A two-stage dynamic group decision making method for processing ordinal information

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Article history:
Received 19 October 2013
Received in revised form 24 June 2014
Accepted 25 June 2014
Available online 4 July 2014

Keywords:
Power Average operator
Support function
Dominance-based rough set approach
Ordinal preference
Group decision making

A B S T R A C T

In group decision making (GDM) problems, ordinal data provide a convenient way of articulating preferences from decision makers (DMs). A number of GDM models have been proposed to aggregate such kind of preferences in the literature. However, most of the GDM models that handle ordinal preferences suffer from two drawbacks: (1) it is difficult for the GDM models to manage conflicting opinions, especially with a large number of DMs; and (2) the relationships between the preferences provided by the DMs are neglected, and all DMs are assumed to be of equal importance, therefore causing the aggregated collective preference not an ideal representative of the group’s decision. In order to overcome these problems, a two-stage dynamic group decision making method for aggregating ordinal preferences is proposed in this paper. The method consists of two main processes: (i) a data cleansing process, which aims to reduce the influence of conflicting opinions pertaining to the collective decision prior to the aggregation process; as such an effective solution for undertaking large-scale GDM problems is formulated; and (ii) a support degree oriented consensus-reaching process, where the collective preference is aggregated by using the Power Average (PA) operator; as such, the relationships of the arguments being aggregated are taken into consideration (i.e., allowing the values being aggregated to support each other). A new support function for the PA operator to deal with ordinal information is defined based on the dominance-based rough set approach. The proposed GDM model is compared with the models presented by Herrera-Viedma et al. An application related to controlling the degradation of the hydrographic basin of a river in Brazil is evaluated. The results demonstrate the usefulness of the proposed method in handling GDM problems with ordinal information.

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1. Introduction

Ordinal information is commonly used to rank the criteria or alternatives in group decision making (GDM) problems [22,48,50,53]. As an example, when a customer is asked to compare different flavors of cakes, it is natural for him/her to give an ordinal preference. Unlike the numerical-based measure, ordinal information is unable to specify the degree of importance of the criteria or alternatives exactly. It only provides the order information pertaining to the criteria or alternatives. This leads to a number of difficulties in aggregating ordinal information presented by a group of Decision Makers (DMs). In this aspect, a lot of aggregation methods based on ordinal evaluation of individual rankings have been proposed in the literature for achieving a collective group preference. Essentially, the aggregation models for tackling GDM problems with ordinal information can be divided into four categories:

1. majority-based models [6,16,35], which produce the collective preference by using the majority-based approach, e.g., the simple majority rule, the Borda method, the majorities based on differences of votes, and their variants;
2. ranking weight-based models [1,2,25], which aggregate individual ordinal preferences by converting ordinal ranking into numerical-valued weight information;
3. distance-based models [17,26], which generate the collective preference by minimizing the total aggregated disagreement between each individual and the final ranking;
A good review is provided by Cook [15]. Despite the extensive research on GDM problems with ordinal information, there are some weaknesses associated with the available methods. One of them is the necessity of managing conflicting opinions in a large group. In large-scale decision making problems, conflicting opinions are inevitable owing to differences among the preferences provided by the DMs. Such conflicting opinions cannot result in a final consensus. Sometimes, conflicting opinions can lead to disagreement after lengthy discussion [30], because some DMs do not want to alter their initial opinions. A number of studies have been conducted to deal with possible conflicts by discarding the preferences of DMs who do not contribute towards achieving a consensus [36], or penalizing them by reducing their influences on the final collective group decision [51].

In order to overcome the aforementioned shortcomings, a data cleansing process, which aims to eliminate conflicting preferences among the DMs prior to the aggregation process, is employed in this paper. The proposed data cleansing process is motivated by the idea of soft consensus, which can be achieved when most of the participating DMs agree on the most important alternatives. Soft consensus was firstly proposed by Kacprzyk [27]. Due to its ability to guide the consensus process in a flexible way until an agreement (not necessarily a full agreement) is achieved among the DMs, soft consensus-based methods have been widely used in various GDM problems with satisfactory results [7,24,28]. The basic idea of soft consensus-based methods is that it allows a group to obtain a limited agreement among the DMs, which provides the foundation for integrating the data cleansing process into the GDM model. In other word, the limited agreement among the DMs can still be achieved by eliminating conflicting opinions from the DMs which do not contribute towards achieving a consensus.

Another challenge is that most aggregation operators in the existing GDM models that handle ordinal preferences usually neglect the relationship (agreement or disagreement) among the DMs. A variety of aggregation operators, which include the weighted average operator, ordered weighted average (OWA) operator [49], numerical weighting linguistic average operator [43], induced ordered weighted averaging operators [9], type-1 OWA operators [14,29,56,57] and interval valued operators [54], have been developed. However, all these aggregation operators are unable to validate the association between two or more DMs. This is because their aggregation is based on prior information gathered beforehand, without measuring the degree of support among the DMs before reaching a final consensus. To overcome this problem, the Power Average (PA) operator introduced by Yager [52] is employed in this paper. With the aid of the PA operator, we are able to allow the values being aggregated to support and reinforce each other; therefore taking the relationships among the arguments into consideration in the aggregation process. One notable property of the PA operator is that it captures both the features of mode-like methods to find the most typical value, and the averaging-type operator to aggregate the data. The PA operator has since been widely used in many multi-criteria decision making and evaluation problems [33,44]. Recently, Zhou and Chen [55] extended the PA operator to a linguistic environment by combining it with the generalized mean operator, and applied it to a multi-attribute GDM problem. Xu and Cai [46] defined an uncertain power weighted average operator and an uncertain power ordered weighted average operator, and proposed a method for GDM using interval fuzzy preference relations.

However, most aggregation algorithms based on the PA operators treat decision making as a static event, namely, the collective opinions aggregated by using the PA operator are directly regarded as the group’s final decision. This is impractical because a simplistic aggregation process would render the final decision invalid when conflicting opinions that differ widely are forced to conform to a full consensus. As a result, there is a need to embed a revision mechanism in the aggregation process, which allows the group to reach a satisfactory decision. Therefore, we propose an algorithm with an iterative mechanism to assist the DMs to revise their opinions in order to reach a high degree of group consensus in this paper. A number of GDM models with revision (feedback) mechanisms have been proposed in the literature. Some examples are the models proposed by Herrera-Viedma et al. [21,22,32]. These models focus on identification of the preferences that need to be revised, and then provide the corresponding directions of changes. In order to indicate the change rules clearly, other models with advice generation have been proposed by Alonso et al. [3], Meta et al. [12], Wu and Chiclana [39,40]. These models not only provide the DMs with the identification of preference values to be changed, but also with the advice to revise the preference values in the light of additional information too. All these GDM models with feedback mechanisms offer a valuable means to help the DMs in achieving a high degree of consensus. These feedback mechanisms embrace the same purpose, i.e., to generate advice to help the group to achieve a higher degree of consensus among the DMs. The feedback mechanisms also share a similar consensus control strategy, i.e., when the overall degree of consensus is lower than a predetermined threshold, the revision process starts until the overall degree of consensus reaches the threshold. However, the advice generation procedure is different, for example, the ones proposed by Herrera-Viedema et al. [22,21] only suggested the direction of changes, while those proposed by Sergio Alonso et al. [3], Chiclana et al. [12], and Wu and Chiclana [39,40] provided both direction and value of changes to the DMs. However, it is found that they included the influence of the conflicting opinions when they provide the guidance for the revision process. In our proposed feedback mechanism, both direction and value of changes to the DMs are also provided, but the revision is guided by the collective opinions aggregated from the non-conflicting ones only; therefore avoiding bias caused by the conflicting opinions. Additionally, our mechanism is more flexible owing to the predetermined aggregation threshold. Our iterative mechanism assumes that opinions with a satisfying support degree have the power to influence the group’s decision; therefore those DMs’ opinions can be aggregated to contribute to the final group decision. Our algorithm for aggregating the opinions from different DMs consists of two stages. In the first stage, the data cleansing process eliminates the opinions that cause conflicts based on the support degree of each DM. The second stage comprises an iterative process, which allows the DMs to revise their opinions in order to obtain an acceptable support degree, and then arrive at the final group decision by using the PA operator.

Methodologically, defining a proper support function of the PA operator is the core of the aggregation process. The commonly used form is defined by Yager [52], which is based on parameterized formulations. However, parameterized formulations restrict the wide usage of the PA operator since these parameters have to be determined by the DMs, or some metaheuristic techniques. Motivated by the dominance-based rough set approach (DRSA) presented by Greco et al. [19], we propose a new non-parameterized definition of the support function, which is appropriate for coping with ordinal preferences provided by the DMs. It is also easy to understand since it has an explicit explanation.

The rest of the paper is organized as follows. Section 2 reviews the concept of the PA operator and the representation of ordinal preferences, along with an introduction to the general scheme of consensus-based GDM models. Section 3 defines a support function to measure the support degree of the DMs using DRSA.
Section 4 proposes a two-stage algorithm to yield a final collective decision by eliminating conflicts and iteratively aggregating the opinions from different DMs. A comparison between the proposed GDM model and those by Herrera et al. is also included in Section 4. In Section 5, an application of a group decision problem [31] is used to validate the method. Section 6 presents concluding remarks and suggestions for further work.

2. Preliminaries

2.1. Power Average (PA) Operator

Introduced by Yager [52], the PA operator allows the aggregated argument values to support or reinforce each other in the aggregation process. It can be viewed as a generalized nonlinear weighted average aggregation operator. The weight depends upon the argument values to support or reinforce each other in the aggregation. The PA operator is an idempotent, bounded, commutative, and non-monotonic operator. The most valuable manifestation of the PA operator is that it can capture both the features of mode-like methods for finding the most typical value, and the averaging-type operator for fusion of data. Such properties make the PA operator appropriate for aggregating ordinal preferences. The PA operator is defined as follows [52].

Definition 2.1. A PA operator is a mapping defined from $\mathbb{R}^n$ to $\mathbb{R}$, i.e.,

$$P - A(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \left(1 + T(a_i) \right) a_i \left(1 + T(a_i) \right)$$

(1)

where $T(a_i) = \sum_{j=1}^{n} \text{Supp}(a_j, a_i)$. Note that $\text{Supp}(a, b)$ is defined as the support for $a$ from $b$, which has the following properties: (1) $\text{Supp}(a, b) \in [0, 1]$; (2) $\text{Supp}(a, b) = \text{Supp}(b, a)$; (3) $\text{Supp}(a, b) \geq \text{Supp}(x, y)$ if $|a - b| \leq |x - y|$.

Remark 1. The PA operator is a type of weighted average operator.

