Robust receding horizon control for constrained linear fractional transformation parameter-dependent systems

F. Wu and Y. Chen

Abstract: A robust receding horizon control (RHC) scheme is proposed for parameter-dependent linear systems with linear fractional parameter dependency and input–output constraints. The cost function is defined over a moving finite horizon as the quadratic performance for future parameter trajectories. The robust stability of the proposed RHC scheme is guaranteed using a parameter-dependent control Lyapunov function as the terminal penalty term, which is available through off-line synthesis procedure. Moreover, it is shown that the domain of attraction will be enlarged and the controlled performance of the RHC scheme will be gradually improved as the upper bound of performance is monotonically decreasing on-line. Both off-line robust control synthesis and on-line RHC computation are formulated and solved using linear matrix inequality optimisation techniques.

1 Introduction

Receding horizon control (RHC), or so called model predictive control (MPC) technique, has been widely used in process control practice [1]. In principle, RHC solves an on-line optimisation problem at each time step to determine an optimal control policy over finite horizon of future time. Typically a sequence of predicted control moves will be calculated, but only the first one is implemented. At the next sampling time, the optimisation problem is solved again with new measurements, and control input is updated to provide feedback control effect. The key advantage of such a methodology is its ability to incorporate hard input/output constraints directly into the optimisation. For linear time-invariant systems with quadratic objective functions, the on-line optimisation is reduced to a tractable quadratic programming (QP) problem, even in the presence of linear input–output constraints. Although using a numerical optimisation as an integral part of this control scheme allows great flexibility, especially concerning constraint handling, it complicates the analysis of stability and performance properties of RHC considerably [2].

To ensure closed-loop stability of the RHC algorithms, terminal equality or inequality constraints were imposed in [3, 4]. Michalska and Mayne [4] proved that robust stability is achievable by introducing terminal conditions at the end of finite moving horizon. The addition of end constraints greatly simplifies the analysis, but inclusion of these constraints usually lack justification or physical motivation. The other approach to guarantee the stability of RHC was based on a terminal cost term [5–7], which should be chosen as close to the remaining infinite-horizon value function as possible. In [8, 9], it was shown that the control Lyapunov function (CLF) could be utilised to improve stability and performance properties of the RHC algorithms for nonlinear systems.

Other than stability concern, another fundamental problem of RHC is its robustness to modelling uncertainties and external disturbances. As part of the modelling effort, various descriptions of the uncertainty have been proposed. For example, uncertainties on the impulse or step-response coefficients provide a practical description in many applications [10]. Another useful description of model uncertainty is polytopic uncertainty, which can be used as a conservative approach to model a nonlinear system \( x(k + 1) = f(x(k), u(k), k) \), when the Jacobian \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial u} \) lies in a polytope [11]. More elaborated uncertainty descriptions were further exploited in [12] for robust RHC formulation. By modifying the on-line optimisation problem to a ‘min-max’ problem, Campo and Morari [13] studied the worst-case performance for FIR plants, where the uncertain model was given in terms of bounds on the impulse response coefficients. Based on infinite horizon quadratic cost function, robust MPC schemes were also developed in [11, 14] for a general class of uncertain linear systems with time-varying perturbations. The controller design was formulated as an optimisation problem of the worst-case objective function over infinite moving horizon, subject to input–output constraints. Reference [15] generalised the robust RHC scheme presented in [11] by including \( N \) free move terms in the quadratic cost function. Another improved robust RHC scheme was also proposed in [16], where parameter-dependent Lyapunov functions were used to relax the requirement of quadratic stability. An detailed review of research activities on robust MPC algorithms can be found in [10].

In this paper, we would like to solve the robust receding horizon control problem by integrating off-line synthesis of a CLF and on-line RHC computation for a general class of parameter-dependent systems. Our proposed approach preserves robust stability through the use of a parameter-dependent CLF as a terminal cost. The control design objective is to find a control policy under input–output

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The authors are with the Department of Mechanical and Aerospace Engineering, North Carolina State University, Raleigh, NC 27695
E-mail: fwu@eos.ncsu.edu

