Capture Region of 3D PPN Guidance Law for Intercepting High Speed Target

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Abstract—The implementation of a classic three dimensional PPN guidance against a high speed-nonmaneuvering target is analyzed. All the analysis is performed in a line of sight fixed coordinate. By utilizing an auxiliary function inspired from the tangential relative velocity at interception, a necessary and sufficient condition for the airfoil shape capture region for intercepting the target is obtained. In addition, a sufficient condition for the capture region with the finite line of sight turn rate constraint is also presented.

I. INTRODUCTION

Intercepting ballistic targets has been a challenge for missile guidance. It has been studied by intensive simulation and concluded that the near-head-on scenery is the best way for a missile to capture a high speed target [1]. To this end, several works were proposed to achieve the “head-on-attack” [2], [3]. Yet the analytical expression for capture region of a classic guidance law, pure proportional navigation (PPN), for intercepting targets that are of higher speeds than the interceptor remains unexploited. We are going to answer the question in this work.

In the past thirty some years, there were many researchers studied the PPN guidance laws, see for example [4]–[8]. Among those Guelman gave a qualitative analysis of PPN and Becker solved the closed-form solution for nonmaneuvering targets. In addition, Ghawghawe and Ghose extended the analysis of Guelman to time-varying maneuvering target. The above works analyzed the capturability of two dimensional PPN, while Ha and Tyan worked on three dimensional ones. Oh and Ha studied the performance of a 3D PPN law against a high speed maneuvering target by introducing a Lyapunov-like function. The analysis is performed in a so-called LOS plane. The result, however, is limited with the assumption that the “high speed” target therein was referred to a target whose speed is at least $1/\sqrt{2}$ times yet smaller than that of a missile. Furthermore, the capturability concluded from Lyapunov-like approach is conservative, since a Lyapunov function provides only sufficiency. Recently, several works used the Frenet formula to describe the guidance problem. By using the line-of-sight (LOS) fixed Frenet framework, the equations describing relative dynamics of guidance problem become simple and the closed-form solution can be obtained [8], [9].

The above works, either considered the problem in a two dimensional space or held the assumption that the target has a speed smaller than that of the interceptor. Hence, in this work we try to fill up the gap, namely analyzing the capturability of PPN against high speed target in a three dimensional LOS fixed coordinate. A necessary and sufficient condition guaranteeing the caption of a target is obtained, the result is more general than that of a Lyapunov-like approach.

This paper is organized as follows. Section II reviews the relative dynamics of PPN in three dimensional space via a line-of-sight fixed coordinate [10]. In this a LOS fixed coordinate, equations of motion of the relative dynamics between and interceptor implementing a 3D PPN law are derived. The an auxiliary function which is mandatory in determining the range of target aspect angle is introduced and analyzed in section III, then in the same section the capturability of PPN law against a high speed-nonmaneuvering target is analyzed. A necessary and sufficient condition of the capture region is given in the main theorem. In addition, a sufficient condition for the capture region with the constraint of finite LOS turn rate is proposed. After that numerical examples are given to demonstrate the air foil shape capture region for various navigation constants and velocity ratios. Concluding remarks are also drawn in section V to highlight the main results.

II. PRELIMINARIES

A. LOS Fixed Coordinate

To determine the dynamic equations of the relative motion between target and missile, a line of sight (LOS) fixed coordinate system $(e_r, e_t, e_\Omega)$ is adopted, where $e_r$ is a unit vector in the direction of the LOS, $e_t$ is a unit vector in the direction of $\dot{e}_r$, and $e_\Omega$ is determined by $e_r \times e_t$. Due to the definition of $e_t$, it can be shown that
the angular velocity of this coordinate system, \( \omega \), takes the form of
\[
\omega = \omega_r e_r + \Omega e_\Omega,
\]
and the unit vectors satisfy the following dynamic equations,
\[
\frac{d}{dt} \begin{bmatrix} e_r^T \\ e_\Omega^T \end{bmatrix} = A \begin{bmatrix} e_r^T \\ e_\Omega^T \end{bmatrix},
\]
where the skew-symmetric angular velocity matrix
\[
A = \begin{bmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & \omega_r \\ 0 & -\omega_r & 0 \end{bmatrix}.
\]
Note that (1) recovers the famous Frenet-Serret equations [11]. Furthermore, assuming that the angular velocity of LOS, \( \omega_L \), is perpendicular to LOS renders \( \omega_L = e_r \times \hat{e}_r = \Omega e_\Omega \) [8].

