Bifurcation analysis of a circuit-related generalization of the shipmap

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Abstract

In this paper a three-parameter bifurcation analysis of a piecewise-affine map is carried out. Such a map derives from a well-known map which has good features from its circuit implementation point of view and good statistical properties in the generation of pseudo-random sequences. The considered map is a generalization of it and the bifurcation parameters take into account some common circuit implementation non-idealities or mismatches. In view of a robust design of the map, this bifurcation analysis should come before a statistical analysis, to find a set of parameters ensuring both robust chaotic dynamics and robust statistical properties.
1 Introduction

In the last decades, many works concerning engineering applications of chaos have been focused on either the analysis (direct problem) or the design (inverse problem) of nonlinear dynamical systems to be used in telecommunication frameworks. In many cases, the possible dynamics of most of them have been extensively analyzed a priori from a bifurcation point of view, in order to find out the regions of the parameter space ensuring chaotic dynamics. After this preliminary study, both model analysis and system synthesis have been carried out on the basis of the statistical characteristics of the considered systems, used as chaos generators.

Such a kind of working procedure is the most direct way to design a system on the basis of a given mathematical model. However, dealing with the physical implementation of engineering systems (typically, electronic circuits), it is often necessary to re-design/analyse the implemented systems to take into account mismatches with respect to the original models. In such cases, each original mathematical model should be re-defined (introducing new parameters related with the real implementation) and new analyses should be carried out. First of all, a bifurcation analysis should be performed, to find the regions of the new parameter space allowing the presence of a unique invariant chaotic interval. Then, a statistical analysis within such parameter regions should be carried out, to ensure that the statistical distribution of the trajectory points is uniform over such an interval.

In this paper we shall propose a bifurcation analysis of a Piecewise-Affine Map (PWAM), defined starting from a known map and introducing new bifurcation parameters to take into account some common circuit implementation non-idealities (not related to a specific electronic implementation). We shall focus our attention on the piecewise-affine map known as “shipmap” [Callegari et al., 2000], since it has good features from its implementation point of view and good statistical properties in the generation of pseudo-random sequences [Delgado, 1999]. This kind of analysis is a first step towards a robust design of the map and should be the prelude to a statistical analysis, to find regions in the parameter space ensuring both robust chaotic dynamics and robust statistical properties.

The bifurcation analysis will be carried out by varying the parameters within ranges that are chosen just on the basis of the mathematical formulation of the map. So doing, we are guaranteed that the reasonable ranges from a circuit point of view are covered by the analysis: the real mismatch ranges, whose exact estimation depends on the specific circuit implementation and on the adopted technology, will be certainly contained within the considered domain of analysis. The proposed results are based on both theoretical tools
and numerical simulations of the map dynamics.

In Sec. 2, we shall define the considered PWAM and the three bifurcation parameters. Section 3 will be devoted to sketch a two-dimensional bifurcation scenario through brute-force one-dimensional bifurcation diagrams. In Sec. 4 we shall derive analytical expressions for some bifurcation sets. In Sec. 5 some further numerical results will be provided. Finally, some concluding remarks will be discussed in Sec. 6.

2 Map definition

Preliminarily, we recall some basic definitions:

- A subset $A$ of the state space is said invariant for a map $T$ if the set $A$ is mapped onto itself ($T(A) = A$), i.e., if all points of $A$ are images of points of $A$.
- A subset $A$ of the state space is said trapping set for a map $T$ if the set $A$ is mapped into itself ($T(A) \subset A$), i.e., if the images of all points of $A$ belong to $A$.
- A closed invariant set $A$ is called asymptotically stable or attracting or absorbing if a neighborhood $U$ of $A$ exists such that $T(U) \subset U$ and $T^n(x) \rightarrow A$ as $n \rightarrow +\infty$ for any $x \in U$.
- The basin of attraction of an attracting set $A$ is the set of all points that generate trajectories asymptotically converging to $A$: $B(A) = \{ x : T^n(x) \rightarrow A \text{ as } n \rightarrow +\infty \}$. Of course, $A \subset B(A)$.

The shipmap $\tilde{M}(x) : \mathbb{R} \rightarrow \mathbb{R}$ is strictly related to the tent map and can be derived from it through a cutting and flipping technique. The main advantage in doing so is that, for a given parameter set ensuring chaotic dynamics within an invariant interval $I$, the basin of attraction $B(I)$ of $I$ is enlarged, while preserving the main statistical properties of the tent map [Callegari et al., 2000].

A possible sequence of operations (circuit blocks) providing the shipmap is shown in Fig. 1. Such operations can be performed through high-level building blocks like the ones proposed in [Callegari et al., 2000]. A further step in the synthesis procedure is the low-level hardware implementation of such blocks, that usually yields to non-idealities. In particular, in the shipmap case, the main possible implementation non-idealities concern:

- slope of the linear branches;
breakpoint positions;
• quote of the sail maximum;
• positions of the stepwise transitions in the block B6

The parameters introduced by this block scheme can be related also to other architectures. Other parameters could be introduced \textit{a posteriori} on the basis of circuit simulations to model, e.g. mismatches due to circuit dynamics, delay times, parasitic effects [Callegari \textit{et al.}, 2000].

Each block can introduce mismatches with respect to the ideal operation, thus leading to a non-ideal version $M(x)$ of the shipmap, where, besides the mismatch $\delta w$ introduced by the block B6, the fully controllable parameters $p_k$ are replaced by mismatched parameters $P_k = p_k + \delta p_k$. For instance, the block B3 can introduce a slope $P_2$ slightly different from $p_2$, whereas the block B4 can introduce a vertical offset $P_3$ different from $p_3$. In this paper, we shall focus on the non-idealities due to the blocks B3 and B4. In particular, the error $\delta p_3$ introduced by the block B4 influences the parameters $P_4(= P_1 - P_3/P_2)$ and $P_5(= P_1 + P_3/P_2)$, which are assumed to be always symmetric with respect to $P_1$ (i.e., $\delta p_5 = -\delta p_4$). Also the value $M(P_1)$ is influenced by $\delta p_3$, but such a value depends on the block B6 too, then we shall take into account the effect of a parameter $\delta m = \delta p_3 + \delta p_6$ on $M(P_1)$.