If we denote $w_i = \frac{1 - T(a_i)}{\sum_{i=1}^{n} (1 + T(a_i))}$, then $w_i$ satisfies: (1) $w_i \geq 0$, and (2) $\sum_{i=1}^{n} w_i = 1$; therefore, the PA operator is expressed in the form of the weighted average: $P - A(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_i \cdot a_i$. It should be emphasized that the PA operator is a nonlinear weighted average operator since the weights $w_i$ depend on the arguments being aggregated.

Remark 2. The support function is essentially a similarity index. Note that the higher the similarity $\text{Sim}(a, b)$, the closer the values of $a$ and $b$; therefore, a greater degree of support and a greater degree of agreement between them.

Table 2.1

<table>
<thead>
<tr>
<th>Case no.</th>
<th>Values of the arguments</th>
<th>PA value</th>
<th>Average value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10 10 10 10 10 10 10 10 10 10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>5 5 5 10 10 10 10 10 10 10</td>
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</tr>
<tr>
<td>3</td>
<td>5 5 5 5 10 10 10 10 10 10</td>
<td>9.12</td>
<td>8.5</td>
</tr>
<tr>
<td>4</td>
<td>5 5 5 5 5 10 10 10 10 10</td>
<td>8.5</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
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<td>7.5</td>
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<tr>
<td>6</td>
<td>5 5 5 5 5 5 5 10 10 10</td>
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<td>6.5</td>
</tr>
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<td>7</td>
<td>5 5 5 5 5 5 5 5 10 10</td>
<td>5.4</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>5 5 5 5 5 5 5 5 5 10</td>
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<td>6.1</td>
</tr>
<tr>
<td>9</td>
<td>5 5 5 5 5 5 5 5 5 5</td>
<td>5 5 5 5</td>
<td>5</td>
</tr>
</tbody>
</table>

Bold values indicate the differences between the special values.

Remark 3. The PA operator is a generalization of the simple average.

Suppose all support functions have the same value, e.g., $\text{Supp}(a_i, a_j) = k$, $\forall i, j$, where $i \neq j, k \geq 0$ is a constant, then $w_i = \frac{1 - k}{\sum_{j=1}^{n} (1 + T(a_j))} = \frac{1}{n}$; therefore, $P - A(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i$.

Remark 4. $\min_{i=1}^{n} \{a_i\} \leq P - A(a_1, a_2, \ldots, a_n) \leq \max_{i=1}^{n} \{a_i\}$.

Remark 5. The PA operator is a mode-like operator. It enlarges the influence of the argument with the highest degree of support on the final decision, which makes it more appropriate to aggregate ordinal preferences. (e.g., Case 8 in Table 2.1).

Define $\text{Supp}(a, a_j) = \begin{cases} 1, & \text{if } a_j \text{ has the same opinion with } a \quad \forall j \quad \text{if } a_j \end{cases}$ and $\text{Supp}(a_j) = \sum_{i=1}^{n} \text{Supp}(a_i, a_j)$, where $\text{Supp}(a_j)$ be the support of the $i$-th argument obtained from the sequence $\{a_1, a_2, \ldots, a_n\}$. Suppose the $i$-th argument is the most supported one, i.e., $\text{Supp}(a_j) = \max\{\text{Supp}(a_j)\}$. Since the $i$-th argument obtains the largest support, $a_i$ is the value that appears most often in the sequence, i.e., $\text{Mode}(a_1, a_2, \ldots, a_n) = a_i$. Similarly, in the PA operator, the weight of $a_i$ is the largest among all weights of arguments. Therefore, the aggregated result is highly influenced by $a_i$. In other words, the PA operator guarantees that the argument with the largest support has the greatest influence on the final decision. Additionally, the PA operator is non-monotonous, which is a typical property of the mode operator. The following example demonstrates that the PA operator has the characteristics of the mode operator as well as the average operator, which makes it more appropriate to aggregate ordinal information.

Example 2.1. Let $A = \{5, 5.5, 5.5, 5, 10, 10, 10, 10\}$ be the original argument to be aggregated by the PA operator. If we change the ratio of cluster ‘5’ or ‘10’, then we obtain the results in Table 2.1.

Table 2.1 shows that when we decrease the ratio of cluster 10s by decreasing the value and moving it to cluster 5s, the aggregated values based on the PA operator decrease, and move to the value of 5 more dramatically as compared with that from the average operator. Essentially, cluster 5s increases its power in the PA operator when value 5 obtains more support than value 10, as shown in Case 6 to Case 9 where value 5 appears more often than value 10.

Although both the PA and mode operators attempt to recognize the most supported argument(s), there are fundamental differences between them, i.e., the mode operator does not aggregate values, while the PA operator blends the arguments. In other words, the PA operator is able to take the opinions of the minority into account while trying to discover the most supported argument(s). As an example, in Case 7 (Table 2.1), cluster 10s exhibits a small influence on the final PA value since it is a minority. However, in
Case 8, cluster 10s almost loses its influence since it does not obtain any support from other arguments. Additionally, from Case 7 to Case 8, we show the non-monotonicity property of the PA operator, since we have $PA(5, 5, 5, 5, 5, 5, 5, 5, 10, 10) > PA(5, 5, 5, 5, 5, 5, 5, 5, 10, 11)$.

2.2. Ordinal preference

Ordinal information is usually utilized in ranking criteria or alternatives in decision making problems. Unlike the numerical-based measure, ordinal information cannot quantify exactly the importance of a criterion or alternative, since it only provides order information pertaining to the criteria or alternatives. When the alternatives are ranked by each DM from the best to the worst, we obtain the ordinal preference. As an example, a DM provides his/her preference ordering as $x_3 > x_1 > x_2$ on an alternative set, $X = \{x_1, x_2, x_3\}$, which indicates that he/she prefers alternative $x_3$ to $x_1$, and lastly $x_2$. Ordinal preferences can be represented in different formats. One of the most commonly used format is a $n$-dimensional ordered vector, $A = (a_1, a_2, \ldots, a_n)$, where $a_i$ is the rank assigned to alternative $i$. Therefore, an ordered vector can be defined based on the position of the alternatives from each DM. In other words, for each DM, $D_k$, a mapping, $O^k : X \rightarrow \mathbb{R}^n$, is defined by

$$O^k(x_1, x_2, \ldots, x_n) = (o^k(x_1), o^k(x_2), \ldots, o^k(x_n))$$

where $o^k(x_i) \in \{1, 2, \ldots, n\}$ denotes the order position of alternative $x_i$ assigned by $D_k$, $k = 1, 2, \ldots, m$. In the above example, $O^k(x_1, x_2, x_3) = (o^k(x_1), o^k(x_2), o^k(x_3)) = (2, 3, 1)$.

2.3. Consensus-based group decision making models

A consensus can be defined in different ways, ranging from a total agreement (a full consensus) to a partial agreement (e.g., a soft consensus) among the DMs in a flexible way. It should be noted that a full consensus, which requires all the DMs to achieve a total agreement, may not be practical in the real world. Even in cases that a full consensus could be achieved, the cost would be unacceptable. Therefore, some flexible way of consensus has been proposed by considering different degrees of partial agreement among the DMs. One of the most widely used flexible measurements is soft consensus, as proposed by Kacprzyk [27]. It allows a consensus to exist if most DMs agree on the most important alternatives, whereby “most” is represented as a concept of fuzzy linguistic majority. Reaching a consensus in GDM problems entails a dynamic and iterative process, which is normally synchronized by a human moderator. The moderator is responsible for guiding the DMs and providing them with advice to modify their opinions, in order to achieve the required consensus. A general procedure for conducting a consensus-reaching process in GDM models can be described as follows:

1. Collect the preferences: each DM provides the moderator his/her opinions on all potential alternatives;
2. Compute the current degree of consensus: the moderator calculates the current agreement level among each DMs’ opinions and the temporary collective opinions, whereby the similarity measure is normally used to indicate the agreement level. A statistical comparison study of the effects of five commonly used similarity functions for measuring consensus in GDM was examined by Chiclana et al. [13]. They found that some similarity functions contributed greatly to the speed of the consensus process, while other distance functions with less contribution were stable in terms of speed of convergence. Therefore, they presented the valuable decision support rules to guide the revision process by combining different distance functions to control the convergence speed of the consensus;
3. Control the consensus process: the current consensus degree is compared with a preset threshold of agreement desired by the group. If the current consensus degree reaches the preset threshold, then the group obtains the final collective decision. Otherwise, revisions are required;
4. Detect the preferences to be revised: the moderator identifies the furthest opinion from the current collective opinion, and gives those DMs advice and guides them to modify their opinions.

Numerous consensus-based GDM models are available in the literatures [4,7,24,45]. These models usually tackle GDM problems with a small number of DMs, whereby conflicts can be omitted. However, new trends stemming from demands in societal and technological contexts make it necessary to cope with the challenges to deal with a large number of DMs participating in a particular GDM problem [32]. Therefore, it is necessary to design new models and methods that are able to handle conflicts in large-scale GDM problems.

3. Support functions

In this section, we introduce a novel support function based on DRSA for the PA operator. Compared with the traditional support functions in the literature, the proposed support function not only avoids the difficulty of determining the PA parameter, but also provides more information except for the position of the alternative itself. These two advantages are elaborated in the following sub-sections.

3.1. Traditional support functions

As a crucial part of the PA operator, the support function has been widely studied, with various definitions in different applications [47,55]. The commonly used support functions can be divided into three categories: (1) binary support functions; (2) partitioned-type support functions (a generalization of binary support functions); (3) continuous and fuzzy set-based support functions.

1. A binary support function is a mapping defined on $R \times R \rightarrow \{0, 1\}$, i.e.,

$$Supp(a, b) = \begin{cases} K, & \text{if } |a - b| \leq d \\ 0, & \text{if } |a - b| > d \end{cases}$$

where $d$, a threshold parameter to be determined, is used to indicate the range of the support from its neighborhood. Two arguments support each other if the distance between $a$ and $b$ is smaller than or equal to $d$, otherwise they contribute no support. $K$ is the worth of the support.