### B. Relative Dynamics

The relative dynamics of guidance problem can be expressed in the line-of-sight (LOS) fixed coordinate [10]. Assume that missile and target are particles in the space, the LOS vector can be defined as
\[
r = r_T - r_M = \rho e_r,
\]
where \( r_T \) and \( r_M \) are position vectors of target and missile, respectively, \( \rho \) is the range between target and missile. The relative velocity and acceleration are
\[
\dot{r} = \rho e_r + \dot{\rho} e_r = v_T - v_M,
\]
\[
\ddot{r} = \ddot{\rho} e_r + 2\dot{\rho} \dot{e}_r + \rho \ddot{e}_r = a_T - a_M,
\]
where \( v_T, a_T, v_M \) and \( a_M \) are velocity and acceleration vectors of target and missile, respectively. For convenience, let us define the velocity vector of missile and target and the speed ratio as
\[
v_T \triangleq V_T e_T, \quad v_M \triangleq V_M e_M, \quad \nu \triangleq \frac{V_T}{V_M},
\]
respectively, where \( V_M \equiv \|v_M\|, V_T \equiv \|v_T\\) are the unit vectors in the direction of \( v_M, v_T \). Since this work concerns only high speed targets, \( \nu \) is assumed. Similarly, \( a_T \) and \( a_M \) are defined in the \((e_r, e_\Omega)\) coordinate system as
\[
a_T = V_T e_T, \quad a_M = V_M e_M.
\]
From (3) and (4) it is easy to verify the following kinematic equations
\[
\rho = V_T(e_T^T e_T) - V_M(e_\Omega^T e_\Omega),
\]
\[
\rho \Omega = V_T(e_\Omega^T e_T) - V_M(e_\Omega^T e_\Omega),
\]
\[
0 = V_T(e_\Omega^T e_\Omega) - V_M(e_\Omega^T e_\Omega),
\]
\[
\rho \Omega \omega_r = a_T \Omega - a_M \Omega.
\]
where the direction cosines of target and interceptor satisfy the following dynamic equations
\[
\frac{d}{dt} \begin{bmatrix} e_T^T e_T \\ e_\Omega^T e_T \\ e_T^T e_\Omega \\ e_\Omega^T e_\Omega \end{bmatrix} = A \begin{bmatrix} e_T^T e_T \\ e_\Omega^T e_T \\ e_T^T e_\Omega \\ e_\Omega^T e_\Omega \end{bmatrix} + \Omega e_T + \dot{\Omega} e_\Omega,
\]
and the unity constraints
\[
(e_T^T e_T)^2 + (e_\Omega^T e_T)^2 + (e_T^T e_\Omega)^2 = 1,
\]
\[
(e_T^T e_M)^2 + (e_\Omega^T e_M)^2 + (e_T^T e_\Omega)^2 = 1.
\]

In this work a Coriolis type PPN guidance law is implemented to determine the command acceleration of missile, so that
\[
a_M = \beta \omega_L \times v_M,
\]
\[
= \beta V_M \Omega \left[ -(e_T^T e_M) e_r + (e_\Omega^T e_M) e_\Omega \right],
\]
where \( \beta \) is a navigation constant. In addition, the following assumptions are considered:

1) Missile is not subjected to aerodynamic drag, hence, \( V_M \) is a constant, which renders \( \dot{e}_M = \beta \Omega \left[ -(e_T^T e_M) e_r + (e_\Omega^T e_M) e_\Omega \right] \).

Furthermore, for simplicity, no actuator saturation and time lag are taken into account.

2) Target does not maneuver and is not subjected to aerodynamic drag either, namely \( a_T = 0 \). Hence, from (5d) we have the \( e_r \) component of angular velocity of the LOS coordinate \( \omega_L = 0 \).

