MULTI-ATTRIBUTE TARGET-BASED UTILITIES
AND EXTENSIONS OF FUZZY MEASURES

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ABSTRACT. We introduce a formal description of the Target-Based approach to utility theory for the case of \( n > 1 \) attributes and point out the connections with aggregation-based extensions of capacities. Our discussion provides economic interpretations of different concepts of the theory of fuzzy measures. In particular, we analyze the meaning of extensions of capacities based on \( n \)-dimensional copulas. The latter describes stochastic dependence for random vectors of interest in the problem. We also trace the connections between the case of \( \{0,1\} \)-valued capacities and the analysis of "coherent" reliability systems.

KEYWORDS. Decision Analysis, Stochastic dependence, Copulas, Möbius transform, Owen extension, Lovász extension, Reliability-structured utilities, Correlation Aversion.

1. INTRODUCTION

A rich literature has been devoted in the last decade to the Target-Based Approach (TBA) to utility functions and economic decisions (see [4, 5, 8, 9, 34, 35]). This literature is still growing, with a main focus on applied aspects (see, for example, [2, 37, 38]).

Even from a theoretical point of view, however, some issues of interest demand for further analysis. In this direction, the present paper will consider some aspects that emerge in the analysis of the multi-attribute case. Generally TBA can provide probabilistic interpretations of different notions of utility theory. Here we will in particular interpret in terms of stochastic dependence the differences among copula-based extensions of a same fuzzy measure.

In order to explain the basic concepts of the TBA it is, in any case, convenient to start by recalling the single-attribute case. Let \( \Xi := \{X_\alpha\}_{\alpha \in A} \) be a family of real-valued random variables, that are distributed according to probability distribution functions \( F_\alpha \) respectively. Each element \( X_\alpha \in \Xi \) is seen as a prospect or a lottery and a Decision Maker is expected to conveniently select one element out of \( \Xi \) (or, equivalently, \( \alpha \in A \)). Let \( U : \mathbb{R} \to \mathbb{R} \) be a (non-decreasing) utility function, that describes the Decision Maker’s attitude toward risk. Thus, according to the Expected Utility Principle (see [36]), the DM’s choice is performed by maximizing the integral

\[
E[U(X_\alpha)] = \int_{\mathbb{R}} U(x) \, dF_\alpha(x).
\]

In the Target-Based approach one in addition assumes \( U \) to be right-continuous and bounded so that, by means of normalization, it can be seen as a probability distribution function over the real line. This approach suggests looking at \( U \) as at the distribution function \( F_T \) of a random variable \( T \). This variable will be considered as a target, stochastically independent of all the prospects \( X_\alpha \). If \( T \) is a (real-valued) random variable stochastically independent of \( X_\alpha \) in fact, one has

\[
E(F_T(X_\alpha)) = \int \mathbb{P}(T \leq x) \, dF_\alpha(dx) = \mathbb{P}(T \leq X_\alpha),
\]
and then, by setting $U = F_T$, the Expected Utility Principle prescribes a choice of $\alpha \in A$ which maximizes the probability $E[U(X_\alpha)] = P(T \leq X_\alpha)$.

The conceptual organization and formalization of basic ideas have been proposed at the end of nineties of last century by Castagnoli, Li Calzi, and Bordley. Some arguments, that can be regarded nowadays as related with the origins of TBA, had been around however in the economic literature since a long time (see [4, 8] and references therein).

After the publication of these papers, several developments appeared in the subsequent years concerning the appropriate way to generalize the TBA to the case of multi-attribute utility functions, see in particular [5, 34, 35]. As already mentioned such an approach, when applicable, offers probabilistic interpretations of notions of utility theory, and this is accomplished in terms of properties of the probability distribution of a random target. Such interpretations, in their turn, are easily understandable and practically useful. In particular, they can help a Decision Maker in the process of assessing her/his own utility function.

A natural extension of the concept of Target-Based utility from the case $n = 1$ to the case of $n > 1$ attributes is based on a specific principle of individual choice pointed out in [5]. In this paper, we formalize such a principle in terms of the concept of capacity and analyze a TBA multi-attribut utility as a pair $(m, F)$ where $m$ is a capacity over $N = \{1, \ldots, n\}$ and $F$ is an $n$-dimensional probability distribution function. For our purposes it is convenient to use the Sklar decomposition of $F$ in terms of its one-dimensional margins and of its connecting copula. In such a frame, some aspects of aggregation functions and of copula-based extensions of capacities emerge in a straightforward way.