2. For a partitioned-type support function, let $K_j \in [0, 1]$ be the worth of the support, $K_i > K_j$ when $i < j$, where $i, j = 1, 2, \ldots, p$; $d_i$ be a collection of values in $[0, 1]$, which satisfy $d_i \geq 0$ and $d_i < d_j$ if $i < j$, where $i, j = 1, 2, \ldots, p$. The function is defined as:

$$Supp(a, b) = \begin{cases} K_1, & \text{if } |a - b| \leq d_1 \\ K_j, & \text{if } d_{j-1} < |a - b| \leq d_j \text{ for } j = 2, 3, \ldots, p-1 \\ K_p, & \text{if } d_{p-1} < |a - b| \leq d_p. \end{cases}$$

The partitioned-type support function defined in (3) is a step-wise linear function. The value of the support function decreases when the distance between the aggregated values $a$ and $b$ increases.
(3) A continuous transition of the support function is defined as:

\[
\text{Supp}(a, b) = Ke^{-\alpha |a-b|^2}
\]

where \(K \in [0, 1]\) is the worth of the support, and \(\alpha \geq 0\) is an attenuator of the distance. It is easy to see that the value of the support function reaches the maximal allowable support \(K\) when \(|a-b| = 0\). As the distance between \(a\) and \(b\) becomes large, the value of the support function approaches 0.

Clearly, all the support functions comprise at least two parameters. Determining the parameters require extra procedures, e.g., using some learning techniques to provide the proper approximation. Otherwise, these support functions are inadequate to compute the support between two ordinal preferences. Therefore, we define a DRSA-based support function, which is a non-parameterized formulation, and we show that it is more appropriate to reflect the support between two ordinal preferences.

3.2. New support functions based on the dominance-based rough set approach

The traditional Rough Set Theory (RST), as introduced by [34], has been used to approximate useful knowledge by using indiscernible relations. However, the original rough set approach is not able to discover inconsistencies related to the attributes in preference-ordered domains. Therefore, Greco et al. [20] extended the indiscernible relations into a union of dominance, similarity, and indiscernible relations, in order to deal with multi-attribute and criteria sorting problems whereby the decision classes are preference ordered. The integrated concept was used for out-ranking the relation of ordinal attribute subsets by An and Tong [5].

Motivated by the integrated concept, a new support function for the PA operator is proposed based on DRSA in this paper to cope with ordinal and linguistic preferences given by the DMs.

Following the idea of DRSA, we define a support function for the DMs when their preferences are ordinal. Let \(\triangleright_i [x]\) be a set of alternatives that dominates \(x_i\) in \(X\), based on the opinion from \(D_k\), \(R \setminus i = \{x_i, x_j \in X \times X: o_i(x_i) > o_j(x_j)\}\) denotes a dominance relation defined by \(D_k\) on \(X \times X\). The dominance class of alternative \(x_i\) from the viewpoint of \(D_k\) is denoted as \(\triangleright \triangleright_i [x]\) = \([x_i \in X: o_i(x_i) > o_j(x_j)]\) where \(x_i, x_j \in X\). The family set of the dominance classes derived by \(D_k\) is defined by \(X/R \setminus i = \{\triangleright_i [x_i], \triangleright_i [x_j], \ldots \}\) which generally constitutes a covering of \(X\). Based on this dominance relation, the support function between two DMs is defined as follows.

**Definition 3.1.** Let \(X = \{x_1, x_2, \ldots, x_n\}\) be a finite alternative set, \(DM = \{D_1, D_2, \ldots, D_m\}\) be a group of \(m\) DMs. Each DM provides his/her preferences on \(X\) as an individual preference ordering \(O^k = (o^k(x_1), o^k(x_2), \ldots, o^k(x_n))\), where \(o^k(x_i) \in \{1, 2, \ldots, n\}\) is the order of alternative \(x_i\) assigned by \(D_k\), \(k = 1, 2, \ldots, m\). Then, we define the support function of \(D_k\) from \(D_k\) with respect to alternative \(x_i\) by

\[
\text{Supp}(D_k, D_l) = \frac{\|\triangleright_i [x] \cap \triangleright_j [x]\|}{\|\triangleright_i [x] \lor \triangleright_j [x]\|}
\]

where \(\triangleright_i [x]\) is the set of alternatives that dominates \(x_i\) according to \(D_k\) and \(|\cdot|\) denotes the set cardinality. Obviously, \(X = \bigcup^n_{i=1} \triangleright_i [x]\) for any \(k = 1, 2, \ldots, m\).

**Definition 3.2.** The support of \(D_k\) obtained from \(D_k\) on \(X\) is defined by

\[
\text{Supp}(D_k, D_l) = \frac{1}{n} \sum_{i=1}^{n} \text{Supp}(D_k, D_l) = \frac{1}{n} \sum_{i=1}^{n} \|\triangleright_i [x] \cap \triangleright_j [x]\| / \|\triangleright_i [x] \lor \triangleright_j [x]\|
\]

The support function satisfies the following properties:

1. \(0 \leq \text{Supp}(D_k, D_l) \leq 1\);
2. \(\text{Supp}(D_k, D_l) = \text{Supp}(D_l, D_k)\);
3. \(\text{Supp}(D_k, D_l) \geq \text{Supp}(D_k, D_m)\), if \(|o^k(x_i) - o^j(x_j)| < |o^k(x_i) - o^o(x_o)|\) \(\forall i, l, 1, 2, \ldots, n\);
4. \(\text{Supp}(D_k, D_l) = k, 1, 2, \ldots, n\);

**Remark 6.** Property (3) indicates that a higher degree of similarity in opinions from the DMs implies more support between them; therefore a higher degree of agreement between them.

**Definition 3.3.** The overall support degree of \(D_k\) from all other DMs on \(X\) is defined by

\[
\text{Supp}(D_k) = \frac{1}{m-1} \sum_{l=1}^{m} \text{Supp}(D_k, D_l)
\]

\[
= \frac{1}{(m-1)} \sum_{l=1}^{m} \|\triangleright_i [x] \cap \triangleright_j [x]\| / \|\triangleright_i [x] \lor \triangleright_j [x]\|
\]

For ordinal preferences, the support function defined in Eq. (7) provides more information except for the position of the alternative itself.

Consider the example in Table 3.1. Three DMs, \(D_1, D_2,\) and \(D_3\), give their ordinal preferences on four alternatives, respectively. Clearly, all the DMs consider \(A_3\) as the third best choice. However, some differences can be observed: (1) \(D_1\) and \(D_2\) have more similar opinions as compared with those of \(D_1\) and \(D_3\); (2) \(D_1\) and \(D_3\) have more similar opinions as compared with those of \(D_2\) and \(D_3\); and (3) \(D_1\) and \(D_2\) consider \(A_1\) and \(A_2\) to be better than \(A_3\), while \(D_3\) considers \(A_1\) and \(A_2\) to be better than \(A_3\). Using Eq. (7), the support degree among the DMs are: (1) \(\text{Supp}(D_1, D_2) = 3/4\), which is lower than \(\text{Supp}(D_1, D_3) = 1/3\); (2) \(\text{Supp}(D_2, D_3) = 7/16\), which is lower than \(\text{Supp}(D_1, D_3)\); and (3) \(\text{Supp}_{A_1}(D_1, D_2) = 1\), i.e., \(D_1\) and \(D_3\) totally support each other in choosing alternatives better than \(A_3\), while \(\text{Supp}_{A_2}(D_1, D_2) = 1/2\), which is different from \(\text{Supp}_{A_2}(D_1, D_3)\).

To validate the proposed support function, the commonly used distance function proposed by Cook and Seiford [18] is utilized to measure the differences between the ordinal rankings. The distance function is defined as \(d(O^k, O^l) = \sum_{i=1}^{n} |o^k_i - o^l_i|\). Let \(O^k = (a_1^k, a_2^k, \ldots, a_n^k)\) and \(O^l = (a_1^l, a_2^l, \ldots, a_n^l)\) are two ordinal

<table>
<thead>
<tr>
<th>Alternative</th>
<th>(A_1)</th>
<th>(A_2)</th>
<th>(A_3)</th>
<th>(A_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decision maker</td>
<td>(D_1)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(D_2)</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(D_3)</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3.1

An example of ordinal preferences.

<table>
<thead>
<tr>
<th>Distance between DMs</th>
<th>([D_1, D_2])</th>
<th>([D_1, D_3])</th>
<th>([D_2, D_3])</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Cook and Seiford’s method]</td>
<td>2</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Support degree between DMs (Proposed Method)</td>
<td>3/4</td>
<td>1/3</td>
<td>7/16</td>
</tr>
</tbody>
</table>
rankings given by $D_k$ and $D_l$ respectively. The results in Table 3.2 show that the distance function proposed by Cook and Seiford [18] cannot identify the difference between the opinions from $(D_k, D_l)$ and $(D_l, D_k)$. However, our proposed support degree is able to indicate the dissimilarities.

4. A two-stage aggregation algorithm based on the support degree of decision makers

In this section, a two-stage aggregation algorithm based on the support degree of DMs is proposed. The PA operator is selected as the aggregation operator since it is able to aggregate ordinal information, as explained in the previous sections. Before introducing the algorithm, a transformation function for quantifying ordinal preferences is presented, in order to reflect each individual’s preference.

4.1. Ordinal preference and preference relation matrix

While ordinal data provide a convenient way of articulating DMs’ preferences, they impose difficulty on the aggregation process since ambiguity prevails in the preference structure inherent in ordinal data. A key issue in aggregating ordinal data is how to quantify the preference rankings in order to reflect an individual’s preference. Clearly, different ranking positions hold different importance for the DMs. Without any additional information attached to an ordinal preference, it is rational to assume that there exists a mapping, which is dependent on the position of the alternatives, from the ordinal preference to a preference relation set. Considering each DM’s propensity to hold different views, different quantification schemes can be applied under different circumstances [10,23]. For simplicity, a mapping function which transforms ordinal preferences into preference relations defined in [10] is used, as follows.