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constraints through on-line optimisation. Specifically, we are trying to synthesise a state-feedback control sequence on-line that would result in a cost close to the solution of the infinite horizon optimal control problem with reasonable amount of computational effort. Ideally, the terminal weight should be chosen as the minimum of the constrained infinite horizon cost. Then the optimal infinite horizon controller with its stability property will be recovered regardless of the horizon length used. However, for a general uncertain system, it is difficult to find a suitable terminal penalty term. Instead, a CLF derived from a constrained robust control design will be used to ensure the stability of the proposed MPC algorithm. As a global CLF, it has the advantage of being determined off-line, thus reducing on-line computational effort. Moreover, such a Lyapunov function provides an upper bound, though conservative, of the cost index over the remaining infinite horizon. A preliminary version of the paper was published in [17], in which a single quadratic Lyapunov function was used as terminal penalty and the on-line robust RHC was solved by exploiting vertices of the uncertainty set.

The contributions of the paper are mainly in two folds: First, we derived a parameter-dependent CLF for constrained parameter-dependent linear fractional transformation (LFT) systems, which will reduce the conservatism caused by a single quadratic Lyapunov function. Second, the synergistic integration of off-line synthesis of CLF and on-line RHC will provide robust stability and optimised performance for parameter-dependent LFT systems subject to input–output constraints. The potential benefits of standard RHC formulation from the knowledge of a CLF has been suggested in [8, 9]. The idea is to utilise the stabilising properties of CLFs, without sacrificing the performance advantages of RHC.

The notation is fairly standard, and will be defined as needed. $\mathbf{R}$ stands for the set of real numbers, and $\mathbf{R}^n$ represents $n$-dimensional vector set. For two integers $k_1$ and $k_2$, $k_1 < k_2$, we denote $[k_1, k_2] = \{k_1, k_1+1, \ldots, k_2\}$. $\mathbf{R}^{m \times n}$ is the set of real $m \times n$ matrices. In large symmetric matrices, terms denoted $\ast$ will be induced by symmetry. The transpose of a real matrix $M$ is denoted by $M^T$. A block diagonal matrix with matrices $X_1, \ldots, X_p$ on its diagonal will be denoted by $\text{diag}(X_1, \ldots, X_p)$. We use $\Sigma^{n \times n}$ to denote the set of real, symmetric $n \times n$ matrices, and $S^{n \times n}$ for positive definite matrices. $\lambda_{\text{max}}(M)$ denotes the maximum eigenvalues of a positive-definite matrix $M$, and $M^{1/2}$ stands for its square root. For $x \in \mathbf{R}^n$, the Euclidean norm is $\|x\| := (x^T x)^{1/2}$. An infinite sequence $x := [x_1, x_2, \ldots]$, with $x_i \in \mathbf{R}^n$, is said to be in $\ell^2$ if $\sum_{i=1}^{\infty} x_i^2 < \infty$.

The rest of the paper is organised as follows. In Section 2, a CLF is derived for a constrained robust control problem, and it is shown that such a CLF provides an upper bound over infinite horizon performance index. Section 3 is devoted to the formulation of the state-feedback robust RHC problem for LFT-type parameter-dependent systems, and an on-line synthesis scheme is proposed with the stability guarantee. In Section 4, a two mass-spring-damper system with input and state constraints will be utilised to demonstrate the advantages of proposed robust RHC scheme, and compare with other RHC techniques. The conclusions will be drawn from Section 5.

2 Constrained CLF

Lyapunov theory is the most successful and widely used tool in control system analysis and control design. However, there currently does not exist systematic methods of obtaining Lyapunov functions for general dynamic systems subject to input–output constraints. The synthesis condition presented below provides a CLF candidate for robust stabilisation of a parameter-dependent system.

Consider a discrete-time, parameter-dependent LFT system in the form of

$$
\begin{bmatrix}
    x(k+1) \\
    q(k) \\
    y(k)
\end{bmatrix} =
\begin{bmatrix}
    A & B_0 & B_2 \\
    C_0 & D_{00} & D_{02} \\
    C_2 & D_{20} & D_{22}
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    p(k) \\
    u(k)
\end{bmatrix}
$$