The parameter $\delta w$ can change the structure of the map, by introducing two further branches, as shown in Fig. 2. With this in mind, we shall consider the effects of the other three main parameter mismatches ($\delta p_2$, $\delta p_4$, and $\delta m$), to carry out a bifurcation analysis for a four-branches PWAM.

The aforesaid effects can be summarized by the following map definition:

$$M(x) = \begin{cases} -P_2(x - P_4) & x < P_4 \\ P_3 + P_6 - P_2|x - P_1| & P_4 \leq x < P_5 \\ P_2(x - P_5) & x \geq P_5 \end{cases} \quad (1)$$

Henceforth, we shall fix $P_1 = p_1 = 0.5$ (i.e., $\delta p_1 = 0$), $p_2 = 2$, $p_4 = 0.25$, $p_5 = 0.75$, and $p_3 + p_6 = 1$ and we shall carry out our bifurcation analysis with respect to $\delta p_2$, $\delta p_4$, and $\delta m$. For $\delta p_2 = \delta p_4 = \delta m = 0$ the shipmap has been proven to have good statistical properties and to exhibit chaotic
dynamics \cite{Callegari et al., 2000}. Summarizing, the map we shall consider is defined as follows:

\[ M(x) = \begin{cases} 
-(2 + \delta p_2) [x - (0.25 + \delta p_4)] & x < (0.25 + \delta p_4) \\
1 + \delta m - (2 + \delta p_2) |x - 0.5| & (0.25 + \delta p_4) \leq x < (0.75 - \delta p_4) \\
(2 + \delta p_2) [x - (0.75 - \delta p_4)] & x \geq (0.75 - \delta p_4) 
\end{cases} \]

(2)

3 Bifurcation scenarios

First of all, we define a domain on the parameter space \((\delta p_2, \delta p_4, \delta m)\). The maximum allowed value for \(\delta p_4\) should be 0.25, otherwise the map definition (1) looses its validity. For symmetry reasons, we shall assume that the lower limit for \(\delta p_4\) is \(-0.25\). We shall also assume that \(\delta p_2\) and \(\delta m\) range in \([-2, 2]\) and \([-1, 1]\), respectively: so doing, we shall admit a maximum error of 100% in both directions for all the parameters \(p_2, p_4,\) and \(p_3 + p_6\). The admitted region of the parameter space will be henceforth denoted by \(\Omega_p\). Within such a region, the system can have either one or two (disjoint) absorbing sets, according to the parameter values.

3.1 Equilibrium points

In the chosen range for the parameter \(\delta p_4\), the left branch of the map \(M(x)\) always intersects the main diagonal in correspondence of the equilibrium point \(E_1 = \frac{(2+\delta p_2)(1+4\delta p_4)}{4(3+\delta p_2)}\). The following inequalities locate the existence of other equilibria, say \(E_2, E_3,\) and \(E_4,\) originated by the intersections of the other map branches with the diagonal:

- the fixed point \(E_2 = \frac{-2\delta m + \delta p_2}{2(1+\delta p_2)}\) exists in the subregion of \(\Omega_p\) where the following conditions on the parameters are satisfied:
  \[ \left( \delta p_4 \leq \frac{\delta p_2 - 1 - 4\delta m}{4(\delta p_2 + 1)} \right) \cap \left( \delta m \geq -\frac{1}{2} \right) \]

  The boundary of such a subregion of \(\Omega_p\) will be denoted by \(S_2\) and is defined as follows:

  \[ S_2 = \left\{ (\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = \frac{\delta p_2 - 1 - 4\delta m}{4(1 + \delta p_2)} \right\} \]  

(3)

- the fixed point \(E_3 = \frac{4 + 2\delta m + \delta p_2}{2(3+\delta p_2)}\) exists in the subregion of \(\Omega_p\) where the following conditions on the parameters are satisfied:
  \[ \left( \delta p_4 \leq \frac{\delta p_2 + 1 - 4\delta m}{4(\delta p_2 + 3)} \right) \cap \left( \delta m \geq -\frac{1}{2} \right) \]
The boundary of such a subregion of $\Omega_p$ will be denoted by $S_3$ and is defined as follows:

$$S_3 = \left\{ (\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = \frac{1 - 4\delta m + \delta p_2}{4(3 + \delta p_2)} \right\} \quad (4)$$

- the fixed point $E_4 = \frac{(2 + \delta p_2)(3 - 4\delta p_4)}{4(1 + \delta p_2)}$ exists in the subregion of $\Omega_p$ where the following conditions on the parameters are satisfied:

$$\delta p_2 > -1$$

The boundary $\delta p_2 = -1$ of such a subregion of $\Omega_p$ will be denoted by $S_4$.

Since the absolute values of the map branches slopes are equal to $(2 + \delta p_2)$, all the equilibrium points are unstable for $\delta p_2 > -1$.

### 3.2 Bifurcation curves

Even if we find analytical results in terms of three parameters, for the sake of graphical clarity, we will show two-dimensional sections of the significant sets of points, obtained for $\delta m = -1/3$ (see Fig. 3). The results obtained for other values of $\delta m$ within the range $[-1, 1]$ are subsets of these ones. Figure 3 shows some significant bifurcation curves in the considered plane $(\delta p_2, \delta p_4)$. For instance, the bifurcation curves obtained as intersections between the plane $\delta m = -1/3$ and the surfaces $S_2, S_3,$ and $S_4$ are represented by dashed black lines in Fig. 3.

[Figure 3 about here.]

Since the bifurcations are determined by the critical points (relative maximum and minimum points, discontinuity points) and their iterates, for the sake of compactness we shall denote such points as follows:

- $C = P_1; C_k = M^k(C)$
- $D = P_4; D_k = M^k(D)$
- $F = P_5; F_k = M^k(F)$

To be more precise, we should consider both the right and the left limits of $M$ at each discontinuity point. However, according to (2), the left limit of $M$ in $P_4$ (that is 0) is equal to the right limit of $M$ in $P_3$, whereas the right limit of $M$ in $P_4$ (that is $1 + \delta m - (2 + \delta p_2)(0.25 - \delta p_4)$) is equal to the left
limit of $M$ in $P_5$. Then, the definitions of $M$ in $P_4$ and $P_5$ give us the two limit values at both the discontinuities.

