ON THE BEHAVIOUR OF A RUMOUR PROCESS WITH RANDOM STIFLING

ELCIO LEBENSZTAYN, FÁBIO P. MACHADO, AND PABLO M. RODRÍGUEZ

Abstract. We propose a realistic generalization of the Maki-Thompson rumour model by assuming that each spreader ceases to propagate the rumour right after being involved in a random number of stifling experiences. We consider the process with a general initial configuration and establish the asymptotic behaviour (and its fluctuation) of the ultimate proportion of ignorants as the population size grows to $\infty$. Our approach leads to closed formulas so that the limiting proportion of ignorants and the variance can be computed.

1. Introduction

In the past decades, there has been great interest in understanding and modelling different processes for information diffusion in a population. Most of the time, the mathematical theory of epidemics is adapted for this purpose, even though there are differences from the process of spreading an information and the process of spreading a virus or a disease. Speaking of standard versions of the models, the most noticeable differences are between the way spreaders cease to spread the information and the way infected individuals are removed from epidemic processes. Still, some alternative models fit both processes.

Kurtz et al. [7] introduced recently a model in the complete graph in which, as soon as an individual is infected, an anti-virus is set in that individual in such a way that the next time a virus tries to infect it, the virus can be defeated. Besides, a virus can survive up to a fixed amount of $L$ individuals empowered with anti-virus. Individuals are represented by the vertices of the complete graph, while the virus is represented by a moving agent that replicates every time it hits a healthy individual. The authors prove a Weak Law of Large Numbers and a Central Limit Theorem for the proportion of infected individuals after the process has finished.

There are two classical models for the spreading of a rumour in a population, which were formulated by Daley and Kendall [4] and Maki and Thompson [10]. In the model proposed by Maki and Thompson [10], a closed homogeneously mixing population experiences a rumour process. Three classes of individuals are considered: ignorants, spreaders and stiflers. The rumour is propagated through the population by directed contact between spreaders and other individuals, which are governed by the following set of rules. When a
spreader interacts with an ignorant, the ignorant becomes a spreader; whenever a spreader contacts a stifler, the spreader turns into a stifler and when a spreader meets another spreader, the initiating one becomes a stifler. In the last two cases, it is said that the spreader was involved in a *stifling experience*. Observe that the process eventually finishes (when there are no more spreaders in the population).

We show how techniques used by Kurtz et al. [7] in the context of epidemic models can be useful to study a general rumour process. In particular, we propose a generalization of the Maki-Thompson model. In our model, each spreader decides to stop propagating the rumour right after being involved in a random number of stifling experiences.

To define the process, consider a closed homogeneously mixing population of size $N + 1$. Let $R$ be a nonnegative integer valued random variable with distribution given by $P(R = i) = r_i$ for $i = 0, 1, \ldots$, and let $\mu = E[R] > 0$ and $\nu^2 = \text{Var}[R]$. Assign independently to each individual initially ignorant a random variable with the same distribution of $R$. Once an ignorant hears the rumour, the copy of $R$ assigned to him determines the number of stifling experiences the new spreader will have until he stops propagating the rumour. If this random variable equals zero, then the ignorant joins the stiflers immediately after hearing the rumour.

For $i = 1, 2, \ldots$, we say that a spreader is of type $i$ if this individual has exactly $i$ remaining stifling experiences. We denote the number of ignorants, spreaders of type $i$ and stiflers at time $t$ by $X^{(N)}(t)$, $Y_i^{(N)}(t)$ and $Z^{(N)}(t)$, respectively. Let $Y^{(N)}(t) = \sum_{i=1}^{\infty} Y_i^{(N)}(t)$ be the total number of spreaders at time $t$, so $X^{(N)}(t) + Y^{(N)}(t) + Z^{(N)}(t) = N + 1$ for all $t$. Notice that the infinite-dimensional process

$$\{V^{(N)}(t)\}_{t \geq 0} := \{(X^{(N)}(t), Y_1^{(N)}(t), Y_2^{(N)}(t), \ldots)\}_{t \geq 0}$$

is a continuous time Markov chain with transitions and corresponding rates given by

