TWO SHARP INEQUALITIES FOR BOUNDING THE SEIFFERT MEAN BY THE ARITHMETIC, CENTROIDAL, AND CONTRA-HARMONIC MEANS

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Abstract. In the paper, the authors find the best possible constants appeared in two inequalities for bounding the Seiffert mean by the linear combinations of the arithmetic, centroidal, and contra-harmonic means.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Seiffert mean $T(a, b)$ and the centroidal mean $C(a, b)$ are defined respectively by

$$T(a, b) = \frac{a - b}{2 \arctan \left( \frac{a - b}{a + b} \right)} \quad (1.1)$$

and

$$C(a, b) = \frac{2(a^2 + ab + b^2)}{3(a + b)} \quad (1.2)$$

It is well known that

$$A(a, b) = \frac{a + b}{2}, \quad G(a, b) = \sqrt{ab}, \quad S(a, b) = \sqrt{\frac{a^2 + b^2}{2}},$$

$$C(a, b) = \frac{a^2 + b^2}{a + b}, \quad M_p(a, b) = \sqrt[p]{\frac{a^p + b^p}{2}}$$

for $p \neq 0$ are respectively the arithmetic, geometric, root-square, contra-harmonic and $p$-th power means of two positive numbers $a$ and $b$, that the $p$-th power means $M_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and that the inequalities in

$$G(a, b) = M_0(a, b) < A(a, b) = M_1(a, b) < C(a, b)$$

$$< S(a, b) = M_2(a, b) < C(a, b) \quad (1.3)$$

hold for $a, b > 0$ with $a \neq b$. For more information on results of mean values, please refer to, for example, [11, 12, 19, 20, 21] and closely related references therein.

In [22], Seiffert proved the double inequality

$$A(a, b) = M_1(a, b) < T(a, b) < M_2(a, b) = S(a, b) \quad (1.4)$$

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for $a, b > 0$ with $a \neq b$. In [13], Hästö showed that the function $T(\frac{1}{x}, x)$ is increasing with respect to $x \in (0, \infty)$ if $p \leq 1$. In [3, 5], the authors demonstrated that the double inequalities
\begin{equation}
\alpha_1 S(a, b) + (1 - \alpha_1) A(a, b) < T(a, b) < \beta_1 S(a, b) + (1 - \beta_1) A(a, b)
\end{equation}
and
\begin{equation}
C(\alpha_2 a + (1 - \alpha_2) b, \alpha_2 b + (1 - \alpha_2) a) < T(a, b) < C(\beta_2 a + (1 - \beta_2) b, \beta_2 b + (1 - \beta_2) a)
\end{equation}
hold for $a, b > 0$ with $a \neq b$ if and only if
\begin{equation}
\alpha_1 \leq \frac{4 - \pi}{(\sqrt{2} - 1)\pi}, \quad \beta_1 \geq \frac{2}{3}, \quad \alpha_2 \leq \frac{1}{2} \left(1 + \sqrt{\frac{4}{\pi} - 1}\right), \quad \beta_2 \geq \frac{3 + \sqrt{3}}{6}.
\end{equation}

For more information on this topic, please refer to recently published papers [4, 6, 7, 8, 9, 10, 14, 15, 16, 18, 23, 24] and cited references therein.

For positive numbers $a, b > 0$ with $a \neq b$, let
\begin{equation}
J(x) = \overline{C}(xa + (1 - x)a, xb + (1 - x)a)
\end{equation}
on $[\frac{1}{2}, 1]$. It is not difficult to directly verify that $J(x)$ is continuous and strictly increasing on $[\frac{1}{2}, 1]$ and to notice that
\begin{equation}
J\left(\frac{1}{2}\right) = A(a, b) < T(a, b) \quad \text{and} \quad J(1) = \overline{C}(a, b) > T(a, b).
\end{equation}

Therefore, it is much natural to ask a question: What are the best constants $\alpha \geq \frac{1}{2}$ and $\beta \leq 1$ such that the double inequality
\begin{equation}
\overline{C}(\alpha a + (1 - \alpha) b, \alpha b + (1 - \alpha) a) < T(a, b) < \overline{C}(\beta a + (1 - \beta) b, \beta b + (1 - \beta) a)
\end{equation}
holds for $a, b > 0$ with $a \neq b$?

The following Theorem 1.1, the first main result of this paper, gives an affirmative answer to this question.

**Theorem 1.1.** For positive numbers $a, b > 0$ with $a \neq b$, the double inequality (1.10) is valid if and only if
\begin{equation}
\alpha \leq \frac{1}{2} \left(1 + \sqrt{\frac{12}{\pi} - 3}\right) \quad \text{and} \quad \beta = 1.
\end{equation}

In [17] the author posed an unsolved problem: Find the greatest value $\alpha_1$ and the least value $\beta_1$ such that the double inequality
\begin{equation}
\alpha_1 C(a, b) + (1 - \alpha_1) A(a, b) < T(a, b) < \beta_1 C(a, b) + (1 - \beta_1) A(a, b)
\end{equation}
holds for $a, b > 0$ with $a \neq b$.