In the next subsections, we will illustrate the main bifurcations by resorting to one-dimensional bifurcation diagrams, obtained by varying $\delta p_2$ for three fixed values of $\delta p_4$, and to the maps obtained for some significant values of $\delta p_2$. Once the bifurcation scenario will be clearly sketched, we shall find analytical expressions for some of the surfaces in the three-dimensional parameter space corresponding to the bifurcation curves in Fig. 3.

The one-dimensional bifurcation diagrams will be obtained by varying $\delta p_2$ within ranges (depending on the chosen value of $\delta p_4$) such that the asymptotic dynamics of the map is bounded and chaotic everywhere.

We are interested in values of $\delta p_2$ larger than $-1$, as for $-2 < \delta p_2 < -1$ the only invariant sets of the map are stable fixed point(s). For the chosen value of $\delta m$, such fixed points are either $E_1$ (above $S_3$, in Fig. 3) or both $E_1$ and $E_3$ (below $S_3$, in Fig. 3). For different values of $\delta m$ and for $-2 < \delta p_2 < -1$, also $E_2$ could be a stable fixed point, whereas $E_4$ never exists. For $\delta p_2 = -1$ infinite non attracting period-two cycles exist around such equilibrium points.

For $\delta p_2 > -1$, the absolute value of the slopes of the map branches is always larger than 1. This implies that (i) no stable cycle, of any period, can exist, and (ii) the bounded asymptotic dynamics of the system can be only chaotic. In other words, $S_4$ marks the appearance of chaotic dynamics. As we shall see, chaotic intervals may coexist, may be cyclical, and may undergo several global bifurcations. The boundaries of such chaotic intervals will be defined in terms of the critical points $C_k$, $D_k$, and $F_k$.

The grey region in Fig. 3 contains parameter sets either unfeasible (according to the definition of $\Omega_p$) or corresponding to unbounded dynamics (see Sec. 4.1 for details).

### 3.3 One-dimensional bifurcation diagram for $\delta p_4 = -0.16$

Figure 4 shows the one-dimensional bifurcation diagram obtained by varying $\delta p_2$ within the range $[-1, 0.4]$, for $\delta p_4 = -0.16$.

[Figure 4 about here.]

For low values of $\delta p_2$ (i.e., close to $-1$), the map dynamics are confined within the absorbing interval $J_2 = [C_2, C_1]$. Within such an interval, as long as it is absorbing, the map is equivalent (topologically conjugated) to a tent map, then its behaviors are deeply analyzed in [Maistrenko et al., 1993]. As a consequence, for $\delta p_2$ close to $-1$ there is a very large number of cyclic chaotic intervals, whose number decreases and whose amplitude increases by
increasing $\delta p_2$. For the whole map $M$, the mechanisms governing the change in the number of cyclic chaotic intervals will be briefly described in Sec. 5 and is based on the bifurcations that orbits homoclinic to either a fixed or a periodic point (i.e., a fixed point in higher-order iterates of $M$) undergo [Gardini, 1994; Bizzarri et al., 2005].

If we increase $\delta p_2$, the number of chaotic intervals decreases. To locate such intervals, we have to focus on the critical points of the map $M$ and on their iterates. For instance, for $\delta p_2 = -0.85$, we can locate on the map the four cyclic chaotic intervals $J_{2,k}$ ($k = 1, \ldots, 4$) shown in Fig. 5(A1). Of course, if $\delta p_2$ changes, also the chaotic intervals change. The critical points play a significant role in the definition of such intervals. In the considered example, for instance, the first eight iterates of $C$ (i.e., the points $C_k$, $k = 1, \ldots, 8$) give the boundaries of the four cyclic chaotic intervals.

Figure 5(A2) shows the map $M$ for $\delta p_2 = -0.75$, with the two cyclic chaotic intervals $J_{2,1}$ and $J_{2,2}$, and the trapping interval $I = [F_1, C_1]$. The equilibrium point $E_2$ does not exist.

[Figure 5 about here.]

Generally speaking, for any triplet $(\delta p_2, \delta p_4, \delta m)$, the map can have one or more absorbing sets belonging to a given trapping interval $I$. From a circuit point of view, we are interested in finding the trapping interval for any possible triplets in the domain $\Omega_p$, since we have to ensure that the circuit input and output will range within the same intervals, scaled by a proper dimensional constant. For the same reason, we shall assume that the domain of the map $M$ for a given parameter triplet coincides with the corresponding trapping interval $I$. With this caveat in mind, from Fig. 5(A2) it is evident that, since $D_1 > F_2$, the interval $Z = [F_2, D_1] = [P_2 P_4, 1 + \delta m + P_2 (P_1 - 1/2)]$ does not belong to the range of the map. The amplitude $-\delta p_2 / 2 + \delta m$ of such an interval is not affected by $P^4$ (and then by $\delta p_4$), that only influences the vertical position of $Z$.

If we further increase $\delta p_2$, the region $Z$ within $I$ shrinks and disappears at the vertical line

$$S_Z = \{(\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_2 = 2\delta m\}$$

(5)

Moreover, the equilibrium $E_2$ appears (when we cross the dashed curve $S_2$ in Fig. 3) within the interval $[F_1, F_2]$, whereas the two cyclic chaotic intervals $J_{2,1}$ and $J_{2,2}$ approach each other. They have a contact and merge together at the value of $\delta p_2$ such that $C_3 = C_4$ (vertical red line $H_1$ in Fig. 3). At this bifurcation value, there is an homoclinic orbit to the equilibrium $E_3$, since
$C_3 = E_3$. After, the absorbing interval $J_2 = [C_2, C_1]$ becomes chaotic, as pointed out in Figs. 4 and Fig. 5(A3).