Therefore, (6) can be further simplified as
\[
\frac{d}{dt} \begin{bmatrix} e_T^T e_T \\ e_\Omega^T e_T \\ e_T^T e_\Omega \\ e_\Omega^T e_\Omega \end{bmatrix} = \begin{bmatrix} 0 & \Omega & 0 & 0 \\ -\Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega \\ 0 & 0 & -\Omega & 0 \end{bmatrix} \begin{bmatrix} e_T^T e_M \\ e_\Omega^T e_M \\ e_T^T e_\Omega \\ e_\Omega^T e_\Omega \end{bmatrix},
\]
\[
\frac{d}{dt} \begin{bmatrix} e_T^T e_M \\ e_\Omega^T e_M \\ e_T^T e_\Omega \\ e_\Omega^T e_\Omega \end{bmatrix} = -(\beta - 1) \begin{bmatrix} 0 & \Omega & 0 & 0 \\ -\Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega \\ 0 & 0 & -\Omega & 0 \end{bmatrix} \begin{bmatrix} e_T^T e_M \\ e_\Omega^T e_M \\ e_T^T e_\Omega \\ e_\Omega^T e_\Omega \end{bmatrix}.
\]

In the following analysis, we define the normalized closing, transverse velocities and range to go as
\[
\tilde{u} \triangleq \frac{\rho}{V_M}, \quad \tilde{v} \triangleq \frac{\rho \Omega}{V_M}, \quad \tilde{\rho} \triangleq \frac{\rho}{\rho_0},
\]
and adopt the angle traveled by LOS as the independent variable angle,
\[
d \theta = \Omega dt.
\]
and missile can be solved independently as
\[
\begin{bmatrix}
(e^T e_M^e)(\theta) \\
(e^T e_M^e)(\theta)
\end{bmatrix} = R[−Δθ] \begin{bmatrix}
(e^T e_T^e)(θ) \\
(e^T e_T^e)(θ)
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
(e^T e_M^e)(\theta) \\
(e^T e_M^e)(\theta)
\end{bmatrix} = R[(β − 1)Δθ] \begin{bmatrix}
(e^T e_M^e)(θ) \\
(e^T e_M^e)(θ)
\end{bmatrix},
\]
respectively, where Δθ = θ − θ₀ for terseness, and the rotation matrix is defined by 
\[
R[θ] ≜ \begin{bmatrix}
\cos(θ) & -\sin(θ) \\
\sin(θ) & \cos(θ)
\end{bmatrix}.
\]
Use the identities (5a) and (5b), we have
\[
\begin{bmatrix}
\bar{u}(θ) \\
\bar{v}(θ)
\end{bmatrix} = νR[−Δθ] \begin{bmatrix}
(e^T e_T^e)(θ) \\
(e^T e_T^e)(θ)
\end{bmatrix} − R[(β − 1)Δθ] \begin{bmatrix}
(e^T e_M^e)(θ) \\
(e^T e_M^e)(θ)
\end{bmatrix}.
\]

The implementation of (5c) to the constraints (7) indicates that \((e^T e_M^e), (e^T e_M^e), \) and \((e^T e_T^e), (e^T e_T^e), \) satisfy
\[
\begin{align*}
(e^T e_M^e)^2 + 2(e^T e_T^e)^2 &= 1 − ν^2(e^T e_T^e)^2, \\
ν^2(e^T e_T^e)^2 + 2(e^T e_T^e)^2 &= ν^2 − (e^T e_M^e)^2.
\end{align*}
\]
From the above two identities we have the following relationship for \(ν, (e^T e_M^e), \) and \((e^T e_M^e), \)
\[
(e^T e_M^e) ≲ ν^2 ≤ \frac{1}{(e^T e_M^e)^2}.
\]

