Abstract—In this paper extrinsic information transfer (EXIT) charts are proposed to design non-binary low-density parity-check (LDPC) codes for the AWGN channel. A new metric is presented to describe the mutual information of the non-binary messages. The a priori information is modelled using a Gaussian mixture distribution. Analytical expressions are given for the EXIT curves of the variable and check node decoders for both regular and irregular LDPC codes. The analytical expressions are shown to agree well with simulation results. Finally, by matching the variable and check node EXIT curves it is shown that good non-binary LDPC can be designed for the AWGN channel.

Index Terms—Code optimization, iterative decoding, low-density parity-check (LDPC) codes, mutual information.

I. INTRODUCTION

Binary low-density parity-check (LDPC) codes, originally proposed by Gallager [1] in 1963, have recently been rediscovered [2], [3]. Irregular LDPC codes [4] have been shown to approach the Shannon limit and outperform the powerful turbo codes [5] in the AWGN channel for long frame lengths. Binary LDPC codes can be designed using density evolution [4], [6] which involves determining the code threshold by iteratively propagating the message densities through the decoder. This method is computationally intensive, an issue which was addressed in [7]. Recently a more efficient design method, based on extrinsic information transfer (EXIT) charts [8], was proposed in [9]. Here optimal codes were designed by matching the EXIT curves of the variable and check node decoders.

Non-binary LDPC codes [10], [11] are defined over the finite field $GF(q = 2^p)$ and have demonstrated improved performance over binary LDPC codes of the same frame length and rate [11]. By encoding over $GF(q)$ each parity check becomes more complex while the decoding remains tractable. Increasing the field size of a LDPC code is comparable to increasing the memory of a convolutional encoder. Non-binary LDPC codes have recently been applied to the multiple-input, multiple-output (MIMO) channel [12] and the magnetic recording channel [13] with promising results. The design of non-binary LDPC codes has only been addressed in [11].

In this paper a new method, based on EXIT charts, is proposed to design non-binary LDPC codes for the AWGN channel. This method is an extension of [9] to non-binary LDPC codes. To the extent of the authors’ knowledge EXIT charts have not been applied to the analysis or design of non-binary LDPC codes. Good codes are designed by matching the EXIT curves of the variable and check node decoders, which is orders of magnitude more efficient than the design approach employed in [11].

This paper is organised as follows. In section II the encoder, channel model and decoder are described. EXIT charts for regular LDPC codes are presented in section III where a model for the a priori information is proposed and closed form expressions for the EXIT curves are given. This method is then extended to irregular LDPC codes in section IV. In section V non-binary LDPC codes are designed for the AWGN channel. Finally, conclusions are drawn in section VI.

II. SYSTEM MODEL

LDPC codes are linear block codes which are defined by a very sparse parity check matrix $H$ with dimensions $m \times n$. With non-binary LDPC codes the elements of $H$ are defined over finite field $GF(q = 2^p)$. Using Gaussian elimination $H$ can be transformed into the $k \times n$ generator matrix $G$ where $k = n - m$.

The encoding of non-binary LDPC codes is similar to the encoding of binary LDPC codes. The input frame $w$ consisting of $K$ information bits is converted to a frame $v$ of length $k = K/p$ symbols from $GF(q)$. This frame is encoded to give codeword $w = Gv$ of length $n$ symbols. The rate of this encoder is $R = k/n$. The codeword $v$ is converted to a binary codeword $c$ of length of $N = np$ bits. The encoded bits are BPSK modulated and transmitted over the AWGN channel. The received signal after matched filtering is expressed as

$$r_t = (2c_t - 1) + \eta_t.$$  \hfill (1)

Here $t = \{1, \ldots, N\}$ and $\eta_t$ is the AWGN noise which is modelled as an independent, zero-mean Gaussian random sequence with variance $\sigma_n^2 = 1/(2RE_b/N_0)$ where $E_b/N_0$ is the signal-to-noise ratio.