The system undergoes another significant bifurcation when we cross the first green curve $S_5$ in Fig. 3, which marks the locus of points where a second absorbing set, contained within the interval $J_1 = [F_1, F_2]$, appears. This happens when $F_2 = E_2$, i.e., when the repellor $E_2$ exits the interval $J_1$. Before this bifurcation, the interval $J_1$ is not invariant, being $J_1 \subset M(J_1)$. On the contrary, after the bifurcation, $J_1$ becomes invariant and includes two small cyclic chaotic intervals bounded by images of the discontinuity points: $J_{1,1} \cup J_{1,2} = [F_1, D_2] \cup [D_1, F_2] \subset J_1$. The two invariant chaotic sets $(J_{1,1} \cup J_{1,2})$ and $J_2$ (both belonging to $I = [F_1, C_1]$) are pointed out in Fig. 5(A4).

When we cross the red line $H_2$ in Fig. 3, which is the locus of the points $(\delta p_2, \delta p_4)$ such that $D_1 = D_2$, the two cyclic chaotic intervals $J_{1,1}$ and $J_{1,2}$ have a contact and merge together. At this bifurcation value, there is an homoclinic orbit to the equilibrium $E_1$, since $D_1 = E_1$. After, the whole absorbing interval $J_1$ becomes chaotic, as pointed out in Figs. 4 and 6(A5).

After the blue curve $S_6$ in Fig. 3, $D_1$ becomes smaller than $F_1$. This means that the chaotic interval becomes $J_1 = [D_1, D_2]$, and that the trapping interval $I$ (that includes also $J_2 = [C_2, C_1]$) becomes $[D_1, C_1]$ as shown in Figs. 4 and 6(A6).

When we cross the second green curve $S_7$ in Fig. 3, the chaotic attractor corresponding to the absorbing interval $[D_1, D_2]$ disappears. This happens when $D_2 = E_2$, i.e., when the repellor $E_2$ enters the interval $[D_1, D_2]$ (see Figs. 4 and 6(A7)). As a matter of fact, after the bifurcation the interval $[D_1, D_2]$ is no longer invariant ($[D_1, D_2] \subset M([D_1, D_2])$) and the “old” chaotic attractor belonging to such an interval becomes a chaotic repellor. The existence of such a chaotic repellor is proved by the existence of infinitely many homoclinic orbits of $E_2$ belonging to the interval $[D_1, E_2]$ (see, for instance, the cyan homoclinic orbit in 6(A7)).

A further homoclinic bifurcation occurs when $C_2 = E_2$, i.e., when the repellor $E_2$ enters the interval $[C_2, C_1]$, thus causing the “reunion” of the chaotic attractor with the chaotic repellor. After the bifurcation value (vertical red line $H_3$ in Fig. 3), the absorbing chaotic interval $[C_2, C_1]$ has an “explosion” and is extended to $I = [D_1, C_1]$, as shown in Figs. 4 and 6(A8).

The absorbing interval $I$ changes again after the brown curve $S_8$ in Fig. 3, where $C_1 = D_2$. At the right of such a curve, we have $I = [D_1, D_2]$, as shown
in Figs. 4 and 7(A9).

Finally, when $D_2$ overcomes $E_4$, the map cannot admit bounded dynamics, as pointed out in Fig. 7(A10). This happens in correspondence of the black curve $L_1$ in Fig. 3.

To conclude this section, we remark again that the boundaries of the chaotic intervals in the bifurcation diagram shown in Fig. 4 are the points $C_k$ or $D_k$ or $F_k$ (for proper values of $k$), and the same holds also for the next bifurcation diagrams.

### 3.4 One-dimensional bifurcation diagram for $\delta p_4 = 0.03$

Figure 8 shows the one-dimensional bifurcation diagram obtained by varying $\delta p_2$ within the range $[-1, 1.2]$, for $\delta p_4 = 0.03$.

[Figure 8 about here.]

Figure 9(B1) shows the map $M$ for $\delta p_2 = -0.8$, with the two cyclic chaotic intervals $J_{2,1}$ and $J_{2,2}$ (that constitute the chaotic attractor $J_2 = [C_2, C_4] \cup [C_3, C_1]$), inside the trapping interval $I = [F_1, C_1]$. The equilibrium point $E_2$ does not exist, whereas inside the interval $I$ there is the region $Z$.

[Figure 9 about here.]

If we further increase $\delta p_2$, the region $Z$ decreases and disappears at $\delta p_2 = 2\delta m$. The two intervals $J_{2,1}$ and $J_{2,2}$ merge together when we cross the homoclinic bifurcation red curve $H_1$ in Fig. 3, at which $C_3 = E_3$. After the bifurcation, the absorbing interval $J_2 = [C_2, C_1]$ becomes chaotic, as pointed out in Figs. 8 and 9(B2).

The equilibrium $E_2$ appears within the interval $[F_1, F_2]$ when we cross the dashed curve $S_2$ in Fig. 3.

A further homoclinic bifurcation occurs when we cross the vertical red line $H_3$ in Fig. 3, at which $C_2 = E_2$. After the bifurcation value, the chaotic dynamics is extended to the trapping interval $[F_1, C_1]$, within two cyclic chaotic intervals $J_A = [F_1, D_2]$ and $J_B = [D_1, C_1]$, as shown in Figs. 8 and 9(B3).

The absorbing interval $[F_1, C_1]$ becomes chaotic when we cross the homoclinic curve $H_2$ (see Figs. 8 and 9(B4)).

[Figure 10 about here.]

After the cyan curve $S_9$ in Fig. 3, $F_2$ becomes larger than $C_1$. This means that the absorbing interval becomes $[F_1, F_2]$, as shown in Figs. 8 and 10(B5).
When we cross the blue curve $S_6$ in Fig. 3, $D_1$ becomes lower than $F_1$, and the absorbing interval $[F_1, F_2]$ extends to $[D_1, D_2]$ (see Figs. 8 and 10(B6)).

Finally, when $D_2$ overcomes $E_4$, the map cannot admit bounded dynamics, as pointed out in Fig. 10(B7). This happens again in correspondence of the black curve $L_1$ in Fig. 3.

### 3.5 One-dimensional bifurcation diagram for $\delta p_4 = 0.24$

When we overcome the curve $S_3$ in the plane $(\delta p_2, \delta p_4)$, the fixed points $E_2$ and $E_3$ can not exist, while a local maximum persists. As a consequence, the map is no longer equivalent to a skew tent map. In Fig. 11 (obtained for $\delta p_4 = 0.24$), there is a small number of cyclic chaotic intervals (just three) for $\delta p_2 \rightarrow -1$.