(1)

$$
p(k) = \Theta q(k)
$$

(2)

where the state $x \in \mathbf{R}^n$, the control input $u \in \mathbf{R}^{m_u}$, and pseudo-input and -output $p, q \in \mathbf{R}^{m_p}$. $y \in \mathbf{R}^{m_y}$ is the constrained output. All of the state-space matrices have compatible dimensions. Suppose the system is well-posed, that is $I - \Theta D_{00}$ is non-singular for all admissible $\Theta$. In general, $D_{00} \neq 0$ and the state-space data depends on the parameter in an LFT fashion. $\Theta$ is a time-varying parameter matrix which obeys the following structure

$$
\Theta = \{\text{diag}\{\theta_1 I_{r_1}, \theta_2 I_{r_2}, \ldots, \theta_s I_{r_s}\}; \theta_i; \ell_2, \|\theta_i\| \\
\leq 1, \ i = 1, 2, \ldots, s\}
$$

(3)

where the operator norm on $\theta_i$ is the induced $\ell_2$ norm and $\sum_{i=1}^{s} r_i = n_p$. This general uncertainty description includes affine structured uncertainties as a special case. The vertex set of parameter $\Theta$ is denoted by $\Theta^v$, which contains $2^s$ extreme points $\Theta^v_i$.

In the following derivation, we will often resort to an alternative representation of the parameter-dependent LFT system (1)–(2) as

$$
\begin{bmatrix}
    x(k+1) \\
    q(k) \\
    y(k)
\end{bmatrix} = \begin{bmatrix}
    A & B_0 \Theta & B_2 \\
    C_0 & D_{00} \Theta & D_{02} \\
    C_2 & D_{20} \Theta & D_{22}
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    \tilde{q}(k) \\
    u(k)
\end{bmatrix}
$$

(4)

$$
\tilde{q}(k) = I q(k)
$$

(5)

Note that the new formulation has all its state-space data as affine functions of the parameter, and the feedback channel between $q$ and $\tilde{q}$ has unity gain. It is also assumed that

(A1) the state $x$ is measurable in real-time and the values of parameter $\Theta$ over a future horizon $[k, k+N]$ is available at any time instant $k$.

(A2) the LFT system (4)–(5) is parametrically stabilisable, that is, there exist parameter-dependent matrix functions $X_0, X_{\Theta^v}; \mathbf{R}^n \rightarrow S^{n \times n}$, $S_{\Theta} \rightarrow S^{n_y \times n_y}$ and a matrix function $F_{\Theta}; \mathbf{R}^{d_{\Theta}} \rightarrow \mathbf{R}^{n_y \times n_y}$, such that

$$
\begin{bmatrix}
    (A + B_2 F_{\Theta})^T & (C_0 + D_{02} F_{\Theta})^T \\
    \Theta B_0^T & \Theta D_{00}
\end{bmatrix}
\begin{bmatrix}
    X_{\Theta^v}^{-1} & 0 \\
    0 & S_{\Theta}
\end{bmatrix}
\times
\begin{bmatrix}
    A + B_2 F_{\Theta} & B_2 \Theta \\
    C_0 + D_{02} F_{\Theta} & D_{00} \Theta
\end{bmatrix}
\begin{bmatrix}
    X_{\Theta^v}^{-1} & 0 \\
    0 & S_{\Theta}
\end{bmatrix}
< 0
$$

for any $\Theta \in \Theta$.

(A1) assumes the existence of a parameter preview mechanism such that the parameter values over a future time window $[k, k+N]$ are available at any time instant $k$. This is not a very restrictive assumption since many uncertain systems have predictable behaviour in near future, although their long term information is hardly known except belonging to a bounded set. For example a slow dynamic systems (like chemical process) will have this type of characteristics. The
assumption (A2) reduces the conservatism associated with quadratic stability by considering a parameter-dependent Lyapunov function with a parameter-dependent control law.

Given an initial condition \( x(0) \), our objective for constrained control design is to find a parameter-dependent control law \( u(k) = F_{\Theta}(x(k)) \) such that the infinite horizon cost index

\[
J_m = \max_{\Theta \in \Theta} \sum_{k=0}^{\infty} \{ x^T(k)Qx(k) + u^T(k)Ru(k) \},
\]

