Algorithms, nymphs, and shepherds

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Abstract

Computability and complexity are the two major lines along which the science of algorithms has evolved; but the same concepts guide many human activities. We try to catch some glimpses of the connections between these two worlds, to meet the expectations of the audience. We will encounter nymphs and universal Turing Machines, ancient traditions and randomized procedures, all tending to the same end: having fun with algorithms.

Although the formalization of algorithms was essentially unknown before the present century, a precise organization of actions has played an important role in the human behavior much before landing in mathematics. Clearly a similar phenomenon has occurred in all sciences. With the great profundity found in his entire work, Ernst Mach insisted that scientific thought derives from common thought of people; and that the development of scientific thought consists of a continuous and punctual correction of common thought [5].

The integration between scientific and common thought is indeed strong in the field of algorithms, although scientists seem to have been too busy to notice this point (Mach himself made only a few remarks about calculus, but at his times algorithmica was still to be born). We will move in this direction without pursuing philosophic claims, with the sole aim of enlightening the unplanned relevance of some mathematical concepts in human (and, as we shall see, divine) behavior. Due to space constraints, we will restrict our discussion to two specific topics: the role of self-reference in computing a function and the growth of confidence in algorithmic results for an increasing number of independent testimonies.

1. Choe and the diagonal language

“Cretans lie”. This ancient statement stroke the bases of thought more than the population of Crete (that, incidentally, was known for its sense of humor). Epimenides
of Cnossus is credited as being the author of the provocation, as testified even by 
Apostole Paul (Titus I-12). A Cretan himself, Epimenedes thus introduced self-reference 
into the statement: this would have not borne any consequence if not mixed with 
negation (lie), as it instead was. In fact this disruptive combination gave rise to a 
famous paradox: if the sentence were true (Cretans indeed lie) the author would have 
laid, thereby implying that Cretans do not lie; and vice versa.

The same combination of self-reference and negation has been used along centuries 
for many other paradoxical constructions, gradually moving from philosophic specula-
tions to mathematical logic. The most striking example of application in the latter area 
is Gödel’s proof that any formal system containing the arithmetic of natural numbers is 
incomplete [3]. With a clever construction of a mathematical environment, Gödel gave 
a formula on natural numbers essentially equivalent to the statement: “This statement 
cannot be proved”. The statement is provable in a chosen formal system if and only 
if it is false in the same system. Therefore either we prove something untrue (that is 
clearly contradictory) or the system is incomplete in the sense that contains statements 
that are true but non-provable.

However, before proceeding into the theory of computing we shall again search 
through the roots of our knowledge. Self-reference recalls the sorrowful existence of 
Echo and Narcissus, in one of the most delicate stories of Greek mythology. Narcissus 
was a beautiful youth, but inaccessible to love. Nemesis caused him to walk to a 
clear fountain where he could see his own image reflected in the water; and Narcissus 
became so enamoured of himself that he drowned trying to reach his image. Echo (a 
nymph of questionable honor, as she used to divert the attention of Hera while Zeus 
was having fun with the other nympha) had a pure feeling of love for Narcissus. When 
he died she pined away in grief, leaving to us, as sole remains, her voice changed into 
an echo.

Narcissus loved himself in such a lyrical manner to induce us to speculate on the 
nature of love: a subtle question thoroughly examined by Plato [7]. Following his 
analysis we must preliminarily investigate the development of mankind, with crucial 
attention to the geometric shape of their bodies. Originally each person was round 
all over, with four arms and legs, two faces perfectly alike, two privy members, and 
all the other parts in proportion. So there were three sexes: man–man (the male), 
woman–woman (the female), and man–woman (the androgynous, sharing equally in 
male and female). Their round shape gave them incredible vigor and speed, so they got 
to conspire against the gods. In punishment Zeus ordered to split them in two halves. 
Since then each human being seeks desperately the missing half. In Plato’s own words: 
“thus anciently is mutual love ingrained in mankind, reassembling our early estate and 
endeavouring to combine two in one and heal the human sore”. Now all this explains 
the different natures of love. The halves of the original male, or of the female, search 
and love beings of their own sex. The halves of the androgynous, instead, search the 
missing half in the opposite sex, until they find it in happiness, or proceed for ever 
in an adulterous life. Although the sympathy of Plato does not seem to be directed 
to the third gender, much can be learned from heterosexual behavior. While arrogance
has condemned mankind to the perpetual search of a lost nature, the androgynous was happy in his completeness. Much earlier than Narcissus, the androgynous loved himself.

Let us now return to Echo. She loved Narcissus with sincerity, but her questionable behavior in the Zeus business may induce a legitimate suspicion of additional, more relaxed sentiments. So we extend the definition of the nymph to one who loves all the ones who love themselves (Narcissus and the androgynous, among others). But does Echo love herself? The question is logically harmless: since self-reference is not mixed with negation, both answers, yes or no, are consistent with the definition of Echo. The paradox arises with a nymph of opposite feelings: Choe (pronounce ‘kɔː-ˌeɪ), who loves the ones who do not love themselves. Now the question of whether Choe loves herself or not is autocontradictory, as in Epimenides’ paradox.