![Figure 11 about here.]

The number of such intervals increases with $\delta p_2$ when we cross the vertical line $\dots INTRODURRE UNA CURVA DI BIFORCAZIONE??\dots$

Figure 12(C1) shows the map for $\delta p_2 = -0.85$, where the presence of many cyclic chaotic intervals (bounded by the iterates of the critical points within the trapping interval $[F_1, C_1]$) is evident. The equilibrium points $E_2$ and $E_3$ do not exist, whereas there is the region $Z$, whose preimages cannot be reached by the trajectorries and that consequently determines the lace structure of the bifurcation diagram. The borders of the holes in the bifurcation diagram are bounded by iterates of the critical points, as we shall see in Sec. 5.

![Figure 12 about here.]

By increasing $\delta p_2$, the region $Z$ decreases and disappears when we cross the vertical line $S_9$ in Fig. 3, as usual. From this point on, the bifurcation diagram looses its lace structure, as pointed out in Figs. 11 and 12(C2), where the absorbing set is $J_{1,1} \cup J_{1,2} = [F_1, C_2] \cup [C_3, C_1]$.

When we cross the cyan curve $S_9$ in Fig. 3, $F_2$ becomes higher than $C_1$. This means that the trapping interval becomes $[F_1, F_2]$, and $J_{1,1} \cup J_{1,2} = [F_1, F_3] \cup [F_4, F_2]$, as shown in Figs. 11 and 12(C3).

A further homoclinic bifurcation occurs when $F_3 = F_4 = E_1$, which causes the “reunion” of the two cyclic chaotic intervals $J_{1,1}$ and $J_{1,2}$. After the bifurcation value (red line $H_4$ in Fig. 3), the trapping interval $[F_1, F_2]$ becomes chaotic, as shown in Figs. 11 and 12(C4).

Finally, when $D_2$ overcomes $E_4$, the map cannot admit bounded dynamics. This happens in correspondence of the black curve $L_2$ in Fig. 3.
4 Analytical expressions for some bifurcation sets

The analytical expressions for sets of points $S_2$, $S_3$, $S_4$, and $S_Z$ have been already derived in the previous section. The homoclinic bifurcation curves are derived numerically, according to the guidelines briefly described in the previous section, some of which have been introduced also in [Bizzarri et al., 2005].

In this section, we shall derive analytical expressions for the bifurcation sets corresponding to the other curves in Fig. 3.

4.1 Trapping intervals

Generally speaking, for a given triplet $(\delta p_2, \delta p_4, \delta m)$, the trapping interval of the map $M$ is $I = [I_{INF}, I_{SUP}]$, where $I_{INF}$ and $I_{SUP}$ depend on the parameter values. The lower bound is $I_{INF} = \min\{F_1, D_1\}$. In the parameter domain $\Omega_p$, the condition $D_1 \geq 0 (= F_1)$ can be expressed as

$$f_1(\delta m, \delta p_2) = \frac{-2 - 4\delta m + \delta p_2}{4(\delta p_2 + 2)} \leq \delta p_4 \left( \leq \frac{1}{4} \right)$$

(6)

where the constraint introduced in Sec. 3 on the parameter $\delta p_4$ (i.e., $\delta p_4 < 0.25$) has been taken into account. Henceforth, the feasibility constraints on the bifurcation parameters will be highlighted whenever necessary.

Then, the set of points

$$S_6 = \{(\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = f_1(\delta m, \delta p_2)\}$$

(7)

marks the limit for the points in the parameter space such that the lower bound of $I$ is $I_{INF} = D_1(< F_1)$. The section of $S_6$ for $\delta m = -1/3$ is the blue curve in Fig. 3.

The upper bound of the trapping interval is $I_{SUP} = \max\{C_1, M(I_{INF})\}$.

4.1.1 Case 1

If $I_{INF} = F_1$ (i.e., above $S_6$), we can have two cases.

1.a The upper bound is $I_{SUP} = C_1 \geq M(I_{INF}) = F_2$ as long as the parameter set $(\delta p_2, \delta p_4, \delta m)$ fulfils the following inequality:

$$f_2(\delta m, \delta p_2) = \frac{-2 - 4\delta m + \delta p_2}{4(\delta p_2 + 2)} \geq \delta p_4 \left( \geq \frac{-1}{4} \right)$$

(8)
The trapping interval is \([F_1, C_1]\) whenever the parameters belong to the region delimited by the planes corresponding to the boundaries of \(\Omega_p\) and by the sets of points \(S_6\) and

\[
S_9 = \{(\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = f_2(\delta m, \delta p_2)\}
\] (9)

The section of \(S_9\) for \(\delta m = -1/3\) is the cyan curve in Fig. 3. This means that the trapping interval is \(I = [F_1, C_1]\) in the yellow region pointed out in Fig. 13 (for the chosen value of \(\delta m\), the curve \(L_3\) lies outside \(\Omega_p\)).

We point out that, for different values of \(\delta m\), the region of existence of the trapping interval \([F_1, C_1]\) could be limited also by the set of points such that \(C_1\) is smaller than \(E_4\) [Bizzarri et al., 2005], i.e., :

\[
f_3(\delta m, \delta p_2) \doteq -\frac{-2 + 4\delta m + \delta p_2 + 4\delta m\delta p_2}{4(\delta p_2 + 2)} \geq \delta p_4 \left( \geq -\frac{1}{4} \right)
\] (10)

In Fig. 3, the set of points

\[
L_3 = \{(\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = f_3(\delta m, \delta p_2)\}
\] (11)

marks the limit for the points in the parameter space such that the trapping interval \([F_1, C_1]\) exists. This means that the trapping interval is \(I = [F_1, F_2]\) in the pale green region pointed out in Fig. 13.