### III. Capturability Analysis

For convenience let’s define
\[
\bar{r}_M ≜ \sqrt{(e^T e_M^e)^2 + (e^T e_M^e)^2} = \sqrt{1 − (e^T e_M^e)^2},
\]
\[
\bar{r}_T ≜ \sqrt{(e^T e_T^e)^2 + (e^T e_T^e)^2} = \sqrt{1 − (e^T e_M^e)^2},
\]
and
\[
\begin{align*}
\cos(θ_M) &= (e^T e_M^e) / \bar{r}_M, \\
\sin(θ_M) &= (e^T e_M^e) / \bar{r}_M, \\
\cos(θ_T) &= (e^T e_T^e) / \bar{r}_T, \\
\sin(θ_T) &= (e^T e_T^e) / \bar{r}_T.
\end{align*}
\]
Furthermore, define the effective speed ratio and corresponding critical margin as
\[
ν_c ≜ \frac{\bar{v}_T}{\bar{r}_M}, \quad θ_c ≜ \sin^{-1} \left( \frac{1}{ν_c} \right).
\]
Note that \(0 < θ_c < \frac{π}{2}. \) By utilizing the trigonometric function identity, it is easy to verify that the direction cosines \((e^T e_T^e, e^T e_T^e)\) in (13), \((e^T e_M^e, e^T e_M^e)\) in (14) can be written in the form of
\[
\begin{align*}
(e^T e_T^e)(θ) &= \bar{r}_T \cos(θ_T − Δθ), \\
(e^T e_T^e)(θ) &= \bar{r}_T \sin(θ_T − Δθ).
\end{align*}
\]
and
\[
\begin{align*}
(e^T e_M^e)(θ) &= \bar{r}_M \cos(θ_M + (β − 1)Δθ), \\
(e^T e_M^e)(θ) &= \bar{r}_M \sin(θ_M + (β − 1)Δθ),
\end{align*}
\]
respectively. It follows that the relative velocity \((\bar{u}, \bar{v})\) in (15) can be re-expressed as
\[
\begin{align*}
\bar{u} &= ν\bar{r}_T \cos(θ_T − Δθ) − \bar{r}_M \cos(θ_M + (β − 1)Δθ), \\
\bar{v} &= ν\bar{r}_T \sin(θ_T − Δθ) − \bar{r}_M \sin(θ_M + (β − 1)Δθ).
\end{align*}
\]
Owing to the fact that \(\bar{v}(θ) ≥ 0 \) for all \(θ, θ_M, \) and \(θ_M, \)
must obey the following constraint,
\[
\bar{r}_M \sin(θ_M) ≤ ν\bar{r}_T \sin(θ_T).
\]
It is worthy to point out that (22) places an upper bound of \(θ_T\) for a given \(θ_M. \) From the above definitions, we have the following facts.

**Fact 3.1:**
\[
1 − \bar{r}_M^2 = ν^2(1 − \bar{r}_T^2),
\]
\[
1 < ν \text{ if and only if } 1 < ν_c.
\]
From (11) and (12) we have the normalized distance
\[
\bar{ρ}(θ) = \exp \left( \int_{θ_c}^{θ} \frac{\bar{u}}{\bar{v}} dθ \right).
\]
Evidently that for given ν and β, \(\bar{ρ}\) is a function of \((e^T e_T^e)_0, (e^T e_T^e)_0, (e^T e_M^e)_0, (e^T e_M^e)_0. \) Hence, the capture region of PPN law is defined as follows.

**Definition 3.1:** Capture region of PPN law, \(\mathcal{C}_{\mathcal{R}_{PPN}}\), is the set of initial conditions \((e^T e_T^e)_0, (e^T e_T^e)_0, (e^T e_M^e)_0, \) such that there exists a finite \(θ_f\) satisfying the capture condition,
\[
\bar{ρ}(θ_f) ≤ \bar{ρ}_f.
\]
In general, it is not a trivial task to determine the explicit expression of range to go \(\bar{ρ}(θ_f)\). To this end, let us define a Lyapunov-like function
\[
\mathcal{L} ≜ \bar{ρ}^1 − β \bar{v} ≤ 1,
\]
\[
\frac{d\bar{ρ}}{dθ} \mathcal{L} = −β ν \mathcal{L} \bar{v}^2.(e^T e_T^e)^2, \text{ Hence, if } \bar{v}(θ)_0 < 0 \text{ for } θ_0 ≤ θ ≤ θ_f, \text{ then we have } \mathcal{L}(θ_f) > \mathcal{L}(θ_0), \text{ and } \bar{ρ}(θ_f) < \bar{ρ}(θ_0) \bar{v}(θ_f) \bar{v}(θ_0)^{-1}, \text{ equivalent. Under this condition, the capture condition (25) can be replaced by } \bar{v}(θ_f) ≤ \bar{ρ}^{-1}_f \bar{v}_0 \text{ and } \bar{u}(θ_f) < 0. \text{ Therefore, we adopt the modified Definition 3.2 for capture region in this work.}