The LDPC decoder then calculates the most likely estimate of the original codeword given the received information $r$. The decoding of non-binary LDPC codes is not equivalent to the decoding of binary codes as the non-binary decoder operates on the symbol level, not on the bit level. In this paper decoding is performed using the Fourier transform algorithm of [11]. To avoid numerical instability issues this algorithm is implemented in the log domain [13].

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III. EXIT Charts for Regular Codes

The LDPC decoder can be visualised as a bipartite graph. The graph consists of a set of $n$ variable nodes on the left corresponding to the transmitted codeword and a set of $m$ check nodes on the right corresponding to the parity checks. A connection or edge exists between variable node $j$ and check node $i$ if the corresponding entry $H_{ij}$ in the parity check matrix $H$ is non-zero. The number of edges connected to a variable node $j$ is denoted $d_j^v$ and the number edges connected to check node $i$ is denoted $d_i^c$. The decoding algorithm is initialised by calculating the channel messages. The algorithm then proceeds by iteratively updating and passing messages between connected variable and check nodes until a valid codeword is found.

The sets of variable and check nodes are referred to as the variable node decoder (VND) and check node decoder (CND) respectively as illustrated in Fig. 1. The structure is similar to that of an iterative decoder for a serially concatenated code where inner repetition and outer single parity-check codes are used. The notation $I_{ch}$ denotes the mutual information of the channel messages, $I_{ev}$ and $I_{ec}$ denote the mutual information at the VND input and output and $I_{ac}$ and $I_{ec}$ denote the mutual information at the CND input and output.

An EXIT chart is a superposition of the VND and CND EXIT curves on the same set of axes which graphically illustrates the exchange of mutual information between the two decoders. In this way, the convergence behaviour of the LDPC decoder can be determined without the need to run lengthy BER simulations.

In this section a new metric is presented to calculate the mutual information of the non-binary messages. A decoder model is proposed for both the VND and CND and analytical expressions are presented for the corresponding EXIT curves. Both the model and analysis are compared to simulation results to assess their accuracy for $GF(q)$ LDPC codes. However, the model and analytical expressions presented are also valid for $GF(q)$ LDPC codes.

A. Calculation of the Mutual Information

With binary LDPC codes a single message is passed along each edge of the bipartite graph while with non-binary codes $q$ messages are passed along each edge. EXIT charts have been considered for non-binary convolutional codes [14], [15] where the calculation of the mutual information required the numerical evaluation of a multi-dimensional integral. For non-binary LDPC codes over $GF(q)$ this requires evaluating a $q-1$ dimensional integral which is computationally expensive for $q>4$. Furthermore, it was found that using this metric the EXIT curve of a non-binary irregular code is not the average of the component code EXIT curves as was the case in [9] for binary codes. For these reasons a new metric is proposed.

Let $L^a$ be an arbitrary symbol LLR or decoder message where $a \in GF(q)$ and define $L = [L^0, \ldots, L^{q-1}]$. Let $\lambda$ be an arbitrary bitwise LLR and denote the conversion from symbol LLRs to bitwise LLRs as $\lambda = B(L)$. For each vector of symbol LLRs, $p$ bitwise LLRs $\lambda_i$ are calculated which correspond to the $i$th bit of $a$. This conversion was discussed in [16] for serially concatenated convolutional codes and is given by

$$\lambda_i = \max_{a_i=0}^{a_i=q} (L^a) - \max_{a_i=0}^{a_i=q} (L^a)$$

where the $\max$ operator denotes addition in the log domain

$$\max_j(x_j) = \log \left( \sum_{j=1}^{J} e^{x_j} \right).$$

This can be computed recursively by setting $\delta_1 = x_1$ and for $j = \{2, \ldots, J\}$

$$\delta_j = \max (\delta_j, \delta_{j-1}) + \ln [1 + \exp (-|\delta_j - \delta_{j-1}|)].$$

On the final iteration $\delta_j$ is the solution to (3).

