A note on the equivalences between the averages and the $K$-functionals related to the Laplacian

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Abstract

For $\mathbb{R}^d$ or $\mathbb{T}^d$, a strong converse inequality of type A (in the terminology of Ditzian and Ivanov (J. Anal. Math. 61 (1993) 61)) is obtained for the high order averages on balls and the $K$-functionals generated by the high order Laplacian, which answers a problem raised by Ditzian and Runovskii (J. Approx. Theory 97 (1999) 113).

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1. Introduction and main result

Given a function $f \in L(\mathbb{R}^d)$, its Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i x \cdot \xi} \, dx, \quad \xi \in \mathbb{R}^d.$$
For a positive integer $\ell$, the $\ell$th order Laplacian $\Delta^\ell$ is defined, in a distributional sense, by

$$(\Delta^\ell f)(\xi) = (-1)^\ell |\xi|^{2\ell} \hat{f}(\xi).$$

Associated with the operator $\Delta^\ell$, there is a $K$-functional

$$K_{\Delta,\ell}(f, t^{2\ell})_p := \inf \{ \|f - g\|_p + t^{2\ell} \|\Delta^\ell g\|_p : g, \Delta^\ell g \in L^p(\mathbb{R}^d) \},$$

where $t > 0$, $1 \leq p \leq \infty$ and $\|\cdot\|_p$ denotes the usual $L^p$-norm on $\mathbb{R}^d$.

Let $V_d$ denote the volume of the unit ball of $\mathbb{R}^d$. For $t > 0$ and a locally integrable function $f$, we define the average $B_t(f)$ by

$$B_t(f)(x) = \frac{1}{t^d V_d} \int_{\{u \in \mathbb{R}^d : |u| \leq t\}} f(x + u) \, du$$

and the $\ell$th order average $B_{\ell,t}(f)$ (for a given positive integer $\ell$) by

$$B_{\ell,t}(f)(x) = -\frac{2}{(2\ell)} \sum_{j=1}^\ell (-1)^j \left( \frac{2\ell}{\ell - j} \right) B_{jt}(f)(x).$$

We remark that for $\ell > 1$ the operator $B_{\ell,t}$ was first introduced by Ditzian and Runovskii in [DR, p. 117, (2.6)].

For more background information we refer to [Di1,Di2,DR,Di-Iv,To].

Our main goal in this paper is to prove the following strong converse inequality of type $A$ (in the terminology of [Di-Iv]), which was conjectured in [DR, p. 138].

**Theorem 1.** Let $\ell \in \mathbb{N}$, $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^d)$. Then

$$\|f - B_{\ell,t}(f)\|_p \approx K_{\Delta,\ell}(f, t^{2\ell})_p,$$

where $t > 0$ and

$$A(f, t) \approx B(f, t)$$

means that there is a $C > 0$, independent of $f$ and $t$, such that

$$C^{-1} A(f, t) \leq B(f, t) \leq C A(f, t).$$

Theorem 1 for $\ell = 1$ was proved in [DR, p. 133, Theorem 6.1] and for $d = 1$, $\ell$ small, as it was indicated in [DR, p. 138], can be obtained by following the technique developed in [Di-Iv]. For $\ell \geq 2$ and $d \geq 2$, the following strong converse inequality of type $B$ (in the terminology of [Di-Iv]) was obtained in [DR, p. 127, Theorem 4.8 and p. 131, Theorem 5.7]:

$$K_{\Delta,\ell}(f, t^{2\ell})_p \approx \|f - B_{\ell,1}(f)\|_p + \|f - B_{\ell,\rho}(f)\|_p, \quad 1 \leq p \leq \infty$$

for some $\rho > 1$. The proof of our Theorem 1 will be based on this equivalence.

We remark that with a slight modification of the proof below a similar result for the periodic case can also be obtained.
2. Basic lemmas

The following lemma can be easily obtained by a straightforward computation.