**Definition 3.2:** Capture region \(\mathcal{C}_{\mathcal{R}_{PPN}}\) of PPN law, for the case \(β > 1, \) is the set of initial conditions
(θ_{M0}, θ_{T0}) such that there exists a finite θ_f satisfying the following conditions:

\begin{equation}
(e_r^T e_T)(θ) < 0, \quad \text{for } θ_0 ≤ θ ≤ θ_f,
\end{equation}

\begin{equation}
\bar{v}_f ≜ \bar{v}(θ_f) ≤ \bar{ρ}_f^{−1} v_0, \quad \text{and } \bar{u}_f ≜ \bar{u}(θ_f) < 0.
\end{equation}

Here we consider only the case \( \bar{ρ}_f = 0 \), or \( \bar{v}_f = 0 \), equivalently.

### A. Admissible Regions

In this section, we want to determine the admissible region of \( θ_{M0} \) and \( θ_{T0} \) for capturing high speed target. At first, we determine the admissible region for \( θ_{T0} \). At the moment of capturing target, \( \bar{v}_f = 0 \) yields

\[ \sin(θ_{T0} − Δ\theta_f) \leq \frac{1}{\nu_c} \sin(θ_{M0} + (β − 1)Δ\theta_f), \]

which in turn renders the bounds

\[ |\sin(θ_{T0} − Δ\theta_f)| < \sin θ_c. \]  

(28)

In addition, the requirement \( \bar{u}_f < 0 \) (27) implies that

\[ \cos(θ_{T0} − Δ\theta_f) < \frac{\bar{ρ}_M}{\bar{v}_T} \cos(θ_{M0} + (β − 1)Δ\theta_f), \]

\[ = \frac{1}{\nu_c} \sqrt{1 − \sin^2(θ_{M0} + (β − 1)Δ\theta_f)}, \]

\[ = \frac{1}{\nu_c} \sqrt{1 − ν_c^2 \sin^2(θ_{T0} − Δ\theta_f)}. \]  

(29)

Two cases are considered:

1. \( |θ_{T0} − Δ\theta_f| < θ_c \): In this case, we have

\[ ν_c^2 \cos^2(θ_{T0} − Δ\theta_f) < 1 − ν_c^2 \sin^2(θ_{T0} − Δ\theta_f), \]

or \( ν_c^2 < 1 \), equivalently, which contradicts (24).

2. \( |θ_{T0} − Δ\theta_f| ≥ θ_c \): For this case \( \cos(θ_{T0} − Δ\theta_f) < 0 \), (29) is satisfied automatically.

Also recall from the Definition 3.2, it is necessary that for \( 0 ≤ Δ\theta ≤ Δ\theta_f \),

\[ (e_r^T e_T)(θ) = \bar{ρ}_T \cos(θ_{T0} − Δ\theta) < 0, \]  

(30)

which is equivalent to \( |(θ_{T0} − Δ\theta) − π| < \frac{π}{2} \). In summary, for arbitrary \( \bar{r}_M \) and \( \bar{r}_T \) such that \( \bar{u}_f < 0 \), if \( Δ\theta_f \) satisfies

\[ |(θ_{T0} − Δ\theta_f) − π| ≤ π. \]

(31)

While to have \( \bar{v}_0 ≜ \bar{v}(θ_0) ≥ 0 \), the minimum \( v_{\bar{T}} r_T (e_r^T e_T) = −\bar{r}_M \), alternatively, to have \( \bar{v}_f = 0 \), the maximum \( v_{\bar{T}} r_T (e_r^T e_T) = \bar{r}_M \). As a result, we have

\[ 0 ≤ Δ\theta_f ≤ 2θ_c, \]  

(32)

and the angle traveled by the vector, \( v_{\bar{T}} [(e_r^T e_T)e_r + \bar{e}_r] \), is confined in the interval

\[ |θ_{T0} − π| ≤ θ_c. \]

Hence, only the left branch of the admissible region (for \( \bar{v} = 0 \)) shown in Fig. 1 is allowed for the target velocity on the \( (e_r, e_T) \)-plane.

