Fuzzy sets and systems

Short Communication

Lacunary statistical convergence of sequences of fuzzy numbers

Fatih Nuray

Cumhuriyet University, Education Faculty, Sivas, Turkey

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Abstract

The sequence \( X = \{X_k\} \) of fuzzy numbers is statistically convergent to the fuzzy number \( X_0 \) provided that for each \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} \frac{1}{n} \{ \text{the number of } k \leq n: d(X_k, X_0) \geq \varepsilon \} = 0.
\]

In this paper we study a related concept of convergence in which the set \( \{k: k \leq n\} \) is replaced by \( \{k: kr - 1 < k \leq kr\} \)
for some lacunary sequence \( \{k_r\} \). Also we introduce the concept of lacunary statistically Cauchy sequence and show that it is equivalent to the lacunary statistical convergence. In addition, the inclusion relations between the sets of statistically convergent and lacunary statistically convergent sequences of fuzzy numbers are given. © 1998 Published by Elsevier Science B.V. All rights reserved

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1. Introduction

Let \( D \) denote the set of all closed bounded intervals \( A = [A, \bar{A}] \) on the real line \( R \). For \( A, B \in D \) define

\[
A \leq B \iff A \subseteq B \quad \text{and} \quad \bar{A} \subseteq \bar{B},
\]

\[
d(A, B) = \max\{\vert A - \bar{B}\vert, \vert \bar{A} - B\vert\}.
\]

It is easy to see that \( d \) defines a metric on \( D \) and \( (D, d) \) is complete metric space. Also \( \leq \) is a partial order in \( D \). A fuzzy number is a fuzzy subset of real line \( R \) which is bounded, convex and normal. Let \( L(R) \) denote the set of all fuzzy numbers which are upper semicontinuous and have compact support. In other words, if \( X \in L(R) \) then for any \( \alpha \in [0, 1], X^\alpha \) is compact, where

\[
X^\alpha = \begin{cases} t, & X(t) \geq \alpha \quad \text{if} \quad \alpha \in (0, 1], \\ t, & X(t) > 0 \quad \text{if} \quad \alpha = 0. 
\end{cases}
\]

Define \( \tilde{d}: L(R) \times L(R) \to R \) by \( \tilde{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha) \). For \( X, Y \in L(R) \) define \( X \leq Y \) iff \( X^\alpha \leq Y^\alpha \)
for any \( \alpha \in [0, 1] \). It is known that \( L(R) \) is a complete metric space with the metric \( \tilde{d} \) [2].

By a lacunary sequence we mean an increasing integer sequence \( \theta = \{k_r\} \) such that \( k_0 = 0 \) and \( h_r = k_r - k_{r-1} \to \infty \) as \( r \to \infty \). Throughout this paper the intervals determined by \( \theta = \{k_r\} \) will be denoted by \( I_r = (k_{r-1}, k_r] \) and the ratio \( k_r/k_{r-1} \) will be abbreviated as \( q_r \).

Statistically convergent and statistically Cauchy sequences of fuzzy numbers were introduced by
Nuray and Savaş [3]. Lacunary statistically convergent and lacunary statistically Cauchy sequences of real- or complex-valued sequences were introduced by Fridy and Orhan [1]. In this paper we introduce lacunary statistically convergent and lacunary statistically Cauchy sequences of fuzzy numbers.

2. Statistical convergence

A sequence $X = \{X_k\}$ of fuzzy numbers is statistically convergent to the fuzzy number $X_0$ provided that for each $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} \left| \{k \leq n : \overline{d}(X_k, X_0) \geq \varepsilon\} \right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set [3]. In this case we write $X_k \rightarrow X_0(S)$. We shall use $S$ to denote the set of all statistically convergent sequences of fuzzy numbers.