The following Theorem 1.2, the second main result of this paper, solves this problem.

**Theorem 1.2.** For $a, b > 0$ with $a \neq b$, the double inequality (1.12) holds if and only if $\alpha_1 \leq \frac{1}{\pi} - 1$ and $\beta_1 \geq \frac{1}{3}$.
2. Proof of Theorem 1.1

In this section, we supply a proof of Theorem 1.1.

For simplicity, we denote two numbers in (1.11) by \( \lambda \) and \( \mu \) respectively.

It is clear that, in order to prove the double inequality (1.10), it suffices to show

\[
T(a, b) > \underline{C}(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a)
\]  

(2.1)

and

\[
T(a, b) < \underline{C}(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a).
\]

(2.2)

From definitions (1.1) and (1.2) we see that both \( T(a, b) \) and \( \underline{C}(a, b) \) are symmetric and homogenous of degree 1. Hence, without loss of generality, we assume that \( a > b \). If replacing \( \frac{a}{b} > 1 \) by \( t > 1 \) and letting \( p \in \left( \frac{1}{4}, 1 \right) \), then

\[
\underline{C}(pa + (1 - p)b, pb + (1 - p)a) - T(a, b)
\]

\[
= \frac{[pt + (1 - p)]^2 + [pt + (1 - p)][p + (1 - p)t] + [p + (1 - p)t]^2}{6(1 + t) \arctan \frac{t}{t+1}} bf(t),
\]

(2.3)

where

\[
f(t) = 4 \arctan \frac{t - 1}{t + 1} - \frac{3(t^2 - 1)}{[pt + (1 - p)]^2 + [pt + (1 - p)][p + (1 - p)t] + [p + (1 - p)t]^2}.
\]

(2.4)

Standard computations lead to

\[
f(1) = 0,
\]

(2.5)

\[
\lim_{t \to \infty} f(t) = \pi - \frac{3}{p^2 - p + 1},
\]

(2.6)

and

\[
f'(t) = \frac{f_1(t)}{h_1(t)},
\]

(2.7)

where

\[
f_1(t) = (4p^4 - 8p^3 + 18p^2 - 14p + 1)t^4 - 4(4p^4 - 8p^3 + 9p^2 - 5p + 1)t^3
\]

\[
+ 6(4p^4 - 8p^3 + 6p^2 - 2p + 1)t^2 - 4(4p^4 - 8p^3 + 9p^2 - 5p + 1)t
\]

\[
+ 4p^4 - 8p^3 + 18p^2 - 14p + 1,
\]

(2.8)

\[
f_1(1) = 0,
\]

(2.9)

and

\[
h_1(t) = \{[pt + (1 - p)]^2 + [pt + (1 - p)][p + (1 - p)t] + [p + (1 - p)t]^2\}^2 (1 + t^2).
\]

Let

\[
f_2(t) = \frac{f_1(t)}{4}, \quad f_3(t) = \frac{f_1(t)}{3}, \quad \text{and} \quad f_4(t) = \frac{f_1(t)}{2}.
\]

Then, by standard argument, we have

\[
f_2(t) = (4p^4 - 8p^3 + 18p^2 - 14p + 1)t^4 - 3(4p^4 - 8p^3 + 9p^2 - 5p + 1)t^2
\]

\[
+ 3(4p^4 - 8p^3 + 6p^2 - 2p + 1)t - (4p^4 - 8p^3 + 9p^2 - 5p + 1),
\]

(2.10)

\[
f_2(1) = 0,
\]

(2.11)
there exists a point \( t > t_3 > t_2 > t_1 > t_0 > 1 \), such that \( f(t) \) is strictly decreasing on \([1, t_0]\) and strictly increasing on \([t_0, \infty)\). Similarly, by (2.17) and (2.22), there exists a point \( t_1 > t_0 > 1 \) such that \( f_2(t) \) is strictly decreasing on \([1, t_1]\) and strictly increasing on \([t_1, \infty)\), and, by (2.11) and (2.21), there exists a point \( t_2 > t_1 > 1 \) such that \( f_3(t) \) is strictly decreasing on \([1, t_2]\) and strictly increasing on \([t_2, \infty)\). Further, by (2.7), (2.9), and (2.20), there exists a point \( t_3 > t_2 > 1 \) such that \( f(t) \) is strictly decreasing on \([1, t_3]\) and strictly increasing on \([t_3, \infty)\). Finally, by (2.3) and (2.16), it is deduced that the function \( f(t) \) is negative on \((1, \infty)\). The inequality (2.1) is thus proved.

If \( p = \mu = 1 \), then the function (2.8) becomes

\[
f_1(t) = (t - 1)^4 > 0
\]

for \( t > 1 \). Combining this with (2.7) and (2.5) results in that \( f(t) \) is strictly increasing and positive on \((1, \infty)\). Therefore, the inequality (2.2) is obtained.