4.1.2 Case 2

On the other hand, if \(I_{INF} = D_1\), we can have two cases as well.
2.a The parameters sets such that $I_{SUP} = M(I_{INF}) = D_2$ lie in the region defined by the following inequality (and by the parameter ranges):

$$f_5(\delta m, \delta p_2) = \frac{-6 - 12\delta m + \delta p_2 - 4\delta m\delta p_2 + \delta p_2^2}{4(2 + 3\delta p_2 + \delta p_2^2)} \geq \delta p_4 \left( \geq -\frac{1}{4} \right)$$

(14)

Then the trapping interval is $[D_1, C_1]$ whenever the parameters belong to the region delimited by the planes corresponding to the boundaries of $\Omega_p$ and by the sets of points $S_6$ and $S_8 = \{(\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = f_5(\delta m, \delta p_2)\}$

(15)

The surface $S_8$ corresponds to the brown curve in Fig. 3. As in case 1.a, for the chosen value of $\delta m$, the set of points $L_3$ is not relevant. This means that the trapping interval is $I = [D_1, C_1]$ in the pale pink region pointed out in Fig. 13.

2.b When the inequality (14) is not fulfilled, we have $I_{SUP} = D_2$. The region of existence of the trapping interval $[D_1, D_2]$ is the set of points such that $D_2$ is smaller than $E_4$, i.e., :

$$f_6(\delta m, \delta p_2) = \frac{-4 - 4\delta m - 4\delta m\delta p_2 + \delta p_2^2}{4(2 + 2\delta p_2)[\delta p_2]} \geq -\text{sign}(\delta p_2)\delta p_4$$

(16)

Then, the set of points

$$L_1 = \{(\delta p_2, \delta p_4, \delta m) \in \Omega_p : -\text{sign}(\delta p_2)\delta p_4 = f_6(\delta m, \delta p_2)\}$$

(17)

marks the limit for the points in the parameter space such that the trapping interval $[D_1, D_2]$ exists. This means that the trapping interval is $I = [D_1, D_2]$ in the pale cyan region pointed out in Fig. 13.

Also in Fig. 13, the unfeasible parameter region (either outside $\Omega_p$ or at the right of the black curves corresponding to $L_1$ and $L_2$) is represented in light-grey. For any point of such a region, the generic trajectory is divergent, as infinitely many repelling cycles still exist for $M$, but neither invariant interval, nor cyclic chaotic intervals, nor other chaotic dynamics can exist.

4.2 Bifurcation set $S_{10}$

The interval $[C_2, C_1]$ can contain an absorbing set provided that the following conditions hold:
• the equilibrium point $E_3$ exists;

• $C_2$ belongs to the third branch of the map, i.e., $C_2 \leq P_5$, i.e., $\delta p_4 \leq -\frac{1}{4} - \delta m$;

• if $E_2$ exists, $C_2$ is greater than $E_2$, i.e., $\text{sign}(\delta p_2)\delta m \leq -\frac{1}{2}\text{sign}(\delta p_2)$;

The latter condition can be expressed in compact form in terms of $\delta m$ only, by assuming $\delta p_2 < 0$:

$$\delta m \geq -\frac{1}{2}$$

Indeed, for $\delta m < -\frac{1}{2}$, the graph of the map is below the diagonal.

In Fig. 3, the first condition is fulfilled under the dashed curve $S_3$, the second condition is fulfilled under the horizontal dark-green line $S_{10}$, whereas the third condition holds at the left of the vertical red line $H_1$.

### 4.3 Bifurcation sets $S_5$ and $S_7$

The bifurcation sets $S_5$ and $S_7$ are the boundaries of a region where two absorbing sets can coexist. For instance, in Fig. 4, between the two green vertical lines $S_5$ and $S_7$ it is evident the presence of an absorbing set (either $[F_1, F_2]$ or $[D_1, D_2]$) coexisting with $J_2 = [C_2, C_1]$. The absorbing set either $[F_1, F_2]$ or $[D_1, D_2]$ may exist when:

• the equilibrium point $E_2$ exists;

• $M(I_{INF}) > E_2$;

When $I_{INF} = F_1$ (i.e., above $S_6$ in Fig. 3), this occurs for

$$f_7(\delta m, \delta p_2) = -\frac{2 + 4\delta m + \delta p_2 + \delta p_2^2}{4(2 + 3\delta p_2 + \delta p_2^2)} \geq \delta p_4$$

and when $I_{INF} = D_1$ (i.e., below $S_6$ in Fig. 3) for

$$f_8(\delta m, \delta p_2) = \frac{-2 - 4\delta m - 3\delta p_2 - 12\delta m\delta p_2 + 2\delta p_2^2 - 4\delta m\delta p_2^2 + \delta p_2^3}{4(2 + 5\delta p_2 + 4\delta p_2^2 + \delta p_2^3)} \leq \delta p_4$$

Thus,

$$S_5 = \{(\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = f_7(\delta m, \delta p_2)\}$$

and

$$S_7 = \{(\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = f_8(\delta m, \delta p_2)\}$$
4.4 Bifurcation sets $H_k$

In this subsection we only summarize the homoclinic bifurcation conditions derived in Sec. 3. The curves can be derived by expressing the equality conditions involving iterates of the critical points and equilibria stated in the previous section in terms of the bifurcation parameters. Of course, also the conditions of existence of the involved equilibria must be imposed.

The bifurcation set $H_1$ is obtained by imposing that $C_3 = E_3$ and that $E_3$ exists:

$$H_1 = \left\{ (\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_2 = -2 + \sqrt{2}, \delta p_4 \leq \frac{1 - 4\delta m + \delta p_2}{4(3 + \delta p_2)} \right\} \quad (23)$$

The bifurcation set $H_2$ is obtained by imposing that $D_1 = E_1$:

$$H_2 = \left\{ (\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = \frac{-4 + 2\delta p_2 - 12\delta m - 4\delta m\delta p_2 + \delta p_2^2}{4(4 + 4\delta p_2 + \delta p_2^2)} \right\} \quad (24)$$

The bifurcation set $H_3$ is obtained by imposing that $C_2 = E_2$ and that $E_2$ exists:

$$H_3 = \left\{ (\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_2 = 0, \delta p_4 \leq \frac{\delta p_2 - 1 - 4\delta m}{4(1 + \delta p_2)} \right\} \quad (25)$$

Finally, the bifurcation set $H_4$ is obtained by imposing that $F_3 = E_1$:

$$H_4 = \left\{ (\delta p_2, \delta p_4, \delta m) \in \Omega_p : \delta p_4 = -\frac{\delta p_2^2 + 2\delta p_2 - 4}{4(8 + 6\delta p_2 + \delta p_2^2)} \right\} \quad (26)$$

5 Other numerical results

In this section, we shall describe the mechanisms governing the change in the number of cyclic chaotic intervals, that can be summarized as follows. Let us suppose that for a bifurcation value $\delta p_2 = \hat{\delta} p_2$ it is $M^p(C) = M^r(C)$, with $p < r$. The last equality means that $M^p(C)$ is a fixed point for the map $M^{r-p}$, i.e., it is a periodic point of period $r - p$ for the map $M$. As a matter of fact, $M^r(C) = M^{r-p}[M^p(C)]$.