\( Q \in \mathbb{S}^{n \times n}, \quad Q \succeq 0, \quad R \in \mathbb{S}^{r \times r} \)

is minimised with respect to the parameter set \( \Theta \) and input–output constraints. Specifically, we define the \( n_r \)-tuple of integers \( \{ l_1, \ldots, l_{n_r} \}, \sum_{k=1}^{n_r} l_k = n_r \) and the \( n_m \)-tuple of integers \( \{ m_1, \ldots, m_{n_m} \}, \sum_{k=1}^{n_m} m_k = n_m \), and partition the input–output constraints conformably. Then the input–output constraints will be enforced by

\[
\max_{k \geq 0} \| u_k \| \leq \eta_k, \quad \xi \in [1, n_r] \\
\max_{k \geq 0} \| v_k \| \leq \gamma_k, \quad \eta \in [1, n_m]
\]

The input constraints represent physical limits of the actuators while the output constraints can arise from performance requirements and safety considerations. Note that both component-wise peak-bounded and Euclidean norm-bounded input–output constraints are included in above definition [11], which must be satisfied over infinite time horizon.

It is usually difficult to find a CLF for an uncertain system when input–output constraints are imposed. However, the following theorem provides a constructive way to determine such a CLF for the parameter-dependent LFT system that we are interested in.

**Theorem 1:** Given a scalar \( \gamma > 0 \), the LFT system \((1) - (2)\) with the initial condition \( x(0) \) is robustly stabilised by a parameter-dependent state-feedback controller and has its infinite horizon performance index \( J_m < \gamma^T(0)X_{\Theta}(0)x(0) \) if there exist matrices \( X_i \in \mathbb{S}^{n \times n}, \quad S_i \in \mathbb{S}^{n \times n}, M_i \in \mathbb{R}^{r \times r}, T_{ij} \in \mathbb{S}^{n_m \times n_m}, \quad i \in [1, 2], \quad j \in [1, 2] \), \( \xi \in [1, n_r] \), \( \eta \in [1, n_m] \), \( G_1 \in \mathbb{R}^{n \times n} \), \( G_2 \in \mathbb{R}^{n_m \times n_m} \), and \( G_3 \in \mathbb{R}^{n_m \times r} \), such that for \( i, j \in [1, 2] \)

\[
\begin{bmatrix}
1 \\
X(0) \\
\end{bmatrix} \geq 0,
\]

\[
\begin{bmatrix}
 -(G_1 + \Gamma_1^T - X_j) & * & * \\
0 & -(G_2 + \Gamma_2^T - S_j) & * \\
R^{1/2}M_i & 0 & -\gamma L_j \\
AG_1 + B_2 M_i & B_2 \Theta_j^T G_2 & 0 \\
C_0 G_1 + D_{02} M_i & D_{02} \Theta_j^T G_2 & 0 \\
Q^{1/2} G_1 & 0 & 0 \\
* & * & * \\
* & * & * \\
* & * & * \\
-\xi_j & * & * \\
0 & -S_j & * \\
0 & 0 & -\gamma L
\end{bmatrix} \leq 0,
\]

where

\[
\begin{bmatrix}
\pi_i^T \xi_i \\
M_i^T \xi_i \\
G_1 + G_2^T - X_j \\
\end{bmatrix} \geq 0, \quad \xi \in [1, n_r],
\]

\[
\begin{bmatrix}
\pi_i^T \eta_i \\
M_i^T \eta_i \\
G_3 + G_3^T - T_{\eta i} \\
C_{2,0} G_1 + D_{20,0} M_i \\
C_0 G_1 + D_{02} M_i \\
D_{02} \Theta_j^T G_3 & 0 \\
T_{\eta i} & 0 \\
\end{bmatrix} \geq 0, \quad \eta \in [1, n_m]
\]

Theorem 1 provides a constructive way to determine such a CLF for the parameter-dependent LFT system that we are interested in.
By Schur complement, the above inequality is equivalent to

\[
\begin{bmatrix}
-\gamma I & 0 \\
0 & \Theta D_{10}^T
\end{bmatrix}
\begin{bmatrix}
X_{\Theta}^{-1} & 0 \\
0 & S_{\Theta}^{-1}
\end{bmatrix}
\begin{bmatrix}
\Theta D_{10}
\end{bmatrix}
\leq 0
\]

Multiplying \([x^T(k) - \tilde{q}^T(k)]\) and its transpose on both sides of (14), we have

\[
\gamma x^T(k + k)X_{\Theta}^{-1}(x(k) + 1) + \tilde{q}^T(k)S_{\Theta}^{-1}(q(k)
\]

\[
+ x^T(k)Qx(k) + u^T(k)Ru(k)
\]

\[
< \gamma x^T(k)X_{\Theta}^{-1}(x(k) + 1) + \tilde{q}^T(k)S_{\Theta}^{-1}(q(k)
\]