<table>
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<tr>
<td>$(-1, 0, 0, \ldots)$</td>
<td>$r_0 \ XY$</td>
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<tr>
<td>$(-1, 0, \ldots, 0, 1, 0, \ldots)$</td>
<td>$r_i \ XY$</td>
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<tr>
<td>$i-1 \ i \ i+1 \ 0, \ldots$</td>
<td>$r_i \ XY$</td>
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<tr>
<td>$i-1 \ i \ i+1 \ 0, \ldots$</td>
<td>$(N + 1 - X) Y_i$</td>
</tr>
<tr>
<td>$0, \ldots, 0, 1, -1, 0, \ldots$</td>
<td>$(N + 1 - X) Y_i$</td>
</tr>
<tr>
<td>$0, -1, 0, 0, \ldots$</td>
<td>$(N + 1 - X) Y_i$</td>
</tr>
</tbody>
</table>

In words, the first case indicates the transition of the process in which a spreader interacts with an ignorant and the ignorant becomes a stifler immediately (which happens with probability $r_0$). The second case indicates the transition in which a spreader interacts with an ignorant and the ignorant becomes a spreader of type $i$ (which happens with probability $r_i$). The third one represents the situation in which a spreader of type $i$ is involved in a stifling experience but remains a spreader (of type $i - 1$), and finally the last transition indicates the event that a spreader of type 1 is involved in a stifling experience, becoming a stifler.
We suppose that the process starts with
\[ X^{(N)}(0) = (N + 1) x_0^{(N)}, \]
\[ Y_i^{(N)}(0) = (N + 1) y_i^{(N)}, \quad \text{for } i = 1, 2, \ldots \]
\[ Z^{(N)}(0) = (N + 1) z_0^{(N)}. \]
That is, \( x_0^{(N)}, y_i^{(N)}, z_0^{(N)} \in [0, 1] \) are the initial proportions of ignorants, spreaders of type \( i \) and stiflers of the population, respectively, so that
\[ x_0^{(N)} + \sum_{i=1}^{\infty} y_i^{(N)} + z_0^{(N)} = 1. \]

In addition, we assume that the following limits exist:
\[ x_0 = \lim_{N \to \infty} x_0^{(N)} > 0 \quad \text{and} \]
\[ y_{i,0} = \lim_{N \to \infty} y_{i,0}^{(N)} \quad \text{for all } i = 1, 2, \ldots, \]
and define
\[ w_0 = \sum_{i=1}^{\infty} i y_{i,0}. \]
As already mentioned, the process eventually ends. Let
\[ \tau^{(N)} = \inf \{ t : Y^{(N)}(t) = 0 \} \]
be the absorption time of the process. Our main purpose is to establish limit theorems for the proportion \( N^{-1} X^{(N)}(\tau^{(N)}) \) of ignorants at the end of the process. For the classical Maki-Thompson model, this problem was first studied rigorously by Sudbury [12], who proved, by using martingale arguments, that this proportion converges in probability to 0.203. This result was generalized later by Watson [13], using normal asymptotic approximation. Lefevre and Picard [9] derived the exact joint distribution of the final number of people who heard the rumour and the total personal time units during which the rumour was spread. In Belen and Pearce [1], the authors present an analysis of the proportion of the population never hearing the rumour starting from a general initial condition. See also Chapter 5 of Daley and Gani [3] for an excellent account on rumour models.