**Definition 2.1.** Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $X = \{X_k\}$ of fuzzy numbers is said to be lacunary statistically convergent to the fuzzy number $X_0$, written as $sto - \lim X_k = X_0$, if for every $\varepsilon > 0$

$$\lim_{r \to \infty} \frac{1}{h_r} \left| \{k \in I_r : \overline{d}(X_k, X_0) \geq \varepsilon\} \right| = 0.$$

In this case we write $X_k \rightarrow X_0(S_\theta)$. We shall use $S_\theta$ to denote the set of all lacunary statistically convergent sequences of fuzzy numbers.

**Definition 2.2.** Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $X = \{X_k\}$ of fuzzy numbers is said to be lacunary statistically Cauchy sequence if there is a subsequence $\{X_{k'(r)}\}$ of $X$ such that $k'(r) \in I_r$ for each $r$, $lim_{r \to \infty} X_{k'(r)} = X_0$ and for every $\varepsilon > 0$

$$\lim_{r \to \infty} \frac{1}{h_r} \left| \{k \in I_r : \overline{d}(X_k, X_{k'(r)}) \geq \varepsilon\} \right| = 0.$$

**Theorem 2.1.** The sequence $X = \{X_k\}$ of fuzzy numbers is lacunary statistically convergent if and only if $X = \{X_k\}$ is lacunary statistically Cauchy sequence.

**Proof.** Let $sto - \lim X_k = X_0$, and write $K^{(j)} = \{k \in N : \overline{d}(X_k, X_0) < 1/j\}$ for each $j \in N$. Hence for each $j$, $K^{(j)} \supseteq K^{(j+1)}$ and

$$\frac{|K^{(j)} \cap I_r|}{h_r} \to 1 \quad \text{as} \quad r \to \infty.$$

Choose $m(1)$ such that $r > m(1)$ implies $|K^{(1)} \cap I_r|/h_r > 0$, i.e., $K^{(1)} \cap I_r \neq \emptyset$. Next choose $m(2) > m(1)$ so that $r > m(2)$ implies $K^{(2)} \cap I_r \neq \emptyset$. Then for each $r$ satisfying $m(1) \leq r \leq m(2)$, choose $k'(r) \in I_r$ such that $k'(r) \in I_r \cap K^{(2)}$, i.e., $\overline{d}(X_{k'(r)}, X_0) < 1$. In general, choose $m(p + 1) > m(p)$ such that $r > m(p + 1)$ implies $I_r \cap K^{(p+1)} \neq \emptyset$. Then for all $r$ satisfying $m(p) \leq r \leq m(p + 1)$, choose $k'(r) \in I_r \cap K^{(p)}$ i.e.,

$$\overline{d}(X_{k'(r)}, X_0) < \frac{1}{p}, \quad (**)$$

Using the assumptions that $sto - \lim X_k = X_0$ and $k'(r) \in I_r$, (**)) implies that $lim_{r \to \infty} X_{k'(r)} = X_0$. Furthermore, we have, for every $\varepsilon > 0$,

$$\frac{1}{h_r} \left| \{k \in I_r : \overline{d}(X_k, X_{k'(r)}) \geq \varepsilon\} \right| \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \overline{d}(X_k, X_0) \geq \varepsilon \right\} \right| + \frac{1}{h_r} \left| \left\{ k \in I_r : \overline{d}(X_{k'(r)}, X_0) \geq \varepsilon \right\} \right|.$$

Hence, we get $k'(r) \in I_r$ and (**)) implies that $lim_{r \to \infty} X_{k'(r)} = X_0$. Furthermore, we have, for every $\varepsilon > 0$,

$$\frac{1}{h_r} \left| \{k \in I_r : \overline{d}(X_k, X_0) \geq \varepsilon\} \right| \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \overline{d}(X_k, X_{k'(r)}) \geq \varepsilon \right\} \right| + \frac{1}{h_r} \left| \left\{ k \in I_r : \overline{d}(X_{k'(r)}, X_0) \geq \varepsilon \right\} \right|.$$