Next, we determine the admissible region for \( θ_{M0} \). By definition we know that \( \bar{v}(θ) ≥ 0 \) for \( θ_0 ≤ θ ≤ θ_f \), to have \( \bar{v}_f = 0 \), we need \( \frac{∂^2 \bar{v}}{∂θ^2}(θ_f) ≤ 0 \), while from (15) and (21) we have

\[ \frac{∂^2 \bar{v}}{∂θ}(θ_f) = −\bar{u}_f − \bar{r}_M β \cos(θ_{M0} + (β − 1)Δ\theta_f) ≤ 0, \]

which implies that \( \cos(θ_{M0} + (β − 1)Δ\theta_f) ≥ \frac{−\bar{u}_f}{\bar{r}_M β} > 0 \) or

\[ |θ_{M0} + (β − 1)Δ\theta_f| ≤ \cos^−1 \left( \frac{−\bar{u}_f}{\bar{r}_M β} \right) < \frac{π}{2}. \]  

(33)

equivalently. Therefore, due to (32), the admissible region of \( θ_{M0} \) is given by

\[ −\frac{π}{2} − 2(β − 1)θ_c ≤ θ_{M0} ≤ \frac{π}{2}. \]

### B. Analysis of Auxiliary Function

For convenience, we define the following auxiliary function to be used in the main theorem. For given values of \( β \) and \( ν_c \), define

\[ ψ(k) ≜ \sin^{-1} k − (β − 1) \sin^{-1} \frac{k}{ν_c}. \]  

(34)

Apparently that \( ψ(k) \) is an anti-symmetric function, that is, \( ψ(k) = −ψ(−k) \). Taking the derivative of \( ψ(k) \) with

\[ |π − (θ_{T0} − Δθ_f)| ≤ θ_c. \]  

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It is obvious that for $1 \leq \beta \leq 2$, we have $\frac{d}{dk}\psi(k) > 0$ for $|k| \leq 1$. This indicates that $\psi(k)$ is a monotonically increasing function in the defined interval. While for $\beta > 2$, $\frac{d}{dk}\psi(k) = 0$ occurs at $k^2 = \frac{(\beta - 1)^2 - \nu_e^2}{\beta^2 - 2\beta}$. Three cases are considered here:

1. $\beta - 1 < \nu_e$: it is easy to see that
   $$\frac{d}{dk}\psi(k) > 0 \quad \text{for} \quad |k| \leq 1.$$  
   
   It is obvious that for $|k| \leq 1$ the maximum value of $\psi(k)$ occurs at $k = 1$,
   $$\psi_{\text{max}} = \psi(1) = \frac{\pi}{2} - (\beta - 1)\theta_e > 0,$$

   and the minimum value of $\psi$ occurs at $k = -1$,
   $$\psi_{\text{min}} = \psi(-1) = -\psi_{\text{max}}.$$

2. $\beta - 1 = \nu_e$: we have
   $$\frac{d}{dk}\psi(k) \begin{cases} 0, & \text{for} \quad k = 0, \\ > 0, & \text{otherwise}. \end{cases}$$

   Similarly, we have $\psi_{\text{max}} = \psi(1)$ and $\psi_{\text{min}} = \psi(-1)$.

3. $\beta - 1 > \nu_e$: it follows that
   $$\frac{d}{dk}\psi(k) \begin{cases} < 0, & \text{for} \quad |k| < k_c, \\ 0, & \text{for} \quad k = \pm k_c, \\ > 0, & \text{for} \quad k < |k| \leq 1, \end{cases}$$

   where $k_c = \sqrt{\frac{(\beta - 1)^2 - \nu_e^2}{\beta^2 - 2\beta}}$. Note that $0 < k_c < 1$ for $\nu_e > 1$. Accordingly, the extreme values are $\psi_{\text{max}} = \psi(-k_c)$ and $\psi_{\text{min}} = \psi(k_c)$.

From the above analysis, we conclude that for cases 1 and 2, $\psi(k)$ is a monotonically increasing function. While for case 3, $\psi(k)$ is monotonically increasing for $|k| > k_c$, and decreasing for $|k| < k_c$. Therefore, we have

$$\psi_{\text{max}} = \begin{cases} \psi(1), & \text{for} \quad \beta - 1 \leq \nu_e, \\ \psi(-k_c), & \text{for} \quad \beta - 1 > \nu_e, \end{cases} \quad (36)$$

and $\psi_{\text{min}} = -\psi_{\text{max}}$.