(15)

Defining a Lyapunov function \(V(x) := \gamma x^T X_{\Theta}^{-1} x\) and applying (12), it is clear that the Lyapunov function is monotonically decreasing. Therefore \(\lim_{k \to \infty} x(k) = 0\) and the LFT system is robustly stable. In addition, summing up \(k = 0\) to \(\infty\), (15) leads to \(J_\infty < V(x(0)) = \gamma x^T(0)X_{\Theta}^{-1} x(0)\) as desired.

Similarly, we get from condition (9) for each partitioned output

\[
\begin{bmatrix}
\tilde{T}_J & 0 \\
0 & \tilde{T}_J^{-1}(\Theta)
\end{bmatrix}
\begin{bmatrix}
(C_{2,\eta} + D_{22,\eta}F_{\Theta})^T & 0 \\
0 & \Theta D_{10}^T
\end{bmatrix}
\begin{bmatrix}
C_{2,\eta} + D_{22,\eta}F_{\Theta} \\
0
\end{bmatrix}
\geq 0
\]

which implies (using (12))

\[
\|y_\eta(k)\|^2 = \|(C_{2,\eta} + D_{22,\eta}F_{\Theta})x(k) + D_{20,\eta}Q\eta(k)\|^2 \leq \tilde{T}_J
\]

for any \(k\) and \(\Theta \in \Theta\). Therefore \(\max_{k \geq 0} \|y_\eta(k)\| \leq \gamma_\eta, \eta \in I_{[1, n_c]}\).

On the other hand, for each partitioned input, we have

\[
\max_{k \geq 0} \|u_\xi(k)\|^2 = \max_{k \geq 0} \|F_{\Theta,\xi} x(k)\|^2 \leq \max_{i \in I, j \leq 1} \|F_{\Theta,\xi} X_{\Theta}^{-1} \|^2
\]

\[
\leq \lambda_{\max}(M_{\Theta,\xi}G_1^{-1}X_{\Theta}^{-1}G_1^{-T}M_{\Theta,\xi}^T)
\]

which is less than \(\tilde{T}_J\) by (8).

The synthesis conditions (6)–(9) are in the form of linear matrix inequalities (LMIs) and can be solved efficiently using interior-point algorithms [19]. From the proof of Theorem 1, it is clear that \(V(x) = \gamma x^T X_{\Theta}^{-1} x\) serves as a control Lyapunov function. It also specifies an estimated domain of attraction

\[
S = \{x: x^T X_{\Theta}^{-1} x < 1, \forall \Theta \in \Theta\}
\]

for the constrained LFT system. In addition, the parameter-dependent CLF provides an upper bound of infinite horizon performance for the LFT system starting from any step \(k\). Nevertheless, the results could be conservative due to infinite horizon formulation of the cost criterion. In the presence of parametric uncertainty, the controller may have a strong dependence on the state of the system [11]. Thus solving the optimisation problem on-line could improve the performance and enlarge its domain of attraction significantly. This motivates the use of on-line optimisation algorithm to improve the controlled system.

3 RHC for LFT systems

Robust control techniques were developed to address modelling uncertainty exclusively, however, no straightforward extension of this control approach is available to deal with input–output constraints. On the other hand, RHC provides a means to incorporate hard input–output constraints into on-line control process. Therefore it is desirable to combine existing RHC and robust control techniques to develop an unified control design framework which can handle both modelling inaccuracy and hard input–output constraints simultaneously.

The knowledge of a CLF has a potential benefit to standard RHC algorithm in that it enhances stability without sacrificing the performance advantages [8]. From optimal control viewpoint, RHC techniques produce local approximation to the value function through on-line optimisations, and use them to generate a control law. However, lack of global information may lead RHC astray in terms of stability. On the other hand, the CLF derived from off-line synthesis provides a crude approximation of the value function in global sense. CLF-based approaches emphasise on the stability guarantee, but may result in poor performance when the CLF does not closely resemble the value function [20]. As such, a proper combination of CLFs and RHC is anticipated to overcome the limitations imposed by individual control techniques.