The approach used to prove our theorems is the theory of density dependent Markov chains, presented in Ethier and Kurtz [5]. To the best of our knowledge, this technique was first used in the context of rumour models in Lebendstyn et al. [8]. In that paper, the authors study a family of rumour processes which includes as particular cases the classical Daley-Kendall and Maki-Thompson models. The results present here are of independent interest, as they refer to a generalization of the Maki-Thompson model with random stifling and general initial configuration.
2. Main results

Definition 2.1. Suppose that $\mu < \infty$ and consider the function $f : (0, x_0] \to \mathbb{R}$ given by
\[
f(x) = w_0 + (1 + \mu)(x_0 - x) + \log \frac{x}{x_0}.
\] (2.1)

We define $x_\infty = x_\infty(\mu, x_0, w_0)$ as the unique root of $f$ in the interval $(0, x_0]$ satisfying $f'(x) \geq 0$.

Notice that $x_\infty$ is the unique root of $f$, except in the case that $x_0 > (1 + \mu)^{-1}$ and $w_0 = 0$. See Figure 1.

Remark 2.2. We can express $x_\infty$ in terms of the Lambert $W$ function, which is the inverse of the function $x \mapsto x e^x$. Indeed, $x_\infty$ satisfies
\[
x_\infty = x_0 e^{-(1 + \mu)(x_0 - x_\infty) - w_0}
\] which can be written as
\[
-x_0(1 + \mu) e^{-x_0(1 + \mu) - w_0} = -x_\infty(1 + \mu) e^{-x_\infty(1 + \mu)}.
\] (2.2)
Then, if $W_0$ denotes the principal branch of the Lambert $W$ function (that is, the branch that satisfies $W(x) \geq -1$), we obtain from (2.2) that
\[
x_{\infty}(\mu, x_0, w_0) = -(1 + \mu)^{-1}W_0(-x_0(1 + \mu)e^{-x_0(1+\mu)-w_0}),
\] (2.3)
by noting that $-e^{-1} < -x_0(1 + \mu)e^{-x_0(1+\mu)-w_0} < 0$. More details about the Lambert function can be found in Corless et al. \cite{2}.

Next, we state the Weak Law of Large Numbers for the proportion of the population never hearing the rumour.

**Theorem 2.3.** If $0 < \mu < \infty$, then
\[
\lim_{N \to \infty} \frac{X(N)(\tau(N))}{N} = x_{\infty} \quad \text{in probability.}
\]

As a consequence of this theorem,

**Corollary 2.4.** If $\mu = \infty$, then
\[
\lim_{N \to \infty} \frac{X(N)(\tau(N))}{N} = 0 \quad \text{in probability.}
\]

**Proof.** Let $R_1$ and $R_2$ be nonnegative integer valued random variables such that $R_1 \leq R_2$, that is, $P(R_1 \geq i) \leq P(R_2 \geq i)$ for all $i \geq 0$. Consider the processes $\{V_1^{(N)}(t)\}_{t \geq 0}$ and $\{V_2^{(N)}(t)\}_{t \geq 0}$ defined as in (1.1) by using the random variables $R_1$ and $R_2$, respectively, and with the same initial conditions. Let $\tau_1^{(N)}$ and $\tau_2^{(N)}$ be the respective absorption times. By a coupling argument, these processes can be constructed in such a way that
\[
X_2^{(N)}(\tau_2^{(N)}) \leq X_1^{(N)}(\tau_1^{(N)}), \quad \text{a.s.} \tag{2.4}
\]
Now suppose that $\{V^{(N)}(t)\}_{t \geq 0}$ is the process defined in (1.1) with the random variable $R$ satisfying $\mu = \infty$. Recall that $r_i = P(R = i)$, $i \geq 0$, and for each $k \geq 1$ define the random variable $R_k$ with distribution given by
\[
P(R_k = i) = r_i \quad \text{if } i < k \quad \text{and} \quad P(R_k = k) = \sum_{j=k}^{\infty} r_j.
\]
By construction, we have that $R_k \overset{D}{\leq} R$ for all $k$. Taking this and (2.4) into account, we conclude that
\[
X^{(N)}(\tau^{(N)}) \leq X_k^{(N)}(\tau_k^{(N)}), \quad \text{a.s.} \tag{2.5}
\]
for all $k$. Then (2.5) and Theorem 2.3 imply that
\[
0 \leq \limsup_{N \to \infty} \frac{X^{(N)}(\tau^{(N)})}{N} \leq x_{\infty}(\mu_k, x_0, w_0) \quad \text{a.s.},
\]
where $\mu_k = E[R_k]$. Since $x_0 > 0$ and $\lim_{k \to \infty} \mu_k = \infty$, we have that $x_0 > (1 + \mu_k)^{-1}$ for large enough $k$ and in this case $x_{\infty}(\mu_k, x_0, w_0)$ (given by (2.3) with $\mu = \mu_k$) goes to 0 as $k \to \infty$. This completes the proof of Corollary 2.4. \qed
We present the Central Limit Theorem for the ultimate proportion of ignorants in the population.