Definition 4.1. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite alternative set, $DM = \{D_1, D_2, \ldots, D_m\}$ be $m$ DMs. Each DM provides a set of preferences on $X$ as an individual preference ordering, $\mathcal{O}^k = (\phi^k(x_1), \phi^k(x_2), \ldots, \phi^k(x_n))$, where $\phi^k(x_i) \in \{1, 2, \ldots, n\}$ is the order position of alternative $x_i$ assigned by $D_k$, $k = 1, \ldots, m$. Then, the value of the preference of alternative $x_i$ over alternative $x_j$ from $D_k$, denoted by $p^k_{ij}$, is defined as:

$$p^k_{ij} = f(\phi^k(x_i), \phi^k(x_j))$$

$$= 1/2 [1 + \phi^k(x_i) - \phi^k(x_j)]$$

(11)

where $\phi(x,y)$ is a function defined in $\mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$, which satisfies:

1. $\phi(x,y) = 0.5, \forall x \in \mathbb{R}$;
2. $\partial \phi(x,y)/\partial x < 0$ and $\partial \phi(x,y)/\partial y \geq 0, \forall x, y \in \mathbb{R}$

A preference relation-based matrix can be obtained from the ordinal preference by using the transformation function. For simplicity, if we define $\phi(x,y) = (y - x)/(n - 1)$, then $\phi(x,y)$ satisfies both conditions in Definition 4.1. Therefore, the value of the preference of alternative $x_i$ over alternative $x_j$ is defined as follows:

Definition 4.2. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, $DM = \{D_1, D_2, \ldots, D_m\}$ be $m$ DMs, $\mathcal{O} = (\phi(x_1), \phi(x_2), \ldots, \phi(x_n))$ be individual preferences from $D_k$. The value of the preference of alternative $x_i$ over alternative $x_j$ from $D_k$, is

$$p^k_{ij} = \frac{1}{2} \left[ 1 + \frac{\phi^k(x_i) - \phi^k(x_j)}{n - 1} \right]$$

(12)

It is straightforward to prove that (1) $p^k_{ij} \in [0, 1]$; (2) $p^k_{ii} = 0.5$; (3) $p^k_{ij} + p^k_{ji} = 1 \forall i, j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, m$, i.e., the matrix constituted by all values of the preference of alternative $x_i$ over alternative $x_j$ from $D_k$ is an additive reciprocal preference matrix [23]. We can also show that $p^k = (p^k_{ij})_{n \times n}$ generated by Eq. (12) is also a consistent matrix. In order to prove this, the definition of a consistent matrix is given in Definition 4.3.

Definition 4.3 [37]. An additive preference matrix $A = (a_{ij})_{n \times n}$ is an additive consistent preference matrix if the following additive transitivity is satisfied.

$$a_{ij} = a_{ik} - a_{jk} + 0.5, \forall i, j, k = 1, 2, \ldots, n.$$  

Remark 7. Different transitivity measures have been proposed to evaluate the consistency of a pairwise comparison matrix [23,38]. Herrera-Viedma et al. [23] suggested the use of additive transitivity to measure the consistency of an fuzzy preference relation, while Chiclana et al. [8] argued that the additive transitivity was inappropriate to measure the consistency owing to the possibility for additive transitivity to violate the requirement that all judgements should be confined to the interval of $[0, 1]$.

Since Definition 4.2 has already guaranteed that all judgements obtained in the form of pairwise are confined to $[0,1]$, therefore, the additive transitivity is employed to measure the consistency of a pairwise comparison matrix in this paper owing to its simplicity in computation.

Theorem 1. The pairwise comparison matrix, $P^k$ ($k = 1, 2, \ldots, n$), which is obtained from ordinal preference $p^k_{ij} = \frac{1}{2} [1 + (\phi^k(x_i) - \phi^k(x_j))/(n - 1)]$ is a consistent preference matrix.

Proof. Using $p^k_{ii} = 0.5$, it is straightforward to prove that

$$p^k_{ij} - p^k_{ji} = \frac{1}{2} \left[ 1 + \frac{\phi^k(x_i) - \phi^k(x_j)}{n - 1} \right] - \frac{1}{2} \left[ 1 + \frac{\phi^k(x_j) - \phi^k(x_i)}{n - 1} \right]$$

$$= \frac{1}{2} \left[ 1 + \frac{\phi^k(x_i) - \phi^k(x_j)}{n - 1} \right] - 0.5 = p^k_{ij} - p^k_{ji}$$

i.e., $p^k_{ij} - p^k_{ji} = -0.5, \forall i, j, k = 1, 2, \ldots, n$. Therefore, $P$ is a consistent matrix according to Definition 4.3. □

Theorem 1 guarantees that the preference matrices obtained from different DMs are reciprocal and consistent, which makes the priority weights of the alternatives derived from the matrix reasonable.

Theorem 2 [42]. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives. A DM provides a pairwise comparison matrix by comparing each pair of alternatives based on some criterion and determining an additive preference relation matrix $A = (a_{ij})_{n \times n}$. Then, a priority weight vector $w = (w_1, w_2, \ldots, w_n)$ of the alternatives can be derived by

$$w_i = \frac{\sum_{j=1}^{n} a_{ij} + n/2 - 1}{n(n - 1)}, i = 1, 2, \ldots, n.$$  

Theorem 2 offers a method for ranking the alternatives in ordinal preferences, which can be used to convert a pairwise comparison preference into an ordinal preference. In order to measure the difference in opinions from the DMs, the similarity degree between two reciprocal preference matrices is defined as follows.
Definition 4.4 [41]. Let \( P^k = \left(p_{ij}^k\right)_{n \times n} \) and \( P^j = \left(p_{ij}^j\right)_{n \times n} \) be two additive reciprocal preference matrices. Then, the similarity degree between \( P^k \) and \( P^j \) is defined as

\[
\text{Sim}(P^k, P^j) = 1 - \frac{1}{n(n-1)/2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| p_{ij}^k - p_{ij}^j \right|
\]

Theorem 3. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set of alternatives, \( O = \{\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)\} \) be an individual ordinal preference, \( w = (w_1, w_2, \ldots, w_n) \) is a priority weight vector of the alternatives derived from the additive preference relation matrix, \( P = \left(p_{ij}\right)_{n \times n} \) which is obtained by the transformation function in Theorem 1. Then, \( w_i \geq w_j \iff \alpha(x_i) \geq \alpha(x_j), \quad i,j = 1, 2, \ldots, n \), where \( w_i \) is the weight of alternative \( x_i \) and \( \alpha(x_i) \) is the order position of alternative \( x_i \).

Proof. Based on Theorem 2, \( w_i = \left( \sum_{j=1}^{n} p_{ij} + n/2 - 1 \right)/(n(n-1)) \) and \( w_j = \left( \sum_{k=1}^{n} p_{kj} + n/2 - 1 \right)/(n(n-1)), \forall i,j \).

Therefore, \( w_i \geq w_j \iff \sum_{j=1}^{n} p_{ij} + n/2 - 1 /(n(n-1)) \geq \left( \sum_{k=1}^{n} p_{kj} + n/2 - 1 \right)/(n(n-1)). \) So, \( w_i \geq w_j \iff \sum_{k=1}^{n} p_{ik} \geq \sum_{k=1}^{n} p_{jk} \forall i,j = 1, 2, \ldots, n. \)

Since \( p_{ij} \) is transformed from the ordinal preference by \( p_{ij} = 1/2[1 + \alpha(x_i) - \alpha(x_j)]/(n(n-1)), \) we have \( \sum_{k=1}^{n} p_{ik} \geq \sum_{k=1}^{n} p_{jk} \iff \sum_{k=1}^{n} \left( 1 + \alpha(x_i) - \alpha(x_j) \right)/n(n-1) \geq \sum_{k=1}^{n} \left( 1 + \alpha(x_i) - \alpha(x_k) \right)/n(n-1). \) In other words, \( w_i \geq w_j \iff \sum_{k=1}^{n} \left( 1 + \alpha(x_i) - \alpha(x_k) \right)/n(n-1) \geq \sum_{k=1}^{n} \left( 1 + \alpha(x_j) - \alpha(x_k) \right)/n(n-1). \)

Consequently, \( w_i \geq w_j \iff \sum_{k=1}^{n} (\alpha(x_i) - \alpha(x_k)) \geq 0, \) i.e., \( w_i \geq w_j \iff \sum_{k=1}^{n} (\alpha(x_i) - \alpha(x_k)) = 0. \)

Therefore, \( w_i \geq w_j \iff \alpha(x_i) \geq \alpha(x_k), \) for any \( i,j = 1, 2, \ldots, n. \)

Remark 8. Theorem 3 ensures that the ranking order induced by the additive preference relation matrix, \( P = \left(p_{ij}\right)_{n \times n} \) obtained from the ordinal preference by \( p_{ij} = 1/2[1 + \alpha(x_i) - \alpha(x_j)]/(n(n-1)), \) coincides with the original ordinal preference.

Remark 9. Theorem 3 shows that the higher the degree of \( w_i \), the more important \( x_i \) is, and the lower of its corresponding order position.

Remark 10. If we take \( w_i = \left( \sum_{j=1}^{n} w_j + n/2 - 1 \right)/(n(n-1)), \) \( i = 1, 2, \ldots, n \) as a specific expression of the definition of QGDO or QGNDPO provided by Chiclana et al. [11], then the sufficient part of Theorem 3 constitutes a special case of Proposition 2 in [11].

4.2. A two-stage aggregation algorithm

In GDM problems, the final group decision is usually more than the average of all individual opinions in the group. It is common to iteratively revise the existing opinions, and generate new results until the most satisfied decision accepted by most DMs is reached. During the consensus-reaching process, opinions that receive a satisfied support degree contribute towards the final group decision. Conversely, opinions with an unsatisfied support degree need to be revised in order to attract more supports (agreements) from the DMs. In addition, opinions that give rise to conflicts (and the DMs refuse to change them) need to be eliminated since such conflicts make no contributions to the final consensus. Sometimes, conflicts can hamper the consensus-reaching process owing to their influence on the collective group decision in the decision making process. As an example, if conflicts are included in the collective decision at the beginning of the decision making process, it is possible for the revision process to be distorted because the temporary collective decision is commonly used as guidance for revising the DMs’ opinions in subsequent rounds in most consensus-based GDM models. To overcome the negative influence of conflicts and drawbacks of the existing GDM models, we propose a two-stage aggregation algorithm for GDM problems in this section.

Fig. 1 shows the framework of the proposed two-stage algorithm. It consists of two parts: (1) identify the DMS who cause conflicts, and delete their opinions since they do not contribute towards achieving a consensus; (2) revise and aggregate the opinions from the DMS to produce the final group decision.

Without any loss of generality, suppose there is only one pair of conflicts from the DMS. In the data cleansing process (the first stage), the DMSs with conflicting opinions are identified. Since the pair of conflicting opinions does not contribute towards achieving a consensus, the temporary collective preference is obtained by aggregating non-conflicting preferences only. Based on the similarities between the pair of conflicting opinions and the temporary collective preference, one of the conflicting opinions is retrieved (which has been previously deleted). Therefore, the temporary collective preference in the first round is obtained by aggregating the non-conflicting preference and the retrieved preference. In other words, the temporary collective group decision in this stage is obtained by excluding the influences of the most conflicting opinion. The temporary collective preference, which is the PA operator-based aggregation of all opinions that potentially produces a consensus, is able to properly reflect the opinions of most DMSs. As a result, this temporary collective decision is more accurate to be used for guiding the revision process in the second stage. Algrom 1 is introduced as follows.

Algorithm 1. Input: A set of ordinal preferences from the DMS, \( O_0 = \{O^1, O^2, \ldots, O^m\} \); an empty set TBD = \( \Phi \); a support threshold \( \varepsilon \geq 0 \), which is a small positive real number

Output: A temporary collective preference matrix \( P(c) \) and a set of preference matrices \( P \) without conflicts.