**Lemma 1.** Let $\chi_{B(0,1)}(x)$ denote the characteristic function of the unit ball

$$B(0, 1) := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 \leq 1\}.$$

$V_d$ denote the volume of $B(0,1)$ and let $I(x) = \frac{1}{V_d} \chi_{B(0,1)}(x)$. Then

$$\hat{I}(x) = \frac{\gamma_d}{d} \int_0^1 \cos(u|x|)(1 - u^2)^{\frac{d-1}{2}} \, du$$

with

$$\gamma_d = \left(\int_0^1 (1 - u^2)^{\frac{d-1}{2}} \, du\right)^{-1}. \quad (4)$$

**Lemma 2.** Let $B_{\ell,t}$ be defined by (2) and $I(x)$ the same as in Lemma 1. Then for $f \in L(\mathbb{R}^d)$,

$$\widehat{B_{\ell,t}(f)}(x) = m_{\ell}(t|x|) \hat{f}(x), \quad (5)$$

where

$$m_{\ell}(|x|) = \frac{-2}{(2\ell)^{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \hat{I}(jx)$$

$$= 1 - A_{\ell}(|x|), \quad (7)$$

$$A_{\ell}(|x|) = \gamma_d \frac{4\ell}{(2\ell)^{\ell}} \int_0^1 (1 - u^2)^{\frac{d-1}{2}} \left(\sin \frac{u|x|}{2}\right)^{2\ell} \, du \quad (8)$$

and $\gamma_d$ is given by (5).

**Proof.** For $t > 0$, we write

$$I_t(x) = \frac{1}{t^d} I(\frac{x}{t}).$$

Then from definition (2), it follows that

$$B_{\ell,t}(f)(x) = \frac{-2}{(2\ell)^{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} (f \ast I_{jt})(x),$$

which implies (6) and (7). Substituting (4) into (7) yields

$$m_{\ell}(|x|) = \frac{-2\gamma_d}{(2\ell)^{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \int_0^1 \cos(ju|x|)(1 - u^2)^{\frac{d-1}{2}} \, du \quad (9)$$

and

$$A_{\ell}(|x|) = \gamma_d \frac{4\ell}{(2\ell)^{\ell}} \int_0^1 (1 - u^2)^{\frac{d-1}{2}} \left(\sin \frac{u|x|}{2}\right)^{2\ell} \, du \quad (10)$$
which, together with the following identity
\[
\left( \sin \frac{x}{2} \right)^{2\ell} = \frac{2^{2\ell}}{4^{\ell}} + \frac{2}{4^{\ell}} \sum_{j=1}^{\ell} (-1)^j \left( \frac{2\ell}{\ell - j} \right) \cos jx,
\]
gives (8) and (9). This completes the proof. □

**Lemma 3.** Let \( m_\ell(u) \) be the same as in Lemma 2. Then for \( j \in \mathbb{Z}_+ \) and \( u \geq 0 \),
\[
\left| \left( \frac{d}{du} \right)^j m_\ell(u) \right| \leq C_{\ell,j} \left( \frac{1}{u + 1} \right)^{\frac{d+1}{2}},
\]
where \( C_{\ell,j} > 0 \) is independent of \( u \).

**Proof.** By identity (10), it suffices to show that for \( j \in \mathbb{Z}_+ \) and \( u \geq 0 \),
\[
\left| \left( \frac{d}{du} \right)^j \int_0^1 \cos(\nu v) (1 - v^2)^{\frac{d-1}{2}} \, dv \right| \leq C_j \left( \frac{1}{u + 1} \right)^{\frac{d+1}{2}}.
\]
(11)
We use formula (4.7.5) of [An-As-R, p. 204] to obtain that
\[
\int_0^1 \cos(\nu v) (1 - v^2)^{\frac{d-1}{2}} \, dv = 2^{\frac{d-2}{2}} \sqrt{\pi} I^\nu \left( \frac{d + 1}{2} \right) \frac{J_{\frac{d}{2}}(u)}{u^{\frac{d}{2}}},
\]
(12)
where \( J_{\alpha}(u) \) denotes the Bessel function of the first kind of order \( \alpha \). Now (11) is a consequence of (12) and the following well-known estimates on Bessel functions:
\[
\frac{d}{du} u^{-\alpha} J_{\alpha}(u) = -u^{-\alpha} J_{\alpha+1}(u), \quad [\text{An-As-R, (4.6.2), p. 202}],
\]
\[
J_{\alpha}(u) = O \left( \frac{1}{(u + 1)^2} \right) \text{ for } u \geq 0 \quad [\text{An-As-R, (4.8.5), p. 209}],
\]
\[
J_{\alpha}(u) = O(u^{-2}) \text{ as } u \to 0 \quad [\text{An-As-R, (4.7.6), p. 218}].
\]
This concludes the proof. □