To depict the whole situation we make use of the matrix of mythological loves (Fig. 1) whose interest is enhanced by the numerous loving habits of those creatures [1] and to which Cantor’s diagonal argument applies naturally. Deities were numerable but virtually infinite in number, as they were personages of past anecdotes and tales yet to be told, materializations of natural events, protectors of homes and fields, inhabitants of fountains, trees and mountains. Each one with infinite variations according to place.
Fig. 1 shows only a part of an infinite table, where \( Y \), or \( N \), appears in cell \((i,j)\) to indicate that the creature in row \( i \) loves, or does not love, the creature in column \( j \). Here we find reciprocal eternal loves, as the one of nymph Arethusa merging with Alpheus, god of rivers, in the confluence of two streams in Siracusa. Alpheus has \( Y \) only for Arethusa, and Arethusa has \( Y \) only for Alpheus. Non-reciprocal violent loves, as the one of Zeuss in the form of a bull who raped Europa. \( Y \) and \( N \) are non-symmetrical in this case. The reflexive loves of Narcissus and Androgynous, with a \( Y \) on the diagonal of the matrix. The love of Echo for them and only for them. Blank cells of the matrix may denote lack of information on our side, as in the case of the cell \((Echo, Echo)\) where both \( Y \) and \( N \) would be legitimate. But blanks also denote a subtler situation. Apollo, for example, may regularly go round with Europa without yet having a precise feeling for her. Nobody knows if he will eventually decide for \( Y \) or \( N \), or go on for ever. A blank here denotes that a procedure is still running. And now, what is the position of Choe into the matrix?

A diagonal argument shows that Choe cannot be a member of the celestial company, because her row would not have a consistent entry at the intersection with the diagonal. That is, although the behavior of Choe is well defined, she could exist only as a living paradox (this is probably the reason why Choe is not found in any book of mythology). If we insist to be living in a world free of paradoxes Choe does not exist, although her behavior does. Indeed behaviors are functions \( D \rightarrow \{Y, N, \Box\} \), where \( D = \{\delta \mid \delta \) is a deity\) is the infinite set of deities, and \( \Box \) denotes a blank; and deities are Turing Machines. In the matrix, behaviors are represented as rows of infinite length, each assumed by the deity specified in the row label. Unlike labels, all possible rows are not numerable: it is then not surprising that there exist legitimate behaviors that cannot be assumed by anybody.

Mathematicians have certainly realized that Choe is the (non-existing) Turing Machine that should recognize the diagonal language \( L_d \) (see [4])

\[
L_d = \{\delta \mid \delta \in D, \delta \text{ does not love } \delta\}.
\]

From this we can rephrase a classical result in terms of mythology. Let

\[
L_u = \{\eta\beta \mid \eta \text{ and } \beta \in D, \eta \text{ loves } \beta\}
\]

be the universal language. We have

**Theorem 1.** \( L_u \) is recursively enumerable and non-recursive.

**Proof.** Let a deity of deities \( \Delta \) be able to understand if another deity \( \eta \) will eventually fall in love with \( \beta \), for arbitrary \( \eta \) and \( \beta \). (We like to think that \( \Delta \) is indeed Hera, mother of all deities and always interested in love stories.) \( \Delta \) can induce \( \eta \) and \( \beta \) to meet, and watch the behavior of \( \eta \). If \( \eta \) gives clear signs of enamourement, or disgust, \( \Delta \) decides for \( Y \) or \( N \). This is enough to prove that \( L_u \) is recursively enumerable. However, if \( \eta \) lingers, \( \Delta \) may not decide by simple observation if \( \eta \) is going to eventually develop
a stable sentiment, or keeps $\beta$ in suspense for ever. Nevertheless, we now prove by contradiction that $\Delta$ has no other means for deciding.

Assume that $\Delta$ can decide in finite time if $\eta$ falls or not in love with $\beta$, or will linger indefinitely. $\eta$ and $\beta$ being arbitrary, $\Delta$ can also decide if $\eta$ falls in love with $\eta$. Complementing the output (where $N$ and perpetual lingering are both seen as negative answers), $\Delta$ would recognize $L_d$ thereby taking the role of Choe, hence proving her existence.

Such subtle arguments, we believe, should rightfully induce the sensation of being fooled in any person of good sense. The responsibility of such an unpleasant condition is totally on the side of the mathematicians, whose arguments are nevertheless to be accepted because they are always validated by distinguished persons.

2. Massud and primality testing

"Nihil enim est tam contrarium rationi et constantiae quam fortuna, ut mihi ne in deum quidem cadere videatur ut sciat quid casu et fortuito futurum sit". (Nothing indeed is as contrary to reason and regularity as chance. In fact not even the gods, I believe, know what will happen by chance and fortune) [2].

Cicero underlined the power of chance with these words. Twenty centuries later, computer scientists started using that power to solve difficult problems in a new original manner.