Now consider the partial derivative of $\bar{v}$ with respect to $\theta$ at $\theta_f$,

$$\frac{\partial \bar{v}}{\partial \theta} \bigg|_{\theta = \theta_f} = \sqrt{\nu_e^2 - k^2 - (\beta - 1)\sqrt{1 - k^2}}.$$

It is easy to see that for $|k| \leq 1$ and $1 < \nu_e$, if $\beta - 1 < \nu_e$ then $\frac{\partial \bar{v}}{\partial \theta} \bigg|_{\theta = \theta_f} > 0$. Alternatively if $\nu_e < \beta - 1$ the minimum value of $\frac{\partial \bar{v}}{\partial \theta} \bigg|_{\theta = \theta_f}$ occurs at $k = 0$, at which

$$\frac{\partial \bar{v}}{\partial \theta} \bigg|_{k=0} = \bar{r}_M \nu_e - (\beta - 1) < 0,$$

and the maximum value of that occurs at $k = \pm 1$. Furthermore, $\frac{\partial \bar{v}}{\partial \theta} \bigg|_{\theta = \theta_f} = 0$ occurs at $k = \pm k_c$, equivalently.

In words, $\frac{\partial \bar{v}}{\partial \theta} \bigg|_{\theta = \theta_f} \leq 0$ if and only if $\nu_e < \beta - 1$ and $|k| \leq k_c$.

### C. Main Result

In the following main result, in the admissible region of $\theta_{T0}$ and $\theta_{M0}$, for each given $\theta_{T0}$ the upper and lower bound of $\theta_{M0}$ is determined.

**Theorem 3.1:** Consider a missile commanded by a PPN law (8) To capture a high speed ($v > 1$), non-navigating target, it is necessary and sufficient that $\beta > \nu_e + 1$ and

$$\mathcal{A}_{PPN} = \left\{(\theta_{M0}, \theta_{T0}) \bigg| (\theta_{M0})_{\text{min}} \leq \theta_{M0} \leq \sin^{-1} \frac{k}{\nu_e}, \quad \pi - \sin^{-1} \frac{k}{\nu_e} \leq \theta_{T0} \leq \pi + \theta_e, \quad \theta_{T0} \leq \pi - \sin^{-1} \left( \frac{1}{\nu_e} \sin \theta_{M0} \right), \quad \theta_{T0} \geq \pi - \frac{1}{\beta - 1} (\theta_{M0} + \psi_{\text{max}}) \right\}, \quad (37)$$

where $\psi_{\text{max}}$ is defined by (36) and $(\theta_{M0})_{\text{min}}$ solves

$$\sin \theta_{M0} = \nu_e \sin \left( \frac{1}{\beta - 1} (\theta_{M0} + \psi_{\text{max}}) \right). \quad (38)$$

**Proof:** To save space, the proof is omitted here. Now the capture region for a high speed target with the finite LOS turn rate constraint is derived in the following theorem.

**Theorem 3.2:** Finite LOS turn rate is retained if $(\theta_{M0}, \theta_{T0}) \in \mathcal{A}_{PPN}$, $\beta \geq 2(v + 1)$ and

$$\theta_{T0} - \pi + \frac{1}{\beta - 1} \theta_{M0} \leq \frac{1}{\beta - 1} \psi(-k_{LOS}), \quad (39)$$

where $k_{LOS} \triangleq \sqrt{1 - \frac{4(v - 1)^2}{(\beta^2 - 4\beta)^2}}$.  

**Proof:** To save space, the proof is omitted here.

### IV. Numerical Example

In this example we demonstrate the capture region for various equivalent speed velocity ratio $\nu_e$ and navigation constant $\beta$. From Fig. 2, 3 we can see that for a fixed $\beta$, larger speed ratio renders a smaller capture region.
Alternatively, for a fixed $\nu_e$, larger navigation constant means larger capture region.

V. CONCLUSIONS

In this work, an auxiliary function was utilized in analyzing the necessary and sufficient condition for the capture region of a classic 3D PPN guidance law against a high speed-nonmaneuvering target, which can be sketched in a two dimensional plane. Through the analysis and numerical example, we can conclude that both larger navigation constant and smaller velocity ratio imply larger capture region, this agrees with our intuition. While a maximum range of $\theta_{vT0}$ does not occurs at a head-on aiming ($\theta_{TM0} = 0$), it surprisingly occurs at $\theta_{TM0} = -\arcsin k_e$ instead.

REFERENCES