**Theorem 2.5.** Suppose that $\nu^2 < \infty$. Assume also that $w_0 > 0$ or that $w_0 = 0$ and $x_0 > (1 + \mu)^{-1}$. Then,

$$\sqrt{n} \left( \frac{X^{(N)}(\tau^{(N)})}{N} - x_\infty \right) \xrightarrow{D} N(0, \sigma^2) \text{ as } n \to \infty,$$

where $\xrightarrow{D}$ denotes convergence in distribution, and $N(0, \sigma^2)$ is the Gaussian distribution with mean zero and variance given by

$$\sigma^2 = \frac{x_\infty (1 - (x_0^{-1} + w_0 + (x_0 - x_\infty)(1 + \mu - \nu^2)) x_\infty)}{(1 - (1 + \mu) x_\infty)^2}. \quad (2.6)$$

**Remark 2.6.** Observe that our results refer to a general initial condition, similar to that considered in the deterministic analysis presented in Belen and Pearson [1]. The process starting with one spreader and $N$ ignorants corresponds to $x_0 = 1$ and $w_0 = 0$, in which case the asymptotic variance reduces to

$$\sigma^2 = \frac{x_\infty (1 - x_\infty)(1 - (1 + \mu - \nu^2) x_\infty)}{(1 - (1 + \mu) x_\infty)^2}.$$

In particular, for $R \equiv \kappa$ ($\kappa \geq 1$ an integer), we have the $\kappa$-fold stifling Maki-Thompson model (so called by [3] in the context of Daley-Kendall model), for which

$$\sigma^2 = \frac{x_\infty (1 - x_\infty)}{1 - (\kappa + 1) x_\infty},$$

where

$$x_\infty = x_\infty(\kappa, 1, 0) = -(\kappa + 1)^{-1} W_0(- (\kappa + 1) e^{-(\kappa + 1)}).$$

Table 1 exhibits the values of $x_\infty$ and $\sigma^2$ in this case for $\kappa = 1, \ldots, 6$. The original Maki-Thompson model is obtained by considering $R \equiv 1$, $x_0 = 1$ and $w_0 = 0$, consequently our theorems generalize classical results presented by Sudbury [12] and Watson [13].

Another interesting variant is obtained by letting $R$ following a geometric distribution with parameter $p$, that is, $r_0 = 0$ and $r_i = p (1 - p)^{i-1}$, $i = 1, 2, \ldots$. In this model, an ignorant always becomes a spreader upon hearing the rumour and each time a spreader meets somebody already informed, he decides with probability $p$ to become a stifler, independently for each spreader and each meeting. Table 2 shows the values of $x_\infty$ and $\sigma^2$ for $x_0 = 1, w_0 = 0$ and some arbitrarily chosen values of $p$.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_\infty$</td>
<td>0.203</td>
<td>0.0595</td>
<td>0.0198</td>
<td>0.00698</td>
<td>0.00252</td>
<td>0.000918</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>0.273</td>
<td>0.0681</td>
<td>0.0211</td>
<td>0.00718</td>
<td>0.00255</td>
<td>0.000923</td>
</tr>
</tbody>
</table>