More precisely, the paper will present the following structure. In the next section, we will introduce the appropriate notation and detail the basic aspects of the multi-criteria Target-Based approach. Starting from the arguments presented in [5], we show how every Target-Based $n$-criteria utility is basically determined by a couple of objects: an $n$-dimensional probability distribution function and a fuzzy measure over $N := \{1, \ldots, n\}$. This discussion will allow us to point out, in Section 3, that some of the results presented by Kolesárová et al. in [21] admit, in a completely direct way, probabilistic interpretations and applications in terms of the TBA. It will in particular turn out that $n$-dimensional copulas, that can be used for the extension of fuzzy measures, describe stochastic dependence among the components of random vectors relevant in the problem. Section 4 will be devoted to the special case of $\{0, 1\}$-valued capacities. We shall see how, under such a specific condition, our arguments are directly related to the field of reliability and of lattice polynomial functions. Some final remarks concerning the relations between the parameters of TBA utilities and economic attitudes of a Decision Maker will be presented in Section 5. The notation we used is motivated by our effort to set a bridge between the two different settings. The term “attribute”, as used in the present paper, is substantially a synonymous of “criteria”.

2. Multi-Attribute Target-Based Utilities

In this section we deal with the TBA form of utility functions with $n > 1$ attributes. As recalled in the introduction, in the single-attribute case, $n = 1$, a TBA utility is essentially a non-decreasing, right-continuous, bounded function that, after suitable normalization, is regarded as the distribution function of a scalar random variable $T$ with the meaning of a target. Actually even more general, non-necessarily increasing, “utilities” can be considered in the TBA when possibility of stochastic dependence is admitted between the target and the prospect (see [4], see also [11]), but our interest here is limited to the case of independence between such two objects.

At a first glance, one could consider the distribution functions $F(x_1, \ldots, x_n)$ as the appropriate objects for a straightforward generalization of the definition of the TBA utilities to the $n$-attributes case. A given $F$ should be interpreted as the joint distribution function of a target vector $\mathbf{T} := (T_1, \ldots, T_n)$. But such a choice would be extremely
restrictive, however. A more convincing definition, on the contrary, can be based on the following principle: in the cases when a single deterministic target $t_i$ ($i = 1, \ldots, n$) has been assessed for any attribute $i$ by the Decision maker, the utility $U_{m,i}(x)$ corresponding to an outcome $x := (x_1, \ldots, x_n)$ depends only on the subset of those targets that are met by $x$ (as in [5], Definition 1). More precisely, we assume the existence of a set function $m : 2^N \to \mathbb{R}_+$ such that

$$U_{m,i}(x) = m(Q(t, x)),$$

where $Q(t, x)$ is the subset of $N$ defined by

$$Q(t, x) := \{i \in N | t_i \leq x_i\}.$$

It is natural to require that the function $m$ is finite, non-negative, and non-decreasing, namely such that

$$0 = m(\emptyset) \leq m(I) \leq m(N) < \infty.$$

Without loss of generality one can also assume that $m$ is scaled, in such a way that

$$m(N) = 1. \quad (3)$$

In other words, we are dealing with a capacity or a fuzzy measure $m : 2^N \to [0, 1]$.

Rather than deterministic targets however, it is generally interesting to admit the possibility that the vector $T$ of the targets is random, as it happens in the single-attribute case. Denoting by $F_T$ the joint distribution function of $T$, we replace the definition of a multi-attribute utility function given in (1) with the following more general

**Definition 1.** A multi-attribute target-based utility function, with capacity $m$ and with a random target $T$ has the form

$$U_{m,F}(x) = \sum_{I \subseteq N} m(I) \mathbb{P}\left(\bigcap_{i \in I} \{T_i \leq x_i\} \cap \bigcap_{i \notin I} \{T_i > x_i\}\right). \quad (4)$$

It is clear that $U_{m,F}(x) = U_{m,i}(x)$ when the probability distribution described by $F_T$ is degenerate over the point $t \in \mathbb{R}^n$. On the other hand the special choice $U_{m,F}(x) = F_T(x)$, mentioned above, is obtained by imposing the condition (3) together with

$$m(I) = 0 \text{ for all } I \subset N \quad (5)$$

This position corresponds then to a Decision Maker who is only satisfied when all the $n$ targets are achieved.