Step 1. Compute the support between two DMSs, \( \text{Supp}(D_i, D_j) \), by using Eq. (6).

If \( \text{Supp}(D_i, D_j) < \varepsilon \), then set \( O_0 = \{O^i, O^j\} \) and \( \text{TBD} = \{O^i, O^j\} \), otherwise, go to Step 2.

Step 2. Transform all ordinal individual rankings, \( O_0 = \{O^1, O^2, \ldots, O^m\} \), into individual preference matrices, \( P_0 = \{P^{(1)}, P^{(2)}, \ldots, P^{(m)}\} \) by Definition 4.2.

Step 3. Aggregate all preference matrices in \( P \) based on the PA operator

\[
p_i^{(c)} = P - A(p_{ij}^{(1)}, p_{ij}^{(2)}, \ldots, p_{ij}^{(n-1)}, p_{ij}^{(n+1)}, \ldots, p_{ij}^{(m)})
\]

where \( P^{(c)} = \left(p_{ij}^{(c)}\right)_{n \times n} \) denotes the collective preference matrix, and \( p_{ij}^{(l)} = \left(p_{ij}^{(l)}\right)_{n \times n} \) denotes the preference matrix of \( D_i \)

\( l \in \{1, 2, \ldots, m\} \) \( \{i, j, \ldots\} \); 

Step 4. Calculate the similarity measures between \( p^{(c)} \) and \( p^{(l)} \) as well as \( p^{(l)} \), respectively.

If \( \text{Sim}(p^{(c)}, p^{(l)}) > \text{Sim}(p^{(c)}, p^{(h)}), \) then add \( p^{(l)} \) to \( P \), otherwise add \( p^{(l)} \) to TBD.

Step 5. Re-aggregate all preference matrices in \( P \) by using the PA operator to obtain a temporary collective preference matrix, \( p_i^{(c)} = P - A(p_{ij}^{(1)}, p_{ij}^{(2)}, \ldots, p_{ij}^{(n-1)}, p_{ij}^{(n+1)}, \ldots, p_{ij}^{(m)}) \).

Step 6. Let \( p^{(c)} = \left(p_{ij}^{(c)}\right)_{n \times n} \). Then \( p^{(c)} \) is the temporary collective group preference, and \( P = P_0 \setminus \text{TBD} \) is the set of all individual preferences without conflicting opinions.
Once the conflicting opinions are eliminated, an iterative algorithm is used to revise and aggregate the opinions in order to obtain the final group preference. In the second stage of the proposed model, a support degree threshold, \( \alpha \), which is pre-determined by the DMs, is used to control the aggregation process. If \( \alpha = 0 \), all opinions participate in the aggregation process and contribute towards the final group decision. If \( \alpha > 0.5 \), only the opinions from the DMs who obtain more than half of the group members’ support are involved in the aggregation process, which makes the aggregation result similar to the one provided by the majority rule. Note that different settings of the parameter provide more flexibility for Algorithm 2.

**Algorithm 2.** Inputs: The temporary collective group preference \( P^{(c)} \) and \( P \), where \( P^{(c)} \) is obtained from Algorithm 1, and \( P = \{ P^{(1)}, P^{(2)}, \ldots, P^{(m')} \} \) is the set of reduced individual preference matrices without conflicts. Parameter \( \alpha \) is the aggregation threshold, and \( \gamma \in [0, 1] \) is the consensus approaching rate.

Outputs: The final group preference, \( P^{(G)} \), and the set of modified individual preferences, \( P \).

Step 1. Let \( h = 0 \) and \( P^{(k)} = \{ p^{(k)}_{ij} \}_{n \times n} \), \( P^{(c)} = \{ p^{(c)}_{ij} \}_{n \times n} \).

Step 2. Calculate the support degree of each DM by using Eq. (7). If all support degrees of the DMs are greater than \( \alpha \), then go to Step 5; otherwise, go to Step 3.

Step 3. Find \( D_s \), with the minimum support degree. Suppose \( \text{Supp}(D_s) = \min\{ \text{Supp}(D_k), k = 1, 2, \ldots, m' \} \), let \( P^{(k)}_{h+1} = \{ p^{(k)}_{ij} \}_{n \times n} \), where

\[
p^{(k)}_{ij} = \begin{cases} (1 - \gamma)p^{(k)}_{ij} + \gamma p^{(c)}_{ij}, & k = \tau \\ p^{(k)}_{ij}, & k \neq \tau \end{cases}
\]

Set \( h = h + 1 \), \( P_h = \{ p^{(k)}_{ij} \}_{k = 1, 2, \ldots, m'} \).

Step 4. Derive the priority weight from the revised preference matrix based on Theorem 2 to produce the ordinal preference for \( D_s \), and go to Step 2.

Step 5. Aggregate all the revised individual preferences based on the PA operator

\[
p^{(G)}_{ij} = P - A(P^{(1)}_{ij}, P^{(2)}_{ij}, \ldots, P^{(m')}_{ij})
\]

Step 6. Set \( P^{(G)} = \{ p^{(G)}_{ij} \}_{n \times n} \), stop.

Fig. 1. A framework of the aggregation process in group decision making.
Remark 11. The iteration Eq. (14) in Algorithm 2 replaces the moderator [24] to accomplish the consensus-reaching process. Normally, the consensus process is guided by a moderator who does not participate in the discussion, but supervises the consensus process towards a success, i.e., to achieve the maximum possible agreement and reduce the number of experts outside the consensus in each new consensus round.

Remark 12. The value of the consensus approaching rate, \( \gamma \), is usually guided by the distance between the current support degree of each DM and the support threshold. Generally, \( \gamma \) is higher when the distance is larger. The higher the consensus approaching rate \( \gamma \), the faster the consensus-reaching process, but less original information is preserved during aggregation.

Remark 13. Step 4 in Algorithm 2 essentially achieves a transformation from a preference relation matrix to an ordinal preference. After obtaining the final group preference matrix, \( P^G \), an ordinal preference can be derived by Theorem 2.

Theorem 4. Let \( P^{(1)}_h, P^{(2)}_h, \ldots, P^{(m)}_h \) be m additive reciprocal preference matrices from the DMs, \( D_1, D_2, \ldots, D_m \), respectively, in round \( h \) of Algorithm 2, and \( P^{(1)}_{h+1}, P^{(2)}_{h+1}, \ldots, P^{(m)}_{h+1} \) be m additive reciprocal preference matrices in the \((h + 1)\)th round, then \( \min\{\text{Supp}(P^{(k)}_h)\} \geq \min\{\text{Supp}(P^{(k)}_{h+1})\} \) for any \( k = 1, 2, \ldots, m \), where \( \text{Supp}(P^{(k)}_h) \) denotes the support degree of \( D_k \) in round \( h \) of Algorithm 2.

Proof. Suppose \( D_{\tau} \) obtains the minimum support from other DMs in the \( h \)th round of Algorithm 2, i.e., \( \text{Supp}(P^{(k)}_h) = min \text{Supp}(P^{(k)}_h), k = 1, 2, \ldots, m \). Then, the preference matrix \( P^{(1)}_h \) from \( D_{\tau} \) should be revised by using Eq. (14), while other preference matrices remain unchanged. Therefore, \( P^{(k)}_{h+1} = (1 - \gamma)P^{(k)}_h + \gamma P^{(k)}_{h+1} \), where

\[
\text{Supp}(P^{(k)}_{h+1}) = \frac{\sum_{i=1}^{n}(1 - \text{Supp}(P^{(k)}_h))p^{(k)}_{ij}}{\sum_{i=1}^{n}(1 - \text{Supp}(P^{(k)}_h))p^{(k)}_{ij}}.
\]

(1) We show that \( \text{Supp}(P^{(k)}_{h+1}) \geq \text{Supp}(P^{(k)}_h) \).

Based on Eq. (14), \( P^{(k)}_{h+1} = (1 - \gamma)P^{(k)}_h + \gamma P^{(k)}_h = \sum_{k=1}^{m} \frac{(1 - \gamma)P^{(k)}_h + \gamma P^{(k)}_{h+1}}{\sum_{i=1}^{n}(1 - \text{Supp}(P^{(k)}_h))p^{(k)}_{ij}} \text{Supp}(P^{(k)}_h) \).

For simplicity, let \( w^k_h = 1 + \text{Supp}(P^{(k)}_h) \), we have \( P^{(k)}_{h+1} = \sum_{k=1}^{m} \frac{w^k_h(1 - \gamma)P^{(k)}_h + \gamma P^{(k)}_{h+1}}{\sum_{i=1}^{n}(1 - \text{Supp}(P^{(k)}_h))p^{(k)}_{ij}} \text{Supp}(P^{(k)}_h) \).

Based on Definition 4.4, \( \forall k \neq \tau, k = 1, 2, \ldots, m \), the similarity degree between \( P^{(k)}_{h+1} \) and \( P^{(k)}_h \) is defined as \( \text{Sim}(P^{(k)}_{h+1}, P^{(k)}_h) = 1 - \frac{1}{\sum_{j=1}^{n}|\sum_{i=1}^{n} |P^{(k)}_{h+1} - P^{(k)}_h||} \text{Supp}(P^{(k)}_h) \), the similarity degree between \( P^{(k)}_{h+1} \) and \( P^{(k)}_h \) is defined as \( \text{Sim}(P^{(k)}_{h+1}, P^{(k)}_h) = 1 - \frac{1}{\sum_{j=1}^{n}|\sum_{i=1}^{n} |P^{(k)}_{h+1} - P^{(k)}_h||} \text{Supp}(P^{(k)}_h) \).

Since \( \text{Supp}(P^{(k)}_h) = min\{\text{Supp}(P^{(k)}_h), k = 1, 2, \ldots, m \} \), i.e.,

\[
\sum_{j=1}^{n} |P^{(k)}_{h+1} - P^{(k)}_h| = \max_{k=1, 2, \ldots, m} \sum_{j=1}^{n} |P^{(k)}_{h+1} - P^{(k)}_h|,
\]

then \( \sum_{j=1}^{n} \sum_{i=1}^{n} |P^{(k)}_{h+1} - P^{(k)}_h| = \sum_{j=1}^{n} \sum_{i=1}^{n} |P^{(k)}_{h+1} - P^{(k)}_h| \),

therefore, \( \text{Sim}(P^{(k)}_{h+1}, P^{(k)}_h) \geq \text{Sim}(P^{(k)}_h, P^{(k)}_h) \). Using property (3) of the support function in Definition 2.1, \( \text{Supp}(P^{(k)}_{h+1}) \geq \text{Supp}(P^{(k)}_h) \).