Combining the inequalities (2.1) and (2.2) with the monotonicity of \( J(x) \) defined by (1.8), the double inequality (1.10) is established for all \( \alpha \leq \lambda \) and \( \beta \geq 1 \).

For any given number \( p \) satisfying \( 1 > p > \lambda \), it is obvious that the limit (2.6) is positive. This positivity together with (2.3) and (2.4) implies that for \( 1 > p > \lambda \) there exists \( T_0 = T_0(p) > 1 \) such that the inequality

\[
\overline{C}(pa + (1 - p)b, pb + (1 - p)a) > T(a, b)
\]

holds for \( \frac{a}{b} \in (T_0, \infty) \). This tells us that the constant \( \lambda \) is the best possible.
For $\frac{1}{2} < p < \mu = 1$, the quantity (2.13) is positive. Accordingly, there exists a number $\delta = \delta(p) > 0$ such that the function $f_3(t)$ is negative on $(1, 1 + \delta)$. This negativity together with (2.3), (2.5), (2.7) and (2.9) implies that for any $\frac{1}{2} < p < \mu = 1$, there exists $\delta = \delta(p) > 0$ such that the inequality

$$T(a, b) > C(pa + (1 - p)b, pb + (1 - p)a)$$

is valid for $\frac{a}{b} \in (T_0, \infty)$. Consequently, the number $\mu$ is the best possible. The proof of Theorem 1.1 is complete.

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following Lemmas.

**Lemma 3.1.** The Bernoulli numbers $B_{2n}$ for $n \in \mathbb{N}$ have the property

$$(-1)^{n-1}B_{2n} = |B_{2n}|,$$

where the Bernoulli numbers $B_i$ for $i \geq 0$ are defined by

$$\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} x^i = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} x^{2i}, \quad |x| < 2\pi. \quad (3.2)$$

**Proof.** In [2, p. 16 and p. 56], it is listed that for $q \geq 1$

$$\zeta(2q) = (-1)^{q-1} \left(\frac{2\pi}{2q}\right)^{2q} \frac{B_{2q}}{2}, \quad (3.3)$$

where $\zeta$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (3.4)$$

From (3.3), the formula (3.1) follows. □

**Lemma 3.2.** For $0 < |x| < \pi$,

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n} |B_{2n}|}{(2n)!} x^{2n-1}. \quad (3.5)$$

**Proof.** This may be derived readily from combining the formula [1, p. 75, 4.3.70] with the identity (3.1). □

**Lemma 3.3.** For $0 < |x| < \pi$,

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!} x^{2(n-1)}. \quad (3.6)$$

**Proof.** Since

$$\frac{1}{\sin^2 x} = \csc^2 x = -\frac{d}{dx} \cot x,$$

the formula (3.6) follows from differentiating (3.5). □

Now we are ready to prove Theorem 1.2. It is easy to see that the double inequality (1.12) is equivalent to

$$\alpha_1 < \frac{T(a, b) - A(a, b)}{C(a, b) - A(a, b)} < \beta_1. \quad (3.7)$$
Without loss of generality, we assume $a > b > 0$ and let $x = \frac{a}{b}$. Then $x > 1$ and

$$
\frac{T(a, b) - A(a, b)}{C(a, b) - A(a, b)} = \frac{x^{-1}}{2 \arctan \frac{x-1}{x+1}} - \frac{x+1}{2}.
$$

Let $t = \frac{x-1}{x+1}$. Then $t \in (0, 1)$ and

$$
\frac{T(a, b) - A(a, b)}{C(a, b) - A(a, b)} = \frac{t}{\arctan t} - 1 - \frac{1}{t^2}.
$$

Let $t = \tan \theta$ for $\theta \in \left(0, \frac{\pi}{4}\right)$. Then

$$
\frac{T(a, b) - A(a, b)}{C(a, b) - A(a, b)} = \frac{\tan \theta}{\tan \theta} - 1 - \frac{1}{\sin^2 \theta} + 1.
$$

By Lemmas 3.2 and 3.3, we have

$$
\frac{\cot \theta}{\sin^2 \theta} - 1 + 1 = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n} |\theta^{2n-2} - \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}}{(2n)!} B_{2n} |\theta^{2n-2}
$$

$$
= 1 - \sum_{n=1}^{\infty} \frac{n2^{2n+1}}{(2n)!} |B_{2n} |\theta^{2n-2}
$$

which is strictly decreasing on $\left(0, \frac{\pi}{4}\right)$. Moreover, by L'Hôpital rule and standard argument, we have

$$
\lim_{x \to 0^+} = \frac{1}{3} \quad \text{and} \quad \lim_{x \to (\pi/4)^-} = \frac{4}{\pi} - 1.
$$

The proof of Theorem 1.2 is complete.

REFERENCES


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