z_{01}^\tau = \begin{bmatrix} 12 & 6 \\ 16 & 5 \\ 7 & 1 \\ 1 & 0 \end{bmatrix} \quad Z_{10}^\tau = \begin{bmatrix} 8 \\ 12 \\ 6 \\ 1 \end{bmatrix}
\]

This concludes the determination of the transition matrix of a given matrix.
CONCLUSIONS

A transition matrix with multiple roots is always observed in physical systems. On the other hand, an analytic matrix function of an $n \times n$ arbitrary matrix with repeated eigenvalue can be expressed in terms of constituent idempotent matrices. However, the calculation of constituent matrices is generally very involved, especially for a large $n$. The proposed approaches can be easily programmed in computer computation with less computational complexity than conventional approaches.

REFERENCES


APPENDIX

(a) Algorithm of repeated synthetic divisions for $W$ and $d$
/* matrix index, (row, column), starting from (0,0) */
/* n = the order of B(s) */
/* r[j] = the order of the j-th factor of B(s) */
/* s[j] = the j-th root of B(s) */
/* m = the number of factor of B(s) */
/* b[n] = the parameters of B[s] */

PROCEDURE SYNdivision(VAR m, n: INTEGER; VAR b, d, r, s: VECTOR; VAR W: MATRIX)
VAR i, j, index, step, column, row: INTEGER;
VAR b_d: VECTOR;
BEGIN
  column := 1;
  FOR j := 1 TO m BY +1 DO
    FOR i := 1 TO n BY +1 DO
      b_d[i] := b[i];
    END;
    step := n;
    FOR order := 1 TO r[j] BY +1 DO
      row := order + 1;
      FOR index := 1 TO step BY +1 DO
        b_d[index + 1] := s[j] * b_d[index] + b_d[index + 1];
      END;
      column++;
      step--;
    END;
    d[column] := b_d[index];
    column++;
  END;
END SYNdivision;

(b) Algorithm of longhand division for $V_1$:

PROCEDURE LNGdivision(VAR r, d: VECTOR; VAR W: MATRIX)
VAR i, column, row, order, index, level, step: INTEGER;
VAR W: MATRIX;
BEGIN
  column := 1;
  FOR i := 1 TO m BY +1 DO
    FOR index := 1 TO r[i] BY +1 DO
      W[r[i] - order - 1][index] := W[index][column + order] / d[column];
      FOR level := order TO r[i] BY +1 DO
        W[index][column + level] := W[index][column + level - 1] - W[index - order - 1][index] * d[column + level - index];
      END;
      step--;
    END;
  END;
  column := r[i] + column;
END LNGdivision;