[Figure 14 about here.]

Figure 14 shows a zoom of Fig. 4. The curves superimposed to the bifurcation diagram represent the $k$-th iterates of the point $C$, with $k = 2, \ldots, 8$, as a function of $\delta p_2$. At the bifurcation value corresponding to the dashed
black line in Fig. 14, two pairs of cyclic chaotic intervals merge together. This happens when \( C_5 = C_7 \), i.e., when \( p = 5 \) and \( r = 7 \). This condition corresponds to the presence of a 2-cycle on the map \( M \). Even if the cycle is unstable, it is reached in five map iterations starting from \( C \). This means that the map \( M \) contains orbits homoclinic to such a cycle [Gardini, 1994], as pointed out in Fig. 15.

[Figure 15 about here.]

[Figure 16 about here.]

Figure 16 shows a zoom of Fig. 14. The curves superimposed to the bifurcation diagram represent the \( k \)-th iterates of the point \( C \), with \( k = 2, \ldots, 8 \). At the bifurcation value corresponding to the dashed black line in Fig. 16, two pairs of cyclic chaotic intervals merge together. This happens when \( C_9 = C_3 \). This condition corresponds to the presence of an unstable 4-cycle on the map \( M \), that is reached in eight map iterations starting from \( C \). A homoclinic orbit to such a cycle is shown in Fig. 17.

[Figure 17 about here.]

A completely similar mechanism is the basis of the lace structure in the bifurcation diagram shown in Fig. 11. Figure 18 shows a zoom of such a diagram together with \( k \)-th iterates of the points \( C \) (magenta curves), \( D \) (blue curves), and \( F \) (cyan curves) as a function of \( \delta p_2 \). In order to have a comprehensible figure, we superimposed to the diagram the most significant iterates only: the curves \( C_k(\delta p_2) \) are drawn for even values of \( k \) from 2 up to 18; the curves \( D_k(\delta p_2) \) are drawn for even values of \( k \) from 2 up to 20 and for odd values of \( k \) from 17 to 39; the curves \( F_k(\delta p_2) \) are drawn for even values of \( k \) from 6 up to 32, for \( k = \{1, 3\} \) and for odd values of \( k \) from 21 to 39.

[Figure 18 about here.]

6 Concluding remarks

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The analysis carried out in this paper provides criteria (at least with respect to the considered bifurcation parameters) for the design of circuits implementing the so-called shipmap. More in general, this paper proposes a re-modelling of dynamical systems to introduce new bifurcation parameters related to circuit implementations mismatches. This is the essential pre-requisite for an effective bifurcation analysis, strictly related to the real
implementations. To apply such a design philosophy to dynamical systems in the framework of telecommunications, the next step would be a statistical analysis of the re-defined model, so as to find regions in the parameters space ensuring both good dynamics (i.e., the structural stability of the system) and good statistical features.

From the analysis carried out in this paper (assuming $p_4 = 0.25$ and $p_2 = 2$), one can deduce that $\delta p_4$ is not a critical parameter if $\delta p_2 < 0$ (a 100% mismatch is allowed). For $\delta p_2 > 0$, the influence of $\delta p_4$ is as more relevant as $\delta p_2$ increases towards 2. The vertical line $\delta p_2 = 0$ marks the transition between a region where the dynamics is confined in disjoint absorbing chaotic intervals and a region where the invariant interval $I$ is completely visited by the asymptotic trajectories. The proposed analysis should be extended to take into account a larger number of bifurcation parameters (i.e., of implementation mismatches).

Riprendere questa frase: From a circuit point of view, we are interested in finding the trapping interval for any possible triplets in the domain $\Omega_p$, since we have to ensure that the circuit input and output will range within the same intervals, scaled by a proper dimensional constant.

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**References**


Figure Captions

1. Block schemes of the circuit architecture. (a) Input/output characteristics. The function \( w(x) \) is 1 for \((p_4 - \delta w) \leq x \leq (p_5 + \delta w)\) and 0 elsewhere.

2. Examples of six-branches maps, obtained for (a) \( \delta w = -0.05 \) and (b) \( \delta w = 0.05 \). The other parameters (\( \delta p_2, \delta p_4, \) and \( \delta m \)) are set to zero.

3. Significant bifurcation curves in the parameter domain.

4. One-dimensional bifurcation diagram for \( \delta p_4 = -0.16 \).

5. Maps \( M \) for \( \delta p_4 = -0.16 \) and \( \delta p_2 = -0.85 \) (A1), \(-0.75 \) (A2), \(-0.58 \) (A3), and \(-0.54 \) (A4). The intervals \( J_1 \) and \( J_2 \) or their cyclic chaotic subintervals are represented by red and purple horizontal segments, respectively. The trapping interval \( I \) is the dark-green horizontal segment (not reported in the zoom of map (A1)), and the interval \( Z \) is the vertical red segment (existing in the maps (A1) and (A2), and visible in the map (A2) only).

6. Maps \( M \) for \( \delta p_4 = -0.16 \) and \( \delta p_2 = -0.5 \) (A5), \(-0.25 \) (A6), \(-0.1 \) (A7), and \( 0.05 \) (A8). The intervals \( J_1 \) and \( J_2 \) or their cyclic chaotic subintervals are represented by red and purple horizontal segments, respectively. The trapping interval \( I \) is the dark-green horizontal segment.

7. Maps \( M \) for \( \delta p_4 = -0.16 \) and \( \delta p_2 = 0.2 \) (A9) and \( 0.72 \) (A10). The trapping interval \( I \) is the dark-green horizontal segment (existing in the map (A9) only).