Let \(x(k + i|k)\), \(y(k + i|k)\) denote the predicted state and output based on the measurement at step \(k\). \(u(k + i|k)\) is the predicted control action for step \(k + i\). In particular, \(x(k + i|k) = x(k), y(k + i|k) = y(k)\) and \(u(k + i|k) = u(k)\). Due to assumption (A1), the information of parameter \(\Theta\) over a finite time window \([k, k + N]\) is available for feedback control use. Thus, the LFT dynamics over the future horizon is specified as an linear time-varying (LTV) system. We will define the performance index associated with this LTV system as

\[
J_N(k) := \sum_{i = 0}^{N-1} \left[ x^T(k + i|k)Qx(k + i|k)
\right.
\]

\[
+ u^T(k + i|k)Ru(k + i|k)\]

\[
+ \gamma x^T(k + N|k)X_{\Theta}^{-1} x(k + N|k)
\]

(16)

where \(\gamma x^T X_{\Theta}^{-1} x\) is the terminal penalty term and is chosen as the CLF derived in Theorem 1. Clearly, the performance index \(J_N(k)\) consists of two parts: the cost over the finite
horizon \([k, k + N]\) and a performance upper-bound over the remaining infinite time horizon \([k + N, \infty)\).

At each sampling time \(k\), the control objective of the robust RHC problem is to minimise the cost function \(J_S(k)\) subject to parametric uncertainty and input–output constraints. This can be formally stated as follows:

**Definition 1 (Finite-horizon robust RHC problem):** Given the LFT system (1)–(2) with its parameter set \(\Theta\), and state measurement at sampling time \(k\) as \(x(k)\), partition the input–output vectors into \(n_x\)-tuple and \(n_u\)-tuple respectively. The finite-horizon robust RHC problem at step \(k\) is defined as the following optimisation problem

\[
\min_{u(k)} J_S(k)
\]

subject to (1)–(2), \(x(k + N) \in S\)

\[
\max_{i \geq 0} \|u_x(k+i|k)\| \leq \eta_x, \quad \xi \in \mathbf{I}[1, n_x]
\]

\[
\max_{i \geq 0} \|y_x(k+i|k)\| \leq \eta_y, \quad \eta \in \mathbf{I}[1, n_u]
\]

The stability requirement of finite-horizon robust RHC problem is embedded in the above formulation, that is, the closed-loop system must be robustly stabilised by constrained control input. In the robust RHC problem, an explicit CLF designed to improve the closed-loop stability at the initial step. This prediction capability allows solving constrained optimal control problem on-line. By treating the constrained LFT system using the parameter-dependent CLF derived in Step 0 as its terminal penalty term. Solving the LMI problem (17) at step \(k\), we obtained an optimal control sequence \(u^*(k)\), \(u^*(k + 1)\), \ldots, \(u^*(k + N - 1)\) over the finite horizon \([k, k + N]\). Since the optimal control policy determined at step \(k\) is feasible but sub-optimal for step \(k + 1\), it is clear that a feasible control sequence for step \(k + 1\) will be

\[
\{u^*(k + 1), \ldots, u^*(k + N - 1)\}, F_{\Theta(k+N)}x(k+N)\}
\]

Using this control sequence, we have

\[
J_S(k+1) - J_S^*(k) = x^T(k+N|k)Qx(k+N|k)
\]

\[
+ u^T(k+N|k)Ru^*(k+N|k)
\]

\[
+ \chi^T(k+N|k)X_{\Theta(k+N)}^{-1}(k+N|k)
\]

\[
+ x(k+N|k) - \chi^T(k+N|k)
\]

\[
+ X_{\Theta(k+N)}^{-1}(k+N|k)x(k+N|k) < 0
\]

Therefore we conclude that \(J_S^*(k+1) \leq J_S^*(k)\) for any future time steps.

Finally, we choose a Lyapunov function candidate \(\tilde{V}(x(k)) = \min_{\Theta(k)} J_S(k)\). Invoking standard Lyapunov stability arguments, it can be easily shown that the LFT system (1)–(2) under the robust RHC scheme is asymptotically stable for all \(\Theta \in \Theta\).