(2) Based on the conclusion \( \text{Supp}(P^{(k)}_{h+1}) \geq \text{Supp}(P^{(k)}_h) \), we prove that \( \text{Supp}(P^{(k)}_h) \geq \text{Supp}(P^{(k)}_h) \) when \( k \neq \tau, k = 1, 2, \ldots, m \).

Since \( \text{Supp}(P^{(k)}_h) = \sum_{j=1}^{m} \sum_{i=1}^{n} |P^{(k)}_{h+1} - P^{(k)}_h| = \sum_{j=1}^{m} \sum_{i=1}^{n} |P^{(k)}_{h+1} - P^{(k)}_h| + \text{Supp}(P^{(k)}_h) \), and \( \text{Supp}(P^{(k)}_{h+1}) \geq \text{Supp}(P^{(k)}_h) \), i.e., \( \text{Supp}(P^{(k)}_{h+1}) \geq \text{Supp}(P^{(k)}_h) \) \( \forall k \neq \tau, k = 1, 2, \ldots, m \).

Combining all the above conclusions, we obtain \( \min\{\text{Supp}(P^{(k)}_{h+1})\} \geq \min\{\text{Supp}(P^{(k)}_h)\} \) for \( k = 1, 2, \ldots, m \). \( \square \)

Theorem 4 guarantees that Algorithm 2 converges, and the support degree of the DMs can reach a high level by using Eq. (14) iteratively for the updating process.

4.3. Comparison between the GDM models proposed by Herrera-Viedma et al. and our proposed model

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set of alternatives, \( DM = \{D_1, D_2, \ldots, D_m\} \) be m DMs, \( P^* = \{p^{(k)}_{ij}\}_{n \times n} \) be the individual preferences from \( D_\tau \). A complete comparison between the GDM models by Herrera-Viedma et al. [21,28] and our proposed model is presented in Table 4.1.

From Table 4.1, a number of observations in regards to the feedback mechanisms between the GDM models by Herrera-Viedma et al. and our proposed model can be made, as follows.

(1) both feedback mechanisms are designed to generate advices to help the group to achieve the required consensus degree among the DMs;
(2) both feedback mechanisms depend on the size of the group of DMs and the set of alternatives;
(3) both feedback mechanisms share the same consensus control strategy.

However, there are some fundamental differences between them:

(i) different advice generation systems: the former uses three rules to suggest the direction of changes while the latter updates the values of the corresponding opinions by using an iterative equation;
(ii) different capability of handling large GDM problems: when the group has a large number of DMs, it is hard for the former to obtain the required consensus level;
(iii) different viewpoints of comparison: the former is based on the alternative aspect, while the latter is based on the viewpoint of the DMs.

In order to further investigate the relationship between both feedback mechanisms, we define the global consensus degree in our proposed method as the average support degree for each DM, \( \overline{\text{Supp}}(D) \). Then, both consensus degrees have the following relationship.
<table>
<thead>
<tr>
<th>Feedback mechanism proposed by Herrera-Viedma et al.</th>
<th>The iterative mechanism proposed in our model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consensus degree calculation procedure</td>
<td>1. Pairs of DMs level: measure the agreement (support) between a pair of DMs pertaining to a certain alternative, denoted by $\text{Supp}_i(D, D_k)$</td>
</tr>
<tr>
<td>1. Pairs of alternative level: measure the agreement among all the DMs pertaining to a pair of alternatives $(x_i, x_j)$, and denote it as $\text{cop}_{ij}$</td>
<td>2. DMs level: measure the agreement among each pair of DMs pertaining to all alternatives, denoted by $\text{Supp}(D_{ij}, D_{k})$</td>
</tr>
<tr>
<td>2. Alternative level: measure the agreement among all DMs pertaining to an alternative, denoted by $c_{ai}$</td>
<td>3. One DM level: measure the collective support (agreement) of a particular DM obtained from other DMs pertaining to all alternatives, denoted by $\text{Supp}(D_i)$</td>
</tr>
<tr>
<td>3. Relation level: measure the global agreement among all DMs, denoted by $CR$</td>
<td></td>
</tr>
</tbody>
</table>

**Consensus degree calculation method**

- Similarity measure based on the Euclidean distance function

**Consensus control strategy**

1. If $CR$ is smaller than the predetermined Euclidean consensus threshold and the number of rounds is below MAXCLE, then start the revision round, otherwise obtain the final collective decision
2. Calculate the temporary collective opinion by using the PA operator, and denote it as $p^t_k$.
3. Select the DM with the minimum agreement degree from other DMs

**Revision advice generation system**

- Rule 1. If the $t$-th DM's opinion is lower than the temporary collective opinion, then increase the corresponding opinion
- Rule 2. If the $t$-th DM's opinion is higher than the temporary collective opinion, then decrease the corresponding opinion
- Rule 3. Otherwise, keep the opinion unchanged

**Stopping criterion**

- The global agreement among all DMs $(CR)$ has reached the pre-determined consensus threshold, or the iteration number has reached MAXCLE

---

**Table 4.1**

Comparison between the feedback mechanism proposed by Herrera-Viedma et al. [21,28] and the iterative mechanism in our proposed model.

<table>
<thead>
<tr>
<th>Feedback mechanism proposed by Herrera-Viedma et al.</th>
<th>The iterative mechanism proposed in our model</th>
</tr>
</thead>
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<tr>
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<td>2. DMs level: measure the agreement among each pair of DMs pertaining to all alternatives, denoted by $\text{Supp}(D_{ij}, D_{k})$</td>
</tr>
<tr>
<td>2. Alternative level: measure the agreement among all DMs pertaining to an alternative, denoted by $c_{ai}$</td>
<td>3. One DM level: measure the collective support (agreement) of a particular DM obtained from other DMs pertaining to all alternatives, denoted by $\text{Supp}(D_i)$</td>
</tr>
<tr>
<td>3. Relation level: measure the global agreement among all DMs, denoted by $CR$</td>
<td></td>
</tr>
</tbody>
</table>

---

**Theorem 5.** Let $CR$ be the measure of the global agreement pertaining to the whole alternative set among all the DMs defined by Herrera-Viedma et al. [21], and $\delta = \sum_{i=1}^{n} \text{Supp}(D_i)$ be the measure of the global agreement pertaining to the whole alternative set among all the DMs on our model. Then, $CR = \delta \sum_{i=1}^{n} \text{Supp}(D_i)$, where $\delta$ is a constant, which indicates the difference between the distance-based and support-based agreement models.

**Proof.** In the models proposed by Herrera-Viedma et al. the global agreement pertaining to the whole alternative set among all the DMs, $CR$, is defined as $CR = \frac{1}{n(n-1)} \sum_{i<j}^{n} \phi(\text{sm}_{ij}^d)$, where $\text{cop}_{ij} = \text{sm}_{ij}$ denotes the consensus degree among all the DMs pertaining to a pair of alternatives $(x_i, x_j)$, $i, j = 1, 2, \ldots, n$, and $\text{sm}_{ij}^d = \phi(\text{sm}_{ij}^d)$, where $\text{sm}_{ij}^d = 1 - |p^t_i - p^t_j|$ denotes the similarity of opinions pertaining to the pair of alternatives $(x_i, x_j)$ provided by $DM_i$ and $DM_j$, $k, l = 1, 2, \ldots, m$. Clearly, $\text{cop}_{ij} = \text{cop}_{ji}$ for all $i = 1, 2, \ldots, n$. Note that $\phi$ is an aggregation function used to obtain the collective similarity value from all the DMs, e.g., the arithmetic mean in the models of Herrera-Viedma et al. [21,28] and the S-OWA OR-LIKE operator in [22]. For simplicity, we use the arithmetic mean in this paper. Therefore, we have

$$CR = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{\text{cop}_{ij}}{n - 1} = \frac{1}{n(n-1)} \sum_{i<j}^{n} \phi(\text{sm}_{ij}^d)$$

$$= \frac{1}{n(n-1)} \sum_{i<j}^{n} \left( \frac{2}{m(m-1)} \sum_{k=1}^{m} \sum_{l=k+1}^{m} \text{sm}_{ij}^d \right)$$

$$= \frac{2}{m(m-1)} \sum_{k=1}^{m} \sum_{l=k+1}^{m} \frac{1}{n(n-1)} \sum_{i<j}^{n} \text{sm}_{ij}^d$$

$$= \frac{2}{m} \sum_{k=1}^{m} \text{Supp}(D_k)$$

Clearly, $\frac{2}{m} \sum_{k=1}^{m} \sum_{i<j}^{n} \text{sm}_{ij}^d$ is the average similarity of opinions pertaining to all alternatives provided by $DM_i$ and $DM_j$, i.e., the agreement degree between $D_i$ and $D_j$ pertaining to all alternatives. Therefore, it is reasonable to treat it as an equivalent form of $\text{Supp}(D_i, D_j)$ from the viewpoint of the distance measure. As such, $\text{Supp}(D_i, D_j) = \frac{\delta}{m-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \text{sm}_{ij}^d$, where $\delta$ is a constant, which indicates the difference between the distance-based and support-based agreement models.

$$CR = \frac{2}{m(n-1)} \sum_{k=1}^{m} \sum_{i<j}^{n} \left( \frac{1}{n(n-1)} \sum_{j>i}^{n} \text{sm}_{ij}^d \right)$$

$$= \frac{2}{m(n-1)} \sum_{k=1}^{m} \sum_{i<j}^{n} \text{Supp}(D_i, D_k)$$

$$= \frac{\delta}{m} \sum_{k=1}^{m} \left( \frac{1}{m-1} \sum_{i<j}^{n} \text{Supp}(D_i, D_k) \right) = \frac{1}{m} \sum_{k=1}^{m} \text{Supp}(D_k).$$

**Remark 14.** Theorem 5 shows that the measures of the global agreement pertaining to the whole alternative set among all the DMs in both feedback mechanisms are equivalent.

**Theorem 6.** In our proposed model, if $\text{Supp}(D_i) \geq \alpha$, for all $i = 1, 2, \ldots, m$, where $\alpha > 0$, then we have $\frac{1}{m} \sum_{i=1}^{n} \text{Supp}(D_i) \geq \alpha$. In other words, if the support degree obtained by each DM is greater than $\alpha$, then the global consensus is higher than $\alpha$. The proof is easily obtained. □

As compared with the feedback mechanism of Herrera-Viedma et al. [21,22,32], the proposed feedback (iterative) mechanism in Algorithm 2 has the following advantages:
(1) It not only provides the direction of changes, but also the specific value for changes by the iterative updating Eq. (14) automatically;
(2) It can accelerate the consensus-reaching process by excluding the influence of conflicting opinions pertaining to the collective group decision;
(3) The temporary collective opinion is more accurate to be used for guiding the revision process, since the influence of conflicts has been eliminated; therefore, the obtained temporary collective preference is able to properly reflect most DMs’ opinions;
(4) The pre-determined parameter makes the proposed two-stage algorithm more flexible to meet different requirements of the DMs, and convergence of the algorithm has been proven.