**Table 1.** $\kappa$-fold stifling model, $x_0 = 1$ and $w_0 = 0$. 
Table 2. $R \sim \text{Geometric}(p)$, $x_0 = 1$ and $w_0 = 0.$

### 3. Proofs

Here are the main ideas to prove Theorems 2.3 and 2.5. First, by means of a suitable
time change of the process, we define a new process $\{\tilde{V}(N)(t)\}_{t \geq 0}$ with the same transitions
as $\{V(N)(t)\}_{t \geq 0}$ so that they finish at the same point of the state space. Next, we work
with a reduced Markov chain obtained from $\{\tilde{V}(N)(t)\}_{t \geq 0}$ in order to apply the theory of
density dependent Markov chains presented in Ethier and Kurtz [5]. As the arguments
follow a path similar to that presented in Kurtz et al. [7], we present a sketchy version of
the proofs.

**Time-changed process.** Since the distribution of $X(N)(\tau(N))$ depends on the process
$\{V(N)(t)\}_{t \geq 0}$ only through the embedded Markov chain, we consider a time-changed version
of the process. Let $\{\tilde{V}(N)(t)\}_{t \geq 0}$ be the infinite-dimensional continuous time Markov chain
$\{(\tilde{X}(N)(t), \tilde{Y}_1(N)(t), \tilde{Y}_2(N)(t), \ldots)\}_{t \geq 0}$
with transitions and corresponding rates given by

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<td>$(N + 1 - \tilde{X}) \tilde{Y}_i(\tilde{Y})^{-1}$</td>
</tr>
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Furthermore, $\{\tilde{V}(N)(t)\}_{t \geq 0}$ can be defined in such a way that it has the same initial state
and the same transitions of $\{V(N)(t)\}_{t \geq 0}$, so they have the same embedded Markov chain. Thus, by defining

$$\tilde{\tau}(N) = \inf \left\{ t : \tilde{Y}(N)(t) = 0 \right\},$$

we have that $X(N)(\tau(N)) = \tilde{X}(N)(\tilde{\tau}(N)).$

**Dimension reduction and deterministic limit.** In order to prove the desired limit
theorems using directly Theorem 11.2.1 of Ethier and Kurtz [5], we work with a reduced
Markov chain. We define

$$\tilde{W}(N)(t) = \sum_{i=1}^{\infty} i \tilde{Y}_i(N)(t),$$
and note that the process \(\{(\tilde{X}^{(N)}(t), \tilde{W}^{(N)}(t))\}_{t \geq 0}\) is a continuous time Markov chain with transitions and rates given by

\[
\begin{align*}
\ell_i &= (-1, i) \quad r_i \tilde{X} \quad i = 0, 1, \ldots \\
\ell_{-1} &= (0, -1) \quad N + 1 - \tilde{X}.
\end{align*}
\tag{3.1}
\]

Now we define, for \(t \geq 0\),

\[\tilde{v}^{(N)}(t) = (\tilde{x}^{(N)}(t), \tilde{w}^{(N)}(t)) = N^{-1}(\tilde{X}^{(N)}(t), \tilde{W}^{(N)}(t)),\]

and consider

\[\beta_{\ell_{-1}}(x, w) = 1 - x \quad \text{and} \quad \beta_{\ell_i}(x, w) = r_i x, \quad i = 0, 1, \ldots\]

Notice that the rates in (3.1) can be written as

\[N \left[ \beta_{\ell_i} \left( \frac{\tilde{X}}{N}, \frac{\tilde{W}}{N} \right) + O \left( \frac{1}{N} \right) \right],\]

so \(\{\tilde{v}^{(N)}(t)\}_{t \geq 0}\) is a density dependent Markov chain with possible transitions in the set \(\{\ell_{-1}, \ell_0, \ell_1, \ldots\}\).