Conversely, suppose that $X$ is a lacunary statistically Cauchy sequence. For every $\varepsilon > 0$, we have

$$\frac{1}{h_r} \left| \{k \in I_r : \overline{d}(X_k, X_0) \geq \varepsilon\} \right| \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \overline{d}(X_k, X_{k'(r)}) \geq \varepsilon \right\} \right| + \frac{1}{h_r} \left| \left\{ k \in I_r : \overline{d}(X_{k'(r)}, X_0) \geq \varepsilon \right\} \right|.$$

from which it follows that $sto - \lim X_k = X_0$. □

**Theorem 2.2.** For any lacunary sequence $\theta$, $S_\theta \subseteq S$ if $\lim \sup \theta_r < \infty$.

**Proof.** If $\lim \sup \theta_r < \infty$, then there is a $K > 0$ such that $\theta_r < K$ for all $r$. Suppose that $X_k \rightarrow X_0(S_\theta)$, and let $N_r = \{|k \in I_r : \overline{d}(X_k, X_0) \geq \varepsilon\}$. By the definition,
given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that

$$\frac{N_r}{h_r} < \varepsilon \quad \text{for all } r > r_0. \quad (****)$$

Now let $M = \max \{N_r: 1 \leq r \leq r_0\}$ and let $n$ be any integer satisfying $k_{r-1} < n \leq k_r$; then we can write

$$\frac{1}{n} \left| \{k \leq n: \bar{d}(X_k, X_0) \geq \varepsilon\} \right|$$

$$\leq \frac{1}{k_{r-1}} \left| \{k \leq k_r: \bar{d}(X_k, X_0) \geq \varepsilon\} \right|$$

$$= \frac{1}{k_{r-1}} \{N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r\}$$

$$\leq M \frac{r_0}{k_{r-1}} + \frac{1}{k_{r-1}} \left\{ h_{r_0+1} \frac{N_{r_0+1}}{h_{r_0+1}} + \cdots + h_r \frac{N_r}{h_r} \right\}$$

$$\leq \frac{r_0 M}{k_{r-1}} + \frac{1}{k_{r-1}} \left( \sup_{r_0 \leq r \leq r_0+1} \left( \frac{N_r}{h_r} \right) \right) \{ h_{r_0+1} + \cdots + h_r\}$$

$$\leq \frac{r_0 M}{k_{r-1}} + \frac{h_r - h_{r_0}}{k_{r-1}}, \quad \text{by } (****)$$

$$\leq r_0 M \frac{h_r}{k_{r-1}} + \varepsilon r_0 \frac{M}{k_{r-1}} + \varepsilon K$$

and the result follows immediately. \(\square\)

**Theorem 2.3.** For any lacunary sequence $\theta$, $S \subseteq S_\theta$ if $\lim \inf_r q_r > 1$.

**Proof.** Suppose that $\lim \inf_r q_r > 1$. Then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large $r$, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1 + \delta}.$$

If $X_k \rightarrow X_0(S)$, then for every $\varepsilon > 0$ and sufficiently large $r$, we have

$$\frac{1}{k_r} \left| \{k \leq k_r: \bar{d}(X_k, X_0) \geq \varepsilon\} \right|$$

$$\geq \frac{1}{k_r} \left| \{k \in I_r: \bar{d}(X_k, X_0) \geq \varepsilon\} \right|$$

$$\geq \frac{\delta}{1 + \delta} \frac{1}{h_r} \left| \{k \in I_r: \bar{d}(X_k, X_0) \geq \varepsilon\} \right|,$$

which proves the theorem. \(\square\)

Combining Theorem 2.2 and 2.3 we have

**Theorem 2.4.** Let $\theta$ be a lacunary sequence; then $S = S_\theta$ if $1 < \lim \inf_r q_r \leq \lim \sup_r q_r < \infty$.

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**References**