**Lemma 4.** Suppose that \( a \) is a \( C^\infty \)-function defined on \([0, \infty)\) with the property that for \( u \geq 0 \) and \( 0 \leq j \leq d + 1 \),
\[
\left| \left( \frac{d}{du} \right)^j a(u) \right| \leq C(a) \left( \frac{1}{1 + u} \right)^{\frac{d+1}{2}}.
\]
(13)
For \( t > 0 \), define the operator \( T_t \), in a distributional sense, by
\[
(T_t(f))^\wedge (\xi) = a(t|\xi|) \hat{f}(\xi), \quad \xi \in \mathbb{R}^d.
\]
Then for \( 1 \leq p \leq \infty \) and \( f \in L^p(\mathbb{R}^d) \),
\[
\sup_{t > 0} \| T_t(f) \|_p \leq C_{p,a} \| f \|_p.
\]
This lemma is well known (see [St]), but for the sake of completeness, we give its proof here.

**Proof.** Let

$$K(x) = \int_{\mathbb{R}^d} e^{ix\xi} a(|\xi|) \, d\xi.$$  \hfill (14)

Since

$$T_t(f)(x) = f \ast K_t(x),$$

with

$$K_t(x) = \frac{1}{t^d} K\left(\frac{x}{t}\right),$$

it is sufficient to prove

$$\|K\|_{L^1(\mathbb{R}^d)} < \infty. \hfill (15)$$

By (14), we get for \(\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}_+^d\),

$$(-x)^\gamma K(x) = \int_{\mathbb{R}^d} e^{ix\xi} \left(\frac{\partial}{\partial \xi}\right)^\gamma a(|\xi|) \, d\xi,$$

which, by (13), implies

$$|x^\gamma K(x)| \leq C \int_{\mathbb{R}^d} \frac{d\xi}{(1 + |\xi|)^{d+1}} < \infty,$$

with \(|\gamma| = \gamma_1 + \cdots + \gamma_d \leq d + 1\). Now taking the supremum over all \(\gamma\) with \(|\gamma| = d + 1\) yields

$$|K(x)| \leq \frac{C}{|x|^{d+1}},$$

which, together with the fact that \(K \in C(\mathbb{R}^d)\), implies (15) and so completes the proof. \(\Box\)

3. **Proof of Theorem 1**

The upper estimate

$$\|f - B_{\ell, t}(f)\|_p \leq C_{\ell, p} K_{\Delta, \ell}(f, t^{2\ell})_p$$

follows directly from (3), which, as indicated in Section 1, was proved in [DR]. Hence it remains to prove the lower estimate

$$\|f - B_{\ell, t}(f)\|_p \geq C_{\ell, p} K_{\Delta, \ell}(f, t^{2\ell})_p.$$
Lemma 3 implies that there is a number $\mu = \mu(\ell, d) > 1$ such that for $u > \mu$,

$$|m_\ell(u)| \leq \frac{1}{2}. \quad (16)$$

We will keep this special number $\mu$ throughout the proof.

Let $\eta$ be a $C^\infty$-function on $[0, \infty)$ with the properties that $\eta(x) = 0$ for $x > 2$, $\eta(x) = 1$ for $0 \leq x \leq 1$, and $0 \leq \eta(x) \leq 1$ for all $x \in [0, \infty)$. For $t > 0$, we define the operator $V_t$ by

$$(V_t(f))(\xi) = \eta(t|\xi|) \hat{f}(\xi), \quad (17)$$

where $f \in L^p(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$.

According to definition (1), the estimates

$$\|f - V_t/2\mu(f)\|_p \leq C_{\ell, p}\|f - B_{\ell, t}(f)\|_p \quad (18)$$

and

$$t^{2\ell}\|\Delta^{\ell}V_t/2\mu(f)\|_p \leq C_{\ell, p}\|f - B_{\ell, t}(f)\|_p \quad (19)$$

will prove

$$K_{\Delta, \ell}(f, t^{2\ell}) \leq \|f - V_t/2\mu(f)\|_p + t^{2\ell}\|\Delta^{\ell}V_t/2\mu(f)\|_p \leq C_{\ell, p}\|f - B_{\ell, t}(f)\|_p$$

and so complete the proof of Theorem 1. Thus, it has remained to prove (18) and (19).