The class of $n$-attributes utilities is of course much wider than the one constituted by the functions of the form (4). The latter class is however wide enough and the choice of a utility function within it is rather flexible, since a single function is determined by the pair $(m(\cdot), F_T)$. Sufficient or necessary conditions, under which a utility function is of the form (4), have been studied by Bordley and Kirkwood in [5]. Several situations, where such utilities can emerge as natural, have also been discussed.

For our purposes, the following notation will be useful. We denote by $M_m : [0, 1] \to \mathbb{R}$ the set-function obtained by letting, for $I \subseteq 2^N$,

$$M_m(I) := \sum_{J \subseteq I} (-1)^{|I \setminus J|} m(J) \quad (6)$$

where $|I|$ indicates the cardinality of the set $I$. The function $M_m(\cdot)$ is the Möbius Transform of $m(\cdot)$ and, as a formula of the inverse Möbius Transform, we also have $m(I) = \sum_{J \subseteq I} M_m(J)$ (see e.g. [27]). For $x \in \mathbb{R}^n$ and $I \subseteq N$, we set

$$x_I := \{u_1, \ldots, u_n\} \quad \text{where } u_j = \begin{cases} x_j & j \in I, \\ +\infty & \text{otherwise.} \end{cases} \quad (7)$$

If $F(x)$ is a probability distribution function over $\mathbb{R}^n$, $F(I)(x_{j_1}, \ldots, x_{j_{|I|}}) = F(x_I)$ will be its $|I|$-dimensional marginal. Now we denote by $G_I(\cdot)$ the marginal distribution of $F$.
for \(i = 1, \ldots, n\) and we assume it to be continuous and strictly increasing. Furthermore we will denote by \(C\) the connecting copula of \(F\):
\[
C(y) := F(G_1^{-1}(y_1), \ldots, G_n^{-1}(y_n)).
\] (8)

Using a notation similar to (7), for \(y \in [0, 1]^n\) we set
\[
y_I := \{v_1, \ldots, v_n\} \quad \text{where} \quad v_j = \begin{cases} y_j & j \in I, \\ 1 & \text{otherwise}. \end{cases}
\]

In this way for the connecting copula \(C_F^{(I)}\) of \(F^{(I)}\) we can write
\[
C_F^{(I)}(y_1, \ldots, y_{|J|}) = C(y_I).
\] (9)

The following result can be seen as an analogue of several results presented in different settings (see in particular [21] and [22]).

**Proposition 2.** The utility function \(U_{m,F}\) can also be written in the equivalent form
\[
U_{m,F}(x) = \sum_{I \subseteq N} M_m(I) \mathbb{P}(T \leq x_I).
\] (10)

**Proof.** The proof amounts to a direct application of the inclusion-exclusion principle. Set \(A_i = \{T_i \leq x_i\}\) and we denote its complement by \(A_i^c\); we also set \(A_T = \cap_{i \in I} A_i\) and \(\bar{A}_T = \cap_{i \in I} A_i^c\). Then Equation (4) can be rewritten as
\[
U_{m,F}(x) = \sum_{I \subseteq N} m(I) \mathbb{P}(A_T \cap \bar{A}_T).
\]

By a direct application of the inclusion-exclusion principle we have
\[
U_{m,F}(x) = \sum_{I \subseteq N} m(I) \sum_{J \subseteq N \setminus I} (-1)^{|J|} \mathbb{P}(A_T \cap A_J),
\]
then
\[
U_{m,F}(x) = \sum_{I \subseteq N} \sum_{H \subseteq I} (-1)^{|H|} m(H) \mathbb{P}(A_I) = \sum_{I \subseteq N} M_m(I) \mathbb{P}(A_I),
\]
which is the right hand side of (10). \(\square\)

We now consider the function \(U_{m,F}(G_1^{-1}(y_1), \ldots, G_n^{-1}(y_n))\). By means of (8) we see that such a function depends on \(F\) only through the connecting copula \(C\) and it will be denoted by \(\hat{U}_{m,C}\). The quantities \(G_1(x_1), \ldots, G_n(x_n)\) can be given the meaning of utilities, thus \(\hat{U}_{m,C}\) becomes the aggregation function of the marginal utilities \(y_1, \ldots, y_n\).