However, the disadvantage of the proposed model is that the automatic updating process of the DM preferences is not explicit to the moderator, which can affect the effectiveness of the moderating process.

5. An illustrative example

We consider an actual GDM from Morais and de Almedida [31]. The task was to control degradation of the hydrographic basin of a river located in Brazil. In this example, five DMs from five sectors considered twelve alternatives as the potential solutions to the causes of basin degradation. After collecting all necessary information from the DMs, the preference functions and respective parameters were combined by using PROMETHEE II to obtain the ranking of alternatives from each DM, as shown in Table 5.1.

Based on the individual rankings in Table 5.1, Morais and de Almedida [31] proposed a method with three exploration phases (filter, veto, and choose) to find the best alternative. Their method can handle ordinal rankings by concentrating on the first and last ranked positions and eliminating the irrelevant alternatives. However, it is worth pointing out that the final result (A8 is the best alternative among twelve alternatives based on Table 5.1) satisfies only two DMs (D1 and D3), while other DMs (e.g., D2 and D5) did not even place A8 as a top 6 alternative. Clearly, conflicts in the preferences were ignored. Using our proposed two-stage algorithm, we are able to extract more information from the group by considering both consensus and conflict. This yields more reliable and reasonable results.

Table 5.1
Individual rankings of twelve alternatives from five decision makers.

<table>
<thead>
<tr>
<th>Decision makers</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>A5 &gt; A2 &gt; A10 &gt; A4 &gt; A6 &gt; A3 &gt; A9 &gt; A7 &gt; A11 &gt; A12 &gt; A8 &gt; A1</td>
</tr>
<tr>
<td>D2</td>
<td>A2 &gt; A3 &gt; A6 &gt; A4 &gt; A10 &gt; A1 &gt; A12 &gt; A9 &gt; A11 &gt; A7</td>
</tr>
<tr>
<td>D3</td>
<td>A5 &gt; A6 &gt; A4 &gt; A10 &gt; A11 &gt; A9 &gt; A12 &gt; A8 &gt; A1</td>
</tr>
<tr>
<td>D4</td>
<td>A9 &gt; A8 &gt; A6 &gt; A2 &gt; A11 &gt; A10 &gt; A12 &gt; A3 &gt; A7</td>
</tr>
<tr>
<td>D5</td>
<td>A5 &gt; A6 &gt; A4 &gt; A12 &gt; A10 &gt; A8 &gt; A2 &gt; A9 &gt; A11</td>
</tr>
</tbody>
</table>

Algorithm 2.

(a) Parameter setting of Algorithm 1
First, we set the support threshold for conflicts as ε = 0.1, O0 = {O(1), O(2), ..., O(5)}, where O(k) is determined by Dk. Note that O(k) = \{o(1)(A1), o(1)(A2), ..., o(1)(A12)\}, k = 1, 2, ..., 5, and TBD = \φ.

Based on Definitions 3.2 and 3.3, the support degrees between two DMs, Supp(Di, Dj), and the overall support degree of each DM are computed, as shown in Table 5.2.

(b) Identifying conflicts
Clearly, it can be observed that the support degrees between the DMs, Supp(Di, Dk) > ε = 0.1, ∀i, j = 1, 2, ..., 5, i.e., no conflicts exist among the five DMs based on Algorithm 1. The ordinal individual rankings of the alternatives are transformed into the individual preference matrix by Definition 4.2, denoted as P(3, 1), P(2, 3), P(4, 1), and P(5, 2) respectively, where P(k) = \{p(k)ij\}i,j,k=1,...,5. The temporary collective group preference matrix is then obtained by using the PA operator, denoted as P(k) = \{p(k)ij\}i,j=1,...,5, where p(k)ij = P – A(p(1)ij), p(k)ij = ΣP(k)ij(1 + Supp(Dk))p(1)ij(1 + Supp(DM)). Based on this temporary collective preference matrix, an iterative revision process of the individual preference is conducted by using Algorithm 2.

(c) Parameter setting of Algorithm 2
Given that in the usual GDM problem, the opinion of a DM is valid only when his/her support degree is more than half. As such, we set the aggregation threshold as α = 0.5. From Table 5.2, D5 obtains the minimum degree of support from other DMs in the group, i.e., Supp(D5) = 0.485, which is close to α = 0.5. Based on this analysis, we set the consensus approaching rate γ = 0.2, which means the original information from D5 is able to be preserved at the level of 80%.

(d) Revising or updating
Based on Algorithm 2, only the preference matrix from D5 needs to be revised according to Eq. (14) since its support degree is smaller than 0.5. Therefore, a new preference matrix of D5 is obtained as P(3), while other preference matrices as P(2), P(3), P(4), remain unchanged after one iteration.

(e) Transforming and calculating the support degree in the new round
After updating the preference matrix of D5, the priority weight is derived from the revised preference matrix P(3) based on Theorem 2. Therefore, the ordinal preference of D5 is obtained as O(3)1, which is expressed by O(3)1 = \{o(3)1(A1), o(3)1(A2), ..., o(3)1(A12)\} =

Table 5.2
Support degrees between DMs and the overall support degree of each DM after revision.

<table>
<thead>
<tr>
<th>Supp (Di, Dk)</th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>D5</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>1</td>
<td>0.72</td>
<td>0.79</td>
<td>0.46</td>
<td>0.39</td>
</tr>
<tr>
<td>D2</td>
<td>0.72</td>
<td>1</td>
<td>0.64</td>
<td>0.52</td>
<td>0.46</td>
</tr>
<tr>
<td>D3</td>
<td>0.79</td>
<td>0.64</td>
<td>1</td>
<td>0.50</td>
<td>0.48</td>
</tr>
<tr>
<td>D4</td>
<td>0.46</td>
<td>0.51</td>
<td>0.50</td>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>D5</td>
<td>0.39</td>
<td>0.46</td>
<td>0.48</td>
<td>0.61</td>
<td>1</td>
</tr>
<tr>
<td>Overall support degree</td>
<td>0.607</td>
<td>0.609</td>
<td>0.606</td>
<td>0.534</td>
<td>0.508</td>
</tr>
</tbody>
</table>

Table 5.3
Support degrees between DMs and the overall support degree of each DM after revision.

<table>
<thead>
<tr>
<th>Supp (Di, Dk)</th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>D5</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>1</td>
<td>0.72</td>
<td>0.79</td>
<td>0.50</td>
<td>0.41</td>
</tr>
<tr>
<td>D2</td>
<td>0.72</td>
<td>1</td>
<td>0.64</td>
<td>0.52</td>
<td>0.49</td>
</tr>
<tr>
<td>D3</td>
<td>0.79</td>
<td>0.64</td>
<td>1</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>D4</td>
<td>0.50</td>
<td>0.51</td>
<td>0.50</td>
<td>1</td>
<td>0.63</td>
</tr>
<tr>
<td>D5</td>
<td>0.41</td>
<td>0.49</td>
<td>0.50</td>
<td>0.63</td>
<td>1</td>
</tr>
<tr>
<td>Overall support degree</td>
<td>0.607</td>
<td>0.609</td>
<td>0.606</td>
<td>0.534</td>
<td>0.508</td>
</tr>
</tbody>
</table>

Table 5.4
Individual rankings of twelve alternatives from six DMs.

<table>
<thead>
<tr>
<th>Decision makers</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>A5 &gt; A6 &gt; A10 &gt; A4 &gt; A9 &gt; A3 &gt; A9 &gt; A7 &gt; A11 &gt; A12 &gt; A8 &gt; A1</td>
</tr>
<tr>
<td>D2</td>
<td>A2 &gt; A3 &gt; A6 &gt; A4 &gt; A10 &gt; A1 &gt; A12 &gt; A9 &gt; A11 &gt; A7</td>
</tr>
<tr>
<td>D3</td>
<td>A5 &gt; A6 &gt; A4 &gt; A10 &gt; A11 &gt; A9 &gt; A12 &gt; A8 &gt; A1</td>
</tr>
<tr>
<td>D4</td>
<td>A9 &gt; A8 &gt; A6 &gt; A2 &gt; A11 &gt; A10 &gt; A12 &gt; A3 &gt; A7</td>
</tr>
<tr>
<td>D5</td>
<td>A5 &gt; A6 &gt; A4 &gt; A12 &gt; A10 &gt; A8 &gt; A2 &gt; A9 &gt; A11</td>
</tr>
<tr>
<td>D6</td>
<td>A1 &gt; A5 &gt; A12 &gt; A7 &gt; A9 &gt; A6 &gt; A8 &gt; A3 &gt; A9 &gt; A10 &gt; A9 &gt; A3 &gt; A11</td>
</tr>
</tbody>
</table>

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Needs from the group, the support degree of each alternative is smaller than 0.485 to 0.508. The support degree of DMs needs to increase his/her opinions pertaining to the following alternatives in Table 5.1, we have 0.59 to 0.638. Therefore, the temporary collective preference matrix is obtained by $P_{ij}^{(c)} = P - A(p_{ij}^{(1)}, p_{ij}^{(2)}, p_{ij}^{(3)}, p_{ij}^{(4)}, p_{ij}^{(5)})$. After removing $D_6$ from the group, the support degree of each DM is re-calculated. The revision and aggregation process with Algorithm 2 is then followed, as explained earlier.

(h) Comparison with the GDM models of Herrera-Viedma et al.

In the GDM models of Herrera-Viedma et al., two consensus criteria are used: (1) a consensus measure which indicates the agreement between the DMs; (2) a proximity measure to compute the distance between the individual opinion and the temporary collective opinion. The former is used to control the consensus process while the latter generates advice for the DMs to change their opinions. Based on the same example, both measures are calculated, as shown in Tables 5.6 and 5.7 (only the measures at or above the level of alternatives are listed due to the limit of space).

Table 5.6

<table>
<thead>
<tr>
<th>Consensus measure at the level of alternative</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
<th>A6</th>
<th>A7</th>
<th>A8</th>
<th>A9</th>
<th>A10</th>
<th>A11</th>
<th>A12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global consensus measure</td>
<td>0.87</td>
<td>0.81</td>
<td>0.84</td>
<td>0.87</td>
<td>0.72</td>
<td>0.86</td>
<td>0.86</td>
<td>0.81</td>
<td>0.82</td>
<td>0.78</td>
<td>0.82</td>
<td>0.82</td>
</tr>
</tbody>
</table>

Table 5.7

The values of proximity pertaining to alternatives of DMs.