The pdfs $p(\lambda|X=0)$ and $p(\lambda|X=1)$ are obtained from the histogram of $\lambda$. The mutual information between the bits $X$ and the bitwise LLRs $\lambda = B(L)$ is [8]

$$I(X; B(L)) = \frac{1}{2} \sum_{x=0,1} \int_{-\infty}^{\infty} p(\lambda|X=x) \cdot \log_2 \frac{2p(\lambda|X=x)}{p(\lambda|X=0) + p(\lambda|X=1)} d\lambda.$$  

B. EXIT Curves for the Variable Node Decoder

1) Decoder Model: Consider an arbitrary variable node of degree $d_v$ which receives $d_v$ messages $L^a_{i,iv}$ from the CND and one message $L^a_{ch}$ from the channel. The variable node computes the $d_v$ output messages as follows for $i = \{1, \ldots, d_v\}$ and $a \in GF(q)$

$$L^a_{i,ov} = L^a_{ch} + \sum_{k \neq i} L^a_{k,iv}.$$  

Assume that the variable node is associated with symbol $x \in GF(q)$ where $x_i \in \{0,1\}$ is the $i$th bit in the binary
representation of $x$ and $l = \{1, \ldots, p\}$. Each channel message $L_{ch}^0$ is calculated from the received signals $r_1, \ldots, r_p$ where

$$L_{ch}^0 = \sum_{l=1}^{l=L_a} 2r_l/\sigma_n^2. \quad (7)$$

Here $r_l = (2x_l - 1) + \eta_l$ (1) and $a_l$ denotes the $l$th bit in the binary representation of $a$. It is more convenient to express (7) in the form

$$L_{ch}^0 = \sum_{l=1}^{l=L_a} \mu_{ch} \cdot (2x_l - 1) + n_{ch,l}. \quad (8)$$

where $\mu_{ch} = \sigma_n^2/2$ and the $n_{ch,l}$ are Gaussian distributed with mean zero and variance

$$\sigma_{ch}^2 = 8R_{ch}E_b / N_0. \quad (9)$$

The input messages of the VND are more difficult to model and their distribution was carefully studied by simulation. A number of observations were made. Firstly, the input messages are independent. Secondly, the messages approach a Gaussian like distribution with an increasing number of iterations where the mean and the variance are related by $\mu = \sigma^2/2$. This approximation has been successfully made in [7], [9] for the case of binary LDPC codes. Thirdly, the messages $L_{i,iv}^0, \ldots, L_{i,iv}^{t-1}$ are correlated due to the normalisation of the VND output messages in the iterative decoder [11]. Based on these observations the following model is proposed for the input messages of the VND:

$$L_{i,iv}^a(\sigma) = \begin{cases} 0, & \text{if } a = 0 \\ \mu + n_i^0 + n_i^a, & \text{if } x = a, a \neq 0 \\ -\mu + n_i^0 + n_i^a, & \text{if } x = 0, a \neq 0 \\ n_i^0 + n_i^a, & \text{otherwise} \end{cases} \quad (10)$$

Here $\mu = \sigma^2/2$ and $n_i^0$ and $n_i^a$ are Gaussian distributed with mean zero and variance $\sigma^2/2$ which ensures the variance of $L_{i,iv}^a(\sigma)$ is $\sigma^2$. The random variable $n_i^0$ is introduced to model the correlation between the messages $L_{i,iv}^0, \ldots, L_{i,iv}^{t-1}$.

The proposed model for the input messages is compared to simulation results in Fig. 2. A regular (3,6) LDPC code over $GF(4)$ is used where $N = 12 000$ bits, $R = 1/2$, $E_b/N_0 = 2$ dB and the decoder is allowed to run for 8 iterations. It can be seen that the proposed model, based on the Gaussian distribution, provides a good fit to the simulated results.