The story of chance is not completely clear. Democritus is often credited as being the initiator, although no ancient thinker was possibly more mechanistic than he was. Only in the sixteenth century Cardano picked up the concept again, until a correspondence between Pascal and Fermat on the game of dice, one century later, gave official birth to the study of probability, and raised the discussion on chance again. Philosophers, biologists, physicists have been deeply interested since then in the habits of this crazy personage who mixes up a well ordered universe. They even discussed ardently if chance existed at all (Carl Gustav Jung had no doubt of its existence as a threat to rational behavior). Mathematicians went on more slowly: for long time they were unable to satisfactorily formalize the rules of the game, until, a few decades ago, Kolmogorov laid down the bases of a sound probability theory.

Discussing this matter is not our purpose. We will illustrate, instead, how the power of chance can be exploited to give satisfactory answers to important and difficult questions. The mathematical aspects of randomization can be studied in [6]. We examine a more concrete fact taken from an ancient chronicle of Central Asia (oral tradition).

Massud, a young shepherd of the steppe, is consumed by the desire of knowing if Pardis, his betrothed, is beautiful as all their relatives swear. By modesty and tradition Pardis always appears in public with a veiled face, and only women can see her in privacy. Massud wishes to inquire delicately with some of them, but speculates on who are the right women to ask. He fears the opinion of relatives to be exceedingly
positive; the judgment of Fatima (the belly dancer who used to be his secret girl friend) to be exaggeratedly negative; the opinions of persons of similar taste and habits to be meaninglessly similar. In essence, Massud wishes to collect reliable opinions from independent witnesses chosen at random.

To proceed with scientific method, Massud establishes some rules based on his knowledge of human psychology. A woman’s opinion that Pardis is unpleasant to the sight must unfortunately be taken for sure. An opinion that Pardis is beautiful, instead, has to be trusted with a certain probability $p$. In the first case Massud will terminate his enquiry, trying to escape his commitment for marriage without major damages. In front of a positive opinion, instead, he will continue the enquiry with other women, until he is convinced that his betrothed is really beautiful. The algorithm is as follows:

**Algorithm 1. TEST FOR BEAUTY**

choose a positive integer $k$ to establish the level of confidence on the answer

while less than $k$ women have expressed an opinion do

choose a new woman $W$ independently at random

if \( \text{OPINION}(W, \text{Pardis}) = \text{ugly} \) then return NO (and run away)

else continue

return YES (and stay).

We have:

**Theorem 2.** If algorithm 1 returns NO, Pardis is certainly ugly, otherwise she is ugly with probability \( (1 - p)^k \).

For example if Massud has decided to interrogate $k = 100$ women, and in his neighborhood $p = \frac{1}{2}$, the probability of marrying a girl who is ugly even if she passed the test for beauty is reduced to \( \frac{1}{2^{100}} \). Massud can wisely accept such a result, thinking that there is a higher probability to lose his own sight at the moment of lifting Pardis’ veil after marriage, for the concurrent attack of two tiger-hornets of the steppe, one for each eye.

Note how powerful the method is. The test runs in time linear with $k$, under the hypothesis that each woman speaks for constant time. If Massud is still not satisfied of the level of confidence on the result reached with a certain value of $k$, he can slightly increase this value to obtain a much higher confidence. The nature of well designed randomized (Monte Carlo) algorithms with a binary answer is thus revealed. In polynomial time and using independent random choices, we reach a certainly correct solution on one of the answers; on the other answer the solution may be wrong, but this happens with a prefixed and arbitrarily small probability. The parameters are then tuned to have this probability smaller than the one that the algorithm fails for any other reason, for example for an hardware crash.

Computer scientists have certainly recognized in algorithm 1 the structure of a famous primality test that marked the birth of randomized algorithms in 1976. Many
other professionals, however, could benefit from the power of chance. In particular judges, who often have chance on their side in courts without even noticing it. With the exception of judge Bridoye who used to give sentences casting dice [8].

We are not so arrogant to teach the judges what to do. But since common people know judicial proceedings from the movies, let us raise for them a scientific argument on the value of proofs. Was Socrates really guilty? Apparently not. But maybe he was and solid proofs against him existed but were never discovered. Or maybe the judges, secretly fond of philosophy, concealed the proofs of guilt to pronounce an apparently unjust sentence upon the philosopher, with the purpose of raising popular sympathy in his favor. Instead of going on with arbitrary speculations we shall take a mathematical approach.

Let us assume that, in a liberal body of laws, a proof in favor of the defendant is sufficient to sentence him not guilty and a proof against him has to be taken with caution. If \( n \) independent testimonies of guilt have been produced and \( p_1, p_2, \ldots, p_n \) are the probabilities of them to be true (the judge assigns these values), the probability that the defendant is unjustly condemned is \((1 - p_1) \times (1 - p_2) \times \cdots \times (1 - p_n)\). The responsibility of the judge is now clear. Since no sentence is absolutely certain, enough testimonies must be accumulated until the probability of error becomes inferior to the one that a different accident renders the sentence vain. For example a blackout that interrupts the execution by electric chair. To condemn an innocent is a very sad fact. But with the aid of mathematics judges can at least do better than Bridoye who, after all, gave a right sentence in every other suit.

References