8. One-dimensional bifurcation diagram for \( \delta p_4 = 0.03 \).

9. Maps \( M \) for \( \delta p_4 = 0.03 \) and \( \delta p_2 = -0.8 \) (B1), \(-0.5 \) (B2), \(0.05 \) (B3), and \( 0.3 \) (B4). The chaotic subintervals (see text) are represented by red and purple horizontal segments. The trapping interval \( I \) is the dark-green horizontal segment, and the interval \( Z \) is the vertical red segment (existing in the map (B1) only).

10. Maps \( M \) for \( \delta p_4 = 0.03 \) and \( \delta p_2 = 0.6 \) (B5), \( 1.2 \) (B6), and \( 1.4 \) (B7). The trapping interval \( I \) is the dark-green horizontal segment.

11. One-dimensional bifurcation diagram for \( \delta p_4 = 0.24 \).
Maps $M$ for $\delta p_4 = 0.24$: $\delta p_2 = -0.85$ (C1), $-0.65$ (C2), $-0.6$ (C3), $-0.5$ (C4). The chaotic subintervals (see text) are represented by purple horizontal segments. The trapping interval $I$ is the dark-green horizontal segment, and the interval $Z$ is the vertical red segment (existing in the map (C1) only).

Subregions of $\Omega_\delta$ where the trapping interval $I$ is $[F_1, C_1]$ (yellow), $[F_1, F_2]$ (pale green), $[D_1, C_1]$ (pale pink), $[D_1, D_2]$ (pale cyan).

Zoom of the one-dimensional bifurcation diagram for $\delta p_4 = -0.16$.

Homoclinic orbit (in magenta) to a 2-cycle (in green) on the map $M$ for $\delta p_4 = -0.16$ and $\delta p_4 \approx -0.8108$.

Four zooms (at decreasing quotes from (a) to (d)) of the one-dimensional bifurcation diagram for $\delta p_4 = -0.16$ and $\delta p_2 \in [-0.915, -0.905]$.

Homoclinic orbit (in magenta) to a 4-cycle (in green) on the map $M$ for $\delta p_4 = -0.16$ and $\delta p_4 \approx -0.9095$.

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Figure 1: Block schemes of the circuit architecture. (a) Input/output characteristics. The function \( w(x) \) is 1 for \((p_4 - \delta w) \leq x < (p_5 + \delta w)\) and 0 elsewhere.
Figure 2: Examples of six-branches maps, obtained for (a) $\delta w = -0.05$ and (b) $\delta w = 0.05$. The other parameters ($\delta p_2$, $\delta p_4$, and $\delta m$) are set to zero.
Figure 3: Significant bifurcation curves in the parameter domain.
Figure 4: One-dimensional bifurcation diagram for $\delta p_4 = -0.16$. 
Figure 5: Maps $M$ for $\delta p_4 = -0.16$ and $\delta p_2 = -0.85$ (A1), $-0.75$ (A2), $-0.58$ (A3), and $-0.54$ (A4). The intervals $J_1$ and $J_2$ or their cyclic chaotic subintervals are represented by red and purple horizontal segments, respectively. The trapping interval $I$ is the dark-green horizontal segment (not reported in the zoom of map (A1)), and the interval $Z$ is the vertical red segment (existing in the maps (A1) and (A2), and visible in the map (A2) only).
Figure 6: Maps $M$ for $\delta p_4 = -0.16$ and $\delta p_2 = -0.5$ (A5), $-0.25$ (A6), $-0.1$ (A7), and $0.05$ (A8). The intervals $J_1$ and $J_2$ or their cyclic chaotic subintervals are represented by red and purple horizontal segments, respectively. The trapping interval $I$ is the dark-green horizontal segment.
Figure 7: Maps $M$ for $\delta p_1 = -0.16$ and $\delta p_2 = 0.2$ (A9) and 0.72 (A10). The trapping interval $I$ is the dark-green horizontal segment (existing in the map (A9) only).
Figure 8: One-dimensional bifurcation diagram for $\delta p_4 = 0.03$. 
Figure 9: Maps $M$ for $\delta p_1 = 0.03$ and $\delta p_2 = -0.8$ (B1), $-0.5$ (B2), $0.05$ (B3), and $0.3$ (B4). The chaotic subintervals (see text) are represented by red and purple horizontal segments. The trapping interval $I$ is the dark-green horizontal segment, and the interval $Z$ is the vertical red segment (existing in the map (B1) only).
Figure 10: Maps $M$ for $\delta p_4 = 0.03$ and $\delta p_2 = 0.6$ (B5), 1.2 (B6), and 1.4 (B7). The trapping interval $I$ is the dark-green horizontal segment.
Figure 11: One-dimensional bifurcation diagram for $\delta p_4 = 0.24$. 


Figure 12: Maps $M$ for $\delta p_4 = 0.24$: $\delta p_2 = -0.85$ (C1), $-0.65$ (C2), $-0.6$ (C3), $-0.5$ (C4). The chaotic subintervals (see text) are represented by purple horizontal segments. The trapping interval $I$ is the dark-green horizontal segment, and the interval $Z$ is the vertical red segment (existing in the map (C1) only).
Figure 13: Subregions of $\Omega_\mu$ where the trapping interval $I$ is $[F_1, C_1]$ (yellow), $[F_1, F_2]$ (pale green), $[D_1, C_1]$ (pale pink), $[D_1, D_2]$ (pale cyan).
Figure 14: Zoom of the one-dimensional bifurcation diagram for $\delta p_4 = -0.16$. 
Figure 15: Homoclinic orbit (in magenta) to a 2-cycle (in green) on the map $M$ for $\delta p_4 = -0.16$ and $\delta p_4 \approx -0.8108$. 
Figure 16: Four zooms (at decreasing quotes from (a) to (d)) of the one-dimensional bifurcation diagram for $\delta p_4 = -0.16$ and $\delta p_2 \in [-0.915, -0.905]$. 
Figure 17: Homoclinic orbit (in magenta) to a 4-cycle (in green) on the map $M$ for $\delta p_4 = -0.16$ and $\delta p_4 \approx -0.9095$. 
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