2) Analytical Expressions: For binary LDPC codes the channel messages, input messages and output messages of the VND can all be modelled using a Gaussian distribution while for non-binary LDPC codes these messages are modelled using different Gaussian mixture distributions given by (8), (10) and (6) respectively. For this reason the analytical expressions for the EXIT curves presented in [9] do not apply directly to non-binary LDPC codes. Therefore, two functions

$$J_0(\sigma) = I[X; B(L_{ch} + L_{iv}(\sigma))] \quad (11)$$

$$J_2(\sigma) = I[X; B(L_{iv}(\sigma))] \quad (12)$$

are defined which relate the standard deviation of the pdf to the mutual information of the messages. In [9] only one
such function is defined. Note that \( J_v(\sigma) \) accounts for the mutual information of the channel messages where \( J_v(0) = I[X; B(L_{ch})] \) which is also the channel capacity.

Using these two functions the analytical expression for the EXIT curve of a degree \( d_v \) variable node is given by

\[
I_{\text{ex}}(I_{av}, d_v) = J_v\left(\sqrt{d_v - 1} \cdot J^{-1}_v(I_{av})\right). \tag{13}
\]

For computer implementation the functions \( J_v(\sigma) \) and \( J^{-1}_v(I_{av}) \) can be computed using a lookup table or the curve fitting approach suggested in [9].

3) Comparison with Simulation Results: In Fig. 3, EXIT curves for a number of variable node degrees are presented for a \( GF(4) \) code with \( R = 1/2 \) and \( E_b/N_0 = 2 \) dB. The EXIT curves obtained from the model (6) and the analysis (13) are compared with simulation results for a “free-running” decoder with a frame length of \( N = 12000 \) bits. For the decoder model the EXIT curve is averaged over \( 10^6 \) messages while for the simulation the snapshot trajectory [8] for a single frame is taken. It is shown that there is good agreement between the model, analysis and simulation.

C. EXIT Curves for the Check Node Decoder

1) Decoder Model: Consider an arbitrary check node of degree \( d_c \) which receives \( d_c \) messages \( L_{i,ic}^a \) from the VND. Define \( h_i \) as the vector of parity checks associated with this check node and \( \bar{x} \) as the vector of encoded symbols which satisfy these checks. These length \( d_c \) vectors contain elements from \( GF(q) \) and satisfy the equality \( h_i \cdot \bar{x} = 0 \) where the arithmetic is performed over \( GF(q) \).

The check node computes the \( d_c \) output messages \( L_{i,oc}^a \) as follows for \( i = 1, \ldots, d_c \) and \( a \in GF(q) \)

\[
\bar{L}_{i,ic} = \mathcal{F}\left[ P_i(L_{i,ic})\right] \tag{14}
\]

\[
\hat{L}_{i,oc}^a(s) = \prod_{k \neq i} \hat{L}_{k,ic}^a(s) \tag{15}
\]

\[
\hat{L}_{i,oc}^a(m) = \sum_{k \neq i} \hat{L}_{k,ic}^a(m) \tag{16}
\]

\[
\hat{L}_{i,oc} = \mathcal{P}_i^{-1}\left[ \mathcal{F}^{-1}\left( \hat{L}_{i,oc}^a \right) \right] \tag{17}
\]

Here \( P_i(L_{i,ic}) \) is the permutation of \( L_{i,ic} \) by \( h_i \) and \( \mathcal{F}(x) \) denotes the Fourier transform of \( x \) as described in [11], [13]. The notation \( L(s) \) and \( L(m) \) denote the sign and magnitude of \( L \).

By simulating the distribution of the CND input messages it was observed that the messages are independent and their pdf is a composite of the pdfs of the channel and input messages of the VND. It is important to note that each VND input message corresponds to the same coded symbol \( x_i \) while each CND input message corresponds to a possibly different coded symbol \( x_i \).