<table>
<thead>
<tr>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
<th>A6</th>
<th>A7</th>
<th>A8</th>
<th>A9</th>
<th>A10</th>
<th>A11</th>
<th>A12</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>0.93</td>
<td>0.89</td>
<td>0.92</td>
<td>0.93</td>
<td>0.82</td>
<td>0.91</td>
<td>0.92</td>
<td>0.91</td>
<td>0.85</td>
<td>0.85</td>
<td>0.92</td>
<td>0.93</td>
</tr>
<tr>
<td>P2</td>
<td>0.94</td>
<td>0.86</td>
<td>0.94</td>
<td>0.91</td>
<td>0.85</td>
<td>0.94</td>
<td>0.94</td>
<td>0.94</td>
<td>0.94</td>
<td>0.94</td>
<td>0.94</td>
<td>0.94</td>
</tr>
<tr>
<td>P3</td>
<td>0.93</td>
<td>0.93</td>
<td>0.90</td>
<td>0.92</td>
<td>0.82</td>
<td>0.87</td>
<td>0.89</td>
<td>0.89</td>
<td>0.85</td>
<td>0.85</td>
<td>0.85</td>
<td>0.85</td>
</tr>
<tr>
<td>P4</td>
<td>0.88</td>
<td>0.79</td>
<td>0.88</td>
<td>0.87</td>
<td>0.87</td>
<td>0.87</td>
<td>0.89</td>
<td>0.84</td>
<td>0.84</td>
<td>0.84</td>
<td>0.84</td>
<td>0.84</td>
</tr>
</tbody>
</table>

The consensus measure degree is obtained by $P_{ij}^{(c)} = P - A(p_{ij}^{(1)}, p_{ij}^{(2)}, p_{ij}^{(3)}, p_{ij}^{(4)}, p_{ij}^{(5)})$. After calculating the similarity between $P^{(c)}$ and $P^{(1)}$, $P^{(6)}$, $P^{(1)}$ is retrieved, and $P^{(6)}$ is finally deleted since $Sim(P^{(c)}, P^{(1)}) > Sim(P^{(c)}, P^{(6)})$, which means that $D_6$ potentially causes the conflicts. This is also clear from the overall support degree of $D_1$ and $D_6$ in Table 5.5, respectively. Therefore, the temporary collective preference matrix is obtained by $P_{ij}^{(c)} = P - A(p_{ij}^{(1)}, p_{ij}^{(2)}, p_{ij}^{(3)}, p_{ij}^{(4)}, p_{ij}^{(5)})$.

(11, 7, 3, 2, 9, 4, 10, 8, 1, 6, 12, 5). Presenting in the format of rankings in Table 5.1, we have $A_9 > A_3 > A_7 > A_9 > A_{12} > A_{10} > A_9 > A_8 > A_9 > A_9 > A_7 > A_11$. Comparing with the original ranking of $D_6$, we notice that the positions of two pairs of alternatives have been changed, i.e., $A_7$ and $A_8$ as well as $A_1$ and $A_7$. This revision improves the support degree of $D_6$ from 0.485 to 0.508, and, at the same time, the support degrees of other four DMs are also improved (see Table 5.3).

(f) Checking and aggregating

After one iteration with Algorithm 2, the overall support degree of each DM reaches the aggregation threshold $\varepsilon = 0.5$. Therefore, the final aggregation matrix, $P^{(1)}$, is obtained by aggregating all the preferences of different DMs based on the PA operator, $P^{(1)} = P - A(p_{ij}^{(1)}, p_{ij}^{(2)}, p_{ij}^{(3)}, p_{ij}^{(4)}, p_{ij}^{(5)})$. Consequently, the final group decision in terms of the ordinal preference of twelve alternatives from the best to the worst, denoted by $A_1 > A_2 > A_3 > A_7 > A_9 > A_8 > A_9 > A_6 > A_10 > A_7 > A_11 > A_9 > A_8 > A_9 > A_9 > A_7 > A_11$. The best alternative selected by the five DMs is $A_5$, which is more reasonable compared with the selection of alternative $A_5$ in [31].

(g) Further discussion

Suppose we add another DM, $D_6$, to the group, whose ranking order of the twelve alternatives is $A_1 > A_9 > A_{12} > A_{11} > A_7 > A_9 > A_3 > A_6 > A_9 > A_{10} > A_2 > A_8$, while other DMs keep their opinions unchanged. Table 5.4 shows the individual rankings. Suppose also the initial thresholds are the same, i.e., the threshold for conflict $\varepsilon = 0.1$, $TBD = \phi$, $\varepsilon = 0.5$, and the consensus approaching rate $\gamma = 0.2$.

Then, the support degrees between the DMs and the overall support degree of each DM are calculated, as shown in Table 5.5, where $\text{Supp}(D_1, D_6) = 0.0833 < \varepsilon$. In other words, conflicts exist between $D_1$ and $D_6$. Based on Algorithm 1, both $D_1$ and $D_6$ are deleted. Aggregating the remaining four preference matrices based on the PA operator yields the temporary collective preference matrix $P^{(c)} = (p_{ij}^{(c)})_{12 \times 12}$, where $p_{ij}^{(c)} = P - A(p_{ij}^{(2)}, p_{ij}^{(3)}, p_{ij}^{(4)}, p_{ij}^{(5)})$. After calculating the similarity between $P^{(c)}$ and $P^{(1)}$ as well as $P^{(c)}$ and $P^{(6)}$, $P^{(1)}$ is retrieved, and $P^{(6)}$ is finally deleted since $Sim(P^{(c)}, P^{(1)}) > Sim(P^{(c)}, P^{(6)})$, which means that $D_6$ potentially causes the conflicts. This is also clear from the overall support degree of $D_1$ and $D_6$ in Table 5.5, respectively. Therefore, the temporary collective preference matrix is obtained by $P_{ij}^{(c)} = P - A(p_{ij}^{(1)}, p_{ij}^{(2)}, p_{ij}^{(3)}, p_{ij}^{(4)}, p_{ij}^{(5)})$.

After removing $D_6$ from the group, the support degree of each DM is re-calculated. The revision and aggregation process with Algorithm 2 is then followed, as explained earlier.

(h) Comparison with the GDM models of Herrera-Viedma et al.

In the GDM models of Herrera-Viedma et al., two consensus criteria are used: (1) a consensus measure which indicates the agreement between the DMs; (2) a proximity measure to compute the distance between the individual opinion and the temporary collective opinion. The former is used to control the consensus process while the latter generates advice for the DMs to change their opinions. Based on the same example, both measures are calculated, as shown in Tables 5.6 and 5.7 (only the measures at or above the level of alternatives are listed due to the limit of space).

Table 5.6 verifies that the most disagreement appears on alternative $A_9$, which is in line with our previous analysis. Suppose that the consensus threshold is set to 0.85, then revision is needed. Based on the explanation of “identification of the preferences to be changed” in Table 4.1, it is found that the consensus measure degrees on alternative $A_3$, $A_2$, $A_9$, $A_{10}$, $A_{11}$, and $A_{12}$ are smaller than the global consensus measure; therefore, the opinions pertaining to these alternatives should be revised. After that, we have to identify those DMs who should change their opinions pertaining to the above alternatives.

Clearly, Table 5.7 shows that the furthest opinion to the collective opinion is $P_5$, which is provided by $D_6$. Therefore, $D_6$ needs to revise his/her opinion in the first round. The result is also consistent with that produced by our algorithm (steps $c$ and $d$). Based on the explanation of “revision advice generation system” in Table 4.1, $D_6$ needs to increase his/her opinions pertaining to the following set of alternative pairs: $\{A_2, A_1\}$, $\{A_2, A_3\}$, $\{A_2, A_4\}$, $\{A_2, A_6\}$, $\{A_2, A_8\}$.
(A2,A5), (A2,A10), (A2,A12), (A5,A1), (A5,A6), (A5,A8), (A5,A9), (A8,A10), (A8,A12), (A11,A1), (A11,A3), (A11,A5), (A11,A12), (A12,A5), (A12,A8), (A12,A10), (A12,A15) and (A15,A12)]. The revision process is omitted since the GDM models by Herrera-Viedma et al. only provided the direction of changes. Obviously, the revision suggested by the GDM models of Herrera-Viedma et al. seems less practical than that from our proposed model for undertaking GDM problems with a large number of DMs, or alternatives.

6. Conclusion

In line with the new trends such as e-democracy processes and social networks, GDM problems involving a large number of DMs are attracting widespread attention. In a large group of DMs, it is common that conflicting opinions exist. In order to tackle this problem, a data cleansing process is coupled with a consensus-reaching process to formulate a two-stage dynamic GDM model for aggregating ordinal preferences in this paper. The proposed model has five advantages: (i) the PA operator-based aggregation method is able to provide more versatility during the information aggregation process by including the relationship between the aggregated support degrees; (ii) the new support function of the PA operator based on DRSA is able to better cope with ordinal preferences given by the DMs; (iii) the PA operator is a non-parameterized average operator, which avoids uncertainty caused by the model parameters; (iv) during the consensus-reaching process, the temporary collective opinion, which is used to guide the revision procedure, is constituted by the opinions with a satisfied support degree; therefore avoiding distortion of the final group opinion caused by conflicts; and (v) the data cleansing process is able to cope with conflicts among the DMs by eliminating the irrelevant or unsupported DMs’ opinions before the iterative revision algorithm is executed; therefore accelerating the process for reaching a consensus.

As the support function is defined based on DRSA, the aggregation method can be easily extended to other quasi-ordinal preference relations. As an example, it can be extended to handle linguistic information or natural language with the internal linear ordinal property. From the viewpoint of the support degree of each DM, the non-parameterized definition of the support function, which is based on DRSA, provides a feasible method to determine the weights of each DM after all the DMs give their opinions pertaining to a specific problem. All these constitute the subject of further investigation.

Acknowledgements

The authors would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions. This research is supported by USM fellowship and by the National Science Fund Project NSFCs (Nos. 61170040, and 1171086), by the Natural Science Foundation of Hebei Province (Nos. F2013201060 and A2013201119) and by the Key Scientific Research Foundation of Education Department of Hebei Province (ZD20131028). The second author would like to express his gratitude to Universiti Sains Malaysia for partially supporting this research under the research Grant Nos. 1001/PMaths/817060 and 1002/PMaths/910306.

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