The proposed robust RHC computational scheme for the parameter-dependent LFT systems with input–output constraints can be summarised in the following steps:

**Step 0** Synthesize a CLF by applying Theorem 1. The resulting CLF \(V(x) = \chi^T X_{\Theta}^{-1} x\) will provide guaranteed stability and a performance upper-bound under parametric uncertainty and input–output constraints.

**Step 1** Formulate the finite-horizon robust RHC problem for the constrained LFT system using the parameter-dependent CLF derived in Step 0 as its terminal penalty term.

**Step 2** At each time step \(k\), determine the optimal control input \(u(k)\), and move to the next sampling time.

A major concern of on-line RHC computational schemes is about their feasibility. The following theorem relates feasibility of the on-line robust RHC algorithm to its solvability at the initial step.

**Theorem 2:** Given an initial condition \(x(0)\), if the finite-horizon robust RHC problem for the LFT system (1)–(2) is solvable at \(k = 0\), then it is always feasible at any step \(k\). The controlled LFT system is asymptotically stable and its performance index \(J_S(k)\) monotonically decreases as \(k \to \infty\). Moreover, the optimal control input at step \(k\) is given by

\[
u(k) = \arg \min_{u(k)} J_S(k)\]
The selection of prediction horizon is dictated by our knowledge of plant dynamics. Whenever it is possible, choosing a larger moving horizon length will be helpful to improve point-wise controlled performance. If \( N = 0 \), the proposed RHC scheme degenerates to a robust control strategy without on-line adaptation. On the other hand, when \( N \to \infty \), the system dynamics is completely known. Then, a large enough prediction horizon will guarantee the stability of RHC algorithm without terminal penalty term [20].

4 Example

In this section, we will demonstrate advantages of the proposed robust RHC scheme using a challenging control problem from [21]. Consider the two-mass-spring-damper system shown in Fig. 1, which is a generic model of a dynamic system with non-collocated sensor and actuator. A control force acts on body 1, and the position of body 2 is measured resulting in a non-collocated control problem. This system can be represented in the state-space form as

\[
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t) \\
    x_4(t)
\end{bmatrix} = \begin{bmatrix}
    0 & 0 & 1 & 0 \\
    -k_1/m_1 & -k_2/m_1 & b/m_1 & 0 \\
    k_1/m_2 & -k_2/m_2 & b/m_2 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t) \\
    x_4(t)
\end{bmatrix} + \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    1/m_1
\end{bmatrix} u(t) + \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix} \triangleq (18)
\]

where \( x_1, x_2 \) are positions of bodies 1, 2, respectively, \( x_3, x_4 \) are the velocities of bodies 1, 2 and \( u \) is control input. The masses 1 and 2, and spring constant are time-varying parameters. We assume that \( 0.8 \text{ kg} \leq m_1, m_2 \leq 1.2 \text{ kg} \) and \( 1 \text{ N/m} < k_1 < 1.5 \text{ N/m} \). There is a small damping effect, that is \( b = 0.01 \text{ Ns/m} \). It is also assumed that all state information can be used by feedback controllers.

Using Euler discretisation scheme with a sampling time \( t_s = 0.1 \text{ s} \), the dynamic equation of mass-spring-damper system (18) can be transformed to a discrete-time LFT system

\[
\begin{bmatrix}
    x(k+1) \\
    q(k)
\end{bmatrix} = \begin{bmatrix}
    A & B_0 & B_2 \\
    C_0 & D_{00} & D_{02}
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    p(k) \\
    u(k)
\end{bmatrix},
\]

\[
p(k) = \text{diag}(\theta_{k_1}, \theta_{m_1}, \theta_{m_2}) q(k)
\]

with nominal values \( k_1 = 1.25 \text{ N/m} \), \( m_1 = 1.0 \text{ kg} \) and \( m_2 = 1.0 \text{ kg} \).

We are mainly interested in minimising quadratic performance of the mass-spring-damper system subject to input–state constraints. The hard input constraint is \( |u| \leq 5.0 \text{ N} \), and the positions \( x_1, x_2 \) are constrained by \( |x_1|, |x_2| \leq 2.0 \text{ m} \). No constraints are imposed on the velocities of masses 1 and 2. The performance index is given by (16) with \( Q = \text{diag} \{1, 1, 1, 1\} \), \( R = 0.01 \). The length of prediction horizon is chosen as \( N = 5 \).