Based on these observations the following model is proposed for the CND input messages:

\[
L_{i,ic}^a(\sigma) = L_{i,c}^a(\sigma) + \sum_{l, a_l = 1} \mu_{ch} \cdot \left( 2x_{i,l} - 1 \right) + n_{ch,i,l} \tag{18}
\]

\[
L_{i,c}^a(\sigma) = \begin{cases} 
0, & \text{if } a = 0 \\
\mu + n_1^0 + n_1^0, & \text{if } x_i = a, a \neq 0 \\
-\mu + n_1^0 + n_1^0, & \text{if } x_i = 0, a \neq 0 \\
n_1^0 + n_1^0, & \text{otherwise}
\end{cases} \tag{19}
\]

where \( x_{i,l} \) is the \( l \)th bit of \( x_i \), \( \mu = \sigma^2/2 \), \( n_{ch,i,l} \) and \( n_i^a \) are Gaussian distributed with mean zero and variance \( \sigma_{\beta_i}^2 \) and \( \sigma^2/2 \) respectively.

2) Analytical Expressions: In [9] the EXIT curve of the CND was expressed in terms of the EXIT curve of the VND based on the duality property of [17]. This approximation is not feasible for \( GF(q > 2) \) because the input and output messages cannot be described by the same Gaussian mixture distribution. However, the EXIT curve of a degree \( d_c \) check node can be well approximated by

\[
I_{ec}(I_{ac}, d_c) = I_{ac}^{(d_c)} I_{ac} + \beta(d_c) \tag{20}
\]

where the constants \( \alpha(d_c) \) and \( \beta(d_c) \) are dependent on the check node degree, field size and SNR. These constants can be calculated by fitting (20) to the CND EXIT curves of interest.

3) Comparison with Simulation Results: EXIT curves for a number of check node degrees are presented in Fig. 4 for a \( GF(4) \) code with \( R = 1/2 \) and \( E_b/N_0 = 2 \) dB. The EXIT curves from the model (14-17) and the analysis (20) are compared with simulation results where the same parameters given for the VND EXIT curves are used. It is observed that the model, analysis and simulation all correspond.

IV. EXIT CHARTS FOR IRREGULAR CODES

For binary LDPC codes, the EXIT curve of an irregular code is simply the average of the component code EXIT curves [9],

Fig. 3. EXIT curves for the variable node decoder of a \( GF(4) \) code with \( R = 1/2 \) and \( E_b/N_0 = 2 \) dB.
[17], [18]. Using the proposed metric this was found to be true for non-binary LDPC codes as well.

The averaging is done with respect to the edge fractions, not the node fractions, because it is the edges which carry the decoder messages. Let $\lambda_i$ ($\rho_i$) denote the fraction of variable (check) nodes of degree $i$ and let $\lambda_i^e$ ($\rho_i^e$) denote the fraction of edges connected to a variable (check) node of degree $i$. The conversion from node fractions to edge fractions is given by $\lambda_i^e = \sum \lambda_i$ and $\rho_i^e = \sum \rho_i$. The VND and CND EXIT curves for an irregular LDPC code are calculated as follows:

$$I_{ev}(I_{av}) = \sum_i \lambda_i^e \cdot I_{ev}(I_{av}, i)$$

$$I_{ec}(I_{ac}) = \sum_i \rho_i^e \cdot I_{ec}(I_{ac}, i)$$

In Fig. 5 the analytical EXIT curves are compared to the average decoding trajectory obtained from simulation where the axes for the CND EXIT curve are swapped. The irregular $GF(4)$ code from [11] is used which is described by the profile $\rho_2 = 0.341$, $\rho_3 = 0.659$, $\lambda_2 = 0.478633$, $\lambda_3 = 0.4085$, $\lambda_8 = 0.000067$, $\lambda_{11} = 0.045233$, $\lambda_5 = 0.067567$. The trajectory is averaged over 20 frames, $N = 12 000$ bits and $E_b/N_0 = 2$ dB. It is observed that the analysis provides a good approximation of the decoding trajectory.