Now we use Theorem 11.2.1 of Ethier and Kurtz \[5\] to conclude that the process \(\{\tilde{v}^{(N)}(t)\}_{t \geq 0}\) converges almost surely as \(N \to \infty\) to a deterministic limit. The drift function defined in Ethier and Kurtz \[5\] by \(F(x, w) = \sum_{i=-1}^{\infty} \ell_i \beta_{\ell_i}(x, w)\) in this case is given by

\[F(x, w) = (-x, (\mu + 1)x - 1).\]

Hence the limiting deterministic system is governed by the following system of ordinary differential equations

\[
\begin{align*}
x'(t) &= -x(t), \\
w'(t) &= (\mu + 1)x - 1
\end{align*}
\]

with initial conditions \(x(0) = x_0\) and \(w(0) = w_0\). The solution of this system is given by \(v(t) = (x(t), w(t))\), where

\[x(t) = x_0 e^{-t} \quad \text{and} \quad w(t) = f(x(t)) = w_0 + (1 + \mu)(x_0 - x(t)) - t.\]

According to Theorem 11.2.1 of Ethier and Kurtz \[5\], we have that on a suitable probability space,

\[\lim_{N \to \infty} \tilde{v}^{(N)}(t) = v(t) \quad \text{a.s.}\]

uniformly on bounded time intervals. In particular, it can be proved that

\[\lim_{N \to \infty} \tilde{x}^{(N)}(t) = x(t) \quad \text{a.s.}\]

\tag{3.2}

uniformly on \(\mathbb{R}\). See Lemma 3.2 in Lebensztayn et al. \[8\] for an analogous detailed proof.
Proofs of Theorems 2.3 and 2.5. To prove both theorems, we use Theorem 11.4.1 of Ethier and Kurtz [5]. We adopt the notations presented there, except by the Gaussian process $V$ defined in p. 458, that we would rather denote by $U = (U_x, U_w)$. Here, $\varphi(x, w) = w$, and

$$\tau_{\infty} = \inf\{t : w(t) \leq 0\} = w_0 + (1 + \mu)(x_0 - x_{\infty}).$$

Moreover,

$$\nabla \varphi(v(\tau_{\infty})) \cdot F(v(\tau_{\infty})) = w'(\tau_{\infty}) = (\mu + 1)x_{\infty} - 1 < 0. \quad (3.3)$$

Proof of Theorem 2.3. We note that $w_0 > 0$ and $(3.3)$ imply that $w(\tau_{\infty} - \varepsilon) > 0$ and $w(\tau_{\infty} + \varepsilon) < 0$ for $0 < \varepsilon < \tau_{\infty}$. Then, the almost sure convergence of $\tilde{w}^{(N)}$ to $w$ uniformly on bounded intervals yields that

$$\lim_{N \to \infty} \tilde{\tau}^{(N)} = \tau_{\infty} \quad \text{a.s.} \quad (3.4)$$

In the case that $w_0 = 0$ and $x_0 > (1 + \mu)^{-1}$, this result is also valid because $w'(0) > 0$ and $(3.3)$ still holds. On the other hand, if $w_0 = 0$ and $x_0 \leq (1 + \mu)^{-1}$, then $w(t) < 0$ for all $t > 0$, and again the almost sure convergence of $\tilde{w}^{(N)}$ to $w$ uniformly on bounded intervals yields that $\lim_{N \to \infty} \tilde{\tau}^{(N)} = 0 = \tau_{\infty}$ almost surely. Therefore, as $X^{(N)}(\tau^{(N)}) = \tilde{X}^{(N)}(\tilde{\tau}^{(N)})$, we obtain Theorem 2.3 from (3.2) and (3.4).

Proof of Theorem 2.5. From Theorem 11.4.1 of Ethier and Kurtz [5], we have that

$$\sqrt{N} (\tilde{x}^{(N)}(\tilde{\tau}^{(N)}) - x_{\infty})$$

converges in distribution as $N \to \infty$ to

$$U_x(\tau_{\infty}) + \frac{x_{\infty}}{(\mu + 1)x_{\infty} - 1} U_w(\tau_{\infty}). \quad (3.5)$$

The resulting normal distribution has mean zero, so to complete the proof of Theorem 2.5 it remains to calculate the corresponding variance.