Let

$$\phi(u) = \left(1 - \eta\left(\frac{u}{2\mu}\right)\right) \frac{(m_\ell(u))^3}{1 - m_\ell(u)} \quad (20)$$

and

$$\psi(u) = \frac{u^{2\ell}\eta\left(\frac{u}{2\mu}\right)}{A_\ell(u)}, \quad (21)$$

with $A_\ell(u)$ and $m_\ell(u)$ the same as in Lemma 2. For $t > 0$, we define two operators $\Phi_t$ and $\Psi_t$ as follows:

$$\left(\Phi_t(f)\right)(\xi) = \phi(t|\xi|) \hat{f}(\xi),$$

$$\left(\Psi_t(f)\right)(\xi) = \psi(t|\xi|) \hat{f}(\xi). \quad (22)$$

It follows from (16), (20) and Lemma 3 that for $u \geq 0$ and $0 \leq j \leq d + 1$,

$$|\phi^{(j)}(u)| \leq C_{\ell, d}\left(\frac{1}{u + 1}\right)^{\frac{3(d+1)}{2}} \quad (23)$$

On the other hand, by (9) and a straightforward computation, we obtain that for $u \geq \frac{\pi}{2}$

$$A_\ell(u) \geq C_{\ell, d}\int_0^{\frac{3}{2}} \left(\sin \frac{u}{2}\right)^{-2\ell} dv \geq C_{\ell, d} > 0 \quad (24)$$
and for $0 < u < \frac{\pi}{2}$

\[
\frac{A_2(u)}{u^{2\ell}} \geq C_{\ell,d} \frac{1}{u^{2\ell}} \int_0^1 (1 - v^2)^{\frac{d-1}{2}} (u v)^{2\ell} dv \geq C_{\ell,d} > 0,
\]

which, together with (21), implies that

\[
\psi \in C^\infty[0, \infty) \quad \text{and} \quad \text{supp} \psi \subset [0, 4\mu].
\]

Now invoking Lemma 4 three times, with $a = \eta, \phi$ and $\psi$, respectively, in view of (23), (26) and the fact that $\eta$ is a $C^\infty$-function with compact support, we obtain from (17) and (22) that for $1 \leq p \leq \infty$,

\[
\sup_{t > 0} \|V_t(f)\|_p + \sup_{t > 0} \|\Phi_t(f)\|_p + \sup_{t > 0} \|\Psi_t(f)\|_p \leq C_p \|f\|_p.
\]

We claim that (18) and (19) follow from (27). In fact, from the identity

\[
(f - V_t/2\mu(f))^\wedge(\xi) = W(t\xi)(f - B_{\ell,t}(f))^\wedge(\xi),
\]

where

\[
W(\xi) := \left(1 - \eta\frac{|\xi|}{2\mu}\right)\left(\frac{m_\ell(|\xi|)}{1 - m_\ell(|\xi|)} + 1 + m_\ell(|\xi|) + (m_\ell(|\xi|))^2\right),
\]

it follows that

\[
f - V_t/2\mu(f) = \Phi_t(f - B_{\ell,t}(f)) + (I - V_t/2\mu)(I + B_{\ell,t} + B^2_{\ell,t})(f - B_{\ell,t}(f)),
\]

where $I$ denotes the identity operator on $L^p(\mathbb{R}^d)$. This, together with (27) and the fact that $\|B_{\ell,t}\|_{(p,p)} \leq C_{\ell,t}$, gives (18).

Similarly, from the identities

\[
(t^{2\ell} \Delta^\ell V_t/2\mu(f))^\wedge(\xi) = \frac{(-1)^\ell t^{2\ell} |\xi|^2 \eta(\xi)}{1 - m_\ell(t|\xi|)} (f - B_{\ell,t}(f))^\wedge(\xi)
\]

it follows that

\[
t^{2\ell} \Delta^\ell V_t/2\mu(f) = (-1)^\ell \Psi_t(f - B_{\ell,t}(f)),
\]

which, again by (27), implies (19). This completes the proof.

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