V. DESIGN OF NON-BINARY LDPC CODES

In [17] it was shown that to design LDPC codes which approach capacity the VND and CND EXIT curves must be matched exactly. This approach was applied in [9] to design LDPC codes for both the AWGN and MIMO channels and in [19] to design repeat-accumulate codes for the MIMO channel. This approach is employed here to design non-binary LDPC codes for the AWGN channel.

The design approach now described closely follows that of [9]. To limit the search space, codes with 3 variable node degrees and 1 check node degree are considered. The code rate is $R = 1/2$ and the EXIT curves are matched at $E_b/N_0 = 0.5$ dB because at this SNR the channel capacity $J_c(0) \approx 0.524$ is fractionally larger than $R$. The check node degree $d_c$ is chosen so that the CND curve is a small distance from the $y$ axis at $I_{ac} = 0.5$. In this way, $I_{ec}$ is not too small on the first iteration which gives the decoder a good start. Due to the constraints on the variable and check node degrees only one $\lambda_i$ can be chosen freely. In practice it was found that best results were obtained by minimising the variance of the distance

$$D(I_{av}) = I_{ec}[I_{ev}(I_{av})] - I_{av}$$

between the VND and CND EXIT curves. The EXIT curves can be efficiently calculated using (21) and (22). The optimised profiles for $GF(4)$ and $GF(8)$ LDPC codes are given in Table I and the corresponding EXIT chart for the $GF(4)$ code is presented in Fig. 6.

The performance of the optimised codes is evaluated in the AWGN channel. Their performance is compared to the $GF(4)$ code of [11] which is denoted by ‘Davey’. Simulation results are presented in Fig. 7 for a frame length of $N = 60 000$ bits since it is for this frame length that the code from [11] is optimised. The number of decoding iterations is set to 100. It can be seen that the codes optimised using the proposed EXIT chart approach achieve superior performance to the code from [11] even though the optimised codes use fewer variable and check node degrees. This indicates that the proposed method is a promising approach for optimising non-binary LDPC codes. Furthermore, the approach used here is considerably less computationally intensive, requiring only function evaluations and table lookups, than the simulation based method used in [11] which requires averaging the number of decoding iterations.

<table>
<thead>
<tr>
<th>Code</th>
<th>Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GF(4)$</td>
<td>$\rho_2 = 1$, $\lambda_2 = 0.50637$, $\lambda_3 = 0.36783$, $\lambda_{11} = 0.12580$</td>
</tr>
<tr>
<td>$GF(8)$</td>
<td>$\rho_6 = 1$, $\lambda_2 = 0.62763$, $\lambda_3 = 0.26776$, $\lambda_3 = 0.10461$</td>
</tr>
</tbody>
</table>

TABLE I

DEGREE PROFILES FOR RATE 1/2 LDPC CODES, OPTIMISED AT 0.5 dB.
Iav, Iev, Iac
VND − Analysis
CND − Analysis

Fig. 6. EXIT chart for the \( GF(4) \) code from Table I at \( E_b/N_0 = 0.5 \) dB.

Fig. 7. Simulation results for the optimised codes from Table I.

VI. CONCLUSIONS

This paper proposed the use of EXIT charts to design non-binary LDPC codes for the AWGN channel. Decoder models were developed for both the VND and CND where the \textit{a priori} information was modelled using a Gaussian mixture distribution. Analytical expressions were given for the VND and CND EXIT curves for both regular and irregular codes. The EXIT curves obtained from the model and analysis closely matched simulation results. It was shown that by matching the EXIT curves of the VND and CND, good non-binary LDPC codes could be designed which achieved superior performance to the code optimised in [11]. Furthermore, the proposed EXIT chart approach is considerably less computationally intensive than the simulation based approach used in [11].

REFERENCES