Using LMI synthesis condition (6)–(9) in Theorem 1, one could find a robust CLF \( V(x) = x^T X_0 x \) with \( X_0 = \sum_{i=1}^{4} \alpha_i X_i \) for the given LFT system. To expand stability region and improve local performance, one can utilise the CLF as terminal penalty term, and solve a finite-horizon robust RHC on-line through the LMI optimisation (17). The feasibility of these optimisation problems are always guaranteed by Theorem 2.

For simulation purpose, we choose

\[
\begin{align*}
    k_1 &= 1.0 + 0.25 \sin k \text{ (N/m)}, \\
    m_1 &= 1.2 + 0.2 \cos 2k \text{ (kg)}, \\
    m_2 &= 0.8 + 0.1(\sin 3k + \cos 4k) \text{ (kg)}
\end{align*}
\]

Starting from several initial positions \( (0.5, 0.2) \text{ m}, (0.3, -0.4) \text{ m}, (1.6, 1.2) \text{ m} \) and \( (0.9, -1) \text{ m} \) with zero initial velocities, the controlled performance and controller input of mass-spring-damper system using the robust RHC are shown in Fig. 2. Note that the initial conditions for cases 3 and 4 were chosen outside the domain of attraction determined from the off-line control design. It can be seen that the input constraint on \( u \) is satisfied over the entire simulation duration. In the first case, the state \( x_1 \) first moves away from origin, and then converges to its equilibrium after 5 s. All of other cases converge to zero quickly.

Next, we compare different RHC schemes with fixed prediction horizon \( N = 5 \). The first one is the proposed
RHC scheme that is derived by solving LMI optimisation problem (17) on-line. The second approach is a robust RHC algorithm with the robust state-feedback law determined at each step $k$ by solving the synthesis conditions in [14] repeatedly for remaining infinite horizon performance. The last control algorithm is a conventional RHC scheme without terminal term. Here we assume the system model used by the conventional RHC scheme is

![Fig. 2](image1.png)

**Fig. 2** Performance of finite-horizon robust RHC on the mass-spring-damper system

$(0.5, 0.2)m$ – solid; $(0.3, -0.4)m$ – dash line; $(1.6, 1.2)m$ – dot line; $(0.9, -1)m$ – dash-dot line

$a$ Position of mass 2

$b$ Control force

![Fig. 3](image2.png)

**Fig. 3** Comparison of proposed robust RHC (solid), robust RHC with infinite horizon performance (dash dot), and conventional RHC (dot)

$a$ Position of mass 1

$b$ Control force

$c$ Performance
different from the actual system. That is there is some discrepancy between assumed model and actual system. To make the performance cost comparable to other approaches, an infinite performance bound rendered by optimal control law is appended to the conventional RHC after finite $N$ steps. For this study, the initial positions of two masses are chosen as 0.5 m and 0.2 m, and their initial velocities are zeros. The state and input constraints are the same as before. As shown in Fig. 3, the conventional RHC algorithm without terminal penalty performed poorly when nominal model had discrepancy from the actual system, while the conventional RHC was capable of stabilising the system with considerable conservatism. The proposed robust RHC scheme performed better than both approaches, and robustly stabilised against parametric uncertainties. This clearly indicates the advantages of robustified on-line optimisations. Moreover, hard input–state constraints have been incorporated into the on-line optimisation thanks to the flexibility of LMI optimisation techniques.

For comparison, Table 1 shows the average computational cost of three RHC algorithms. As can be seen, the conventional RHC has the shortest computational time. The proposed robust RHC has relatively large computational cost compared with infinite horizon robust RHC, but with better controlled performance.

### 5 Concluding remarks

In this paper, a robust RHC scheme was proposed for parameter-dependent LFT systems with input–output constraints. Using a parameter-dependent CLF as the terminal cost, the quadratic cost function consists of performance over a finite moving horizon and that of the remaining infinite horizon. By integrating off-line synthesis of a CLF and on-line RHC computation, our proposed robust RHC approach will enlarge the stability region and improve the controlled performance from off-line control designs. Moreover, LMI optimisation techniques were used to find the CLF for the constrained robust control problem and to conduct the on-line RHC computation. It was shown that asymptotic stability of the robust RHC is achieved if the on-line optimization problem is solvable at the initial step.

### 6 Acknowledgment

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### 7 References