To this end, we have to compute the covariance matrix $\text{Cov}(U(\tau_{\infty}), U(\tau_{\infty}))$, a task that can be accomplished with a mathematical software. The first step is to calculate the matrix of partial derivatives of the drift function $F$ and the matrix $G$. We obtain

$$\partial F(x, w) = \begin{pmatrix} -1 & 0 \\ (\mu + 1) & 0 \end{pmatrix}$$

and

$$G(x, w) = \begin{pmatrix} x & -\mu x \\ -\mu x & (\nu^2 + \mu^2 - 1)x + 1 \end{pmatrix}.$$ 

Next, we compute the solution $\Phi$ of the matrix equation

$$\frac{\partial}{\partial t} \Phi(t, s) = \partial F(x(t), w(t)) \Phi(t, s), \quad \Phi(s, s) = I_2,$$

which is given by

$$\Phi(t, s) = \begin{pmatrix} e^{-(t-s)} & 0 \\ (\mu + 1)(1 - e^{-(t-s)}) & 1 \end{pmatrix}.$$ 

Hence, the covariance matrix of the Gaussian process $U$ at time $t$ is obtained by the formula

$$\text{Cov}(U(t), U(t)) = \int_0^t \Phi(t, s) G(x(s), w(s)) \Phi(t, s)^T ds. \quad (3.6)$$
As the final step to compute Cov($U(\tau_\infty), U(\tau_\infty)$), we have to replace $e^{-t}$ and $t$ in the formula obtained from (3.6) by $x_\infty/x_0$ and $\tau_\infty$, respectively. The resulting formulas are

$$\text{Var}(U_x(\tau_\infty)) = ((x_0 - x_\infty)x_\infty)/x_0,$$

$$\text{Var}(U_w(\tau_\infty)) = (\mu + 1)^2(x_0 - x_\infty)x_\infty/x_0 + \nu^2(x_0 - x_\infty) + (1 - 2(\mu + 1)x_\infty)\tau_\infty,$$

$$\text{Cov}(U_x(\tau_\infty), U_w(\tau_\infty)) = \tau_\infty x_\infty - (\mu + 1)(x_0 - x_\infty)x_\infty/x_0.$$

Using that $\tau_\infty = w_0 + (1 + \mu)(x_0 - x_\infty)$, (3.5) and well-known properties of the variance, we get formula (2.6).

4. Concluding remarks

We propose a general Maki-Thompson model in which an ignorant individual is allowed to have a random number of stifling experiences once he is told the rumour. The assigned number of stifling experiences are independent and identically distributed random variables with mean $\mu$ and variance $\nu^2$. We prove that the ultimate proportion of ignorants converges in probability to an asymptotic value as the population size tends to $\infty$. A Central Limit Theorem describing the magnitude of the random fluctuations around this limiting value is also stated. The asymptotic value and the variance of the Gaussian distribution in the CLT are functions of $\mu$, $\nu^2$ and some constants related to the initial state of the process.

We observe that in fact it is possible to obtain another result, concerning the mean number $m^{(N)}$ of transitions that the process makes until absorption. Using an analogous argument of that presented in Theorem 2.5 of Kurtz et al. [7], it can be proved that, if $\nu^2 < \infty$, then

$$\lim_{N \to \infty} N^{-1} m^{(N)} = \tau_\infty = w_0 + (1 + \mu)(x_0 - x_\infty).$$

As a final remark, we would like to point out the usefulness of the theory of density dependent Markov chains as a tool for studying the limiting behaviour of stochastic rumour processes. This approach constitutes an alternative route for the pgf method and the Laplace transform presented in Daley and Gani [3], Gani [6] and Pearce [11].

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REFERENCES


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