**Corollary 3.** If the one-dimensional distributions \(G_1(x_1), \ldots, G_n(x_n)\) of \(F\) are continuous and strictly increasing, one can also write
\[
\hat{U}_{m,C}(y) = \sum_{I \subseteq N} M_m(I) C(y_I).
\] (11)

For any fixed pair \((m, F)\), we now turn to considering the expected utility corresponding to the choice of a prospect \(X := (X_1, \ldots, X_n)\) distributed according to \(F_X\):
\[
\mathbb{E}_X(U_{m,F}(X)) = \int_{\mathbb{R}^n} U_{m,F}(x) \, dF_X(x) = \sum_{I \subseteq N} M_m(I) \mathbb{P}(T_I \leq X_I).
\] (12)

By taking into account (12) and by interchanging the integration order, we can also write
\[
\mathbb{E}_X(U_{m,F}(X)) = \mathbb{E}_X[\mathbb{E}_T(U_{m,T}(X))] = \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} M_m(I(t, x)) \, dF_X(x) \right] \, dF_T(t).
\] (13)

See also the logic scheme of Figure 1.

The formula (12) points out that, when evaluating the choice of a prospect \(X\), the random vector of interest is \(D = T - X\). Let us assume that the marginal distribution function
of \(D_i\), denoted by \(H_i(\xi)\), is continuous and strictly increasing in \(\xi = 0\) for \(i = 1, \ldots, n\), and put \(\gamma = (\gamma_1, \ldots, \gamma_n)\) with
\[
\gamma_i = H_i(0).
\]
Similarly to (9), let furthermore denote by \(C^{(I)}_{F_D}\) the connecting copula of the marginal distribution corresponding to the coordinates subset \(I \subseteq N\). Then (12) becomes
\[
\tilde{U}_{m,F}(\gamma) := E_X(U_{m,F}(X)) = \sum_{I \subseteq N} M_m(I)C^{(I)}_{F_D}(\gamma) = \sum_{I \subseteq N} M_m(I)C_{F_D}(\gamma_I).
\]
This formula highlights that, concerning the joint distribution of \(D\), we only need to specify the vector \(\gamma\) and \(C_{F_D} = C^{(N)}_{F_D}\), the connecting copula of \(D\). From \(C_{F_D}\), we can derive in fact the family of all marginal copulas \(C^{(I)}_{F_D}\) by means of the formula (9) above.

3. Multi-Attribute TBA and Extensions of Fuzzy Measures

Let a capacity \(m(\cdot)\) over \(2^N\) and a \(n\)-dimensional copula \(C : [0, 1]^n \rightarrow [0, 1]\) be given. For \(y \in [0, 1]^n\), we can consider the aggregation function
\[
V_{m,C}(y) = \sum_{I \subseteq N} M_m(I)C(y_I),
\]
where \(M_m(\cdot)\) denotes the Möbius transform of \(m(\cdot)\) and \(C(y_I)\) is the connecting copula of \(F^{(I)}\), see (9). We remind

**Definition 4.** An aggregation function \(A : [0, 1]^n \rightarrow [0, 1]\) is a function non-decreasing in each component and that satisfies the boundary conditions \(A(0) = 0\) and \(A(1) = 1\).

(see e.g. [7]). By the usual identification of \(\{0, 1\}^n\) with \(2^N\) (where a subset \(I \subseteq N\) is identified with its indicator function) one has, for \(\varphi \in \{0, 1\}^n\) and for any copula \(C\),
\[
V_{m,C}(\varphi) = m(I),
\]
where \(I = \{i \in N | y_i = 1\}\). Thus any aggregation function of the form (16) can be seen as the extension to \([0, 1]^n\) of the capacity \(m(\cdot)\) defined over \(\{0, 1\}^n\). Extensions of a capacity over \(\{0, 1\}^n\) have been of interest in the fuzzy sets literature. Several properties of such extensions have been in particular studied by Kolesárová et al in [21]. In that paper the authors consider extensions of the form (16), where \(C\) is replaced by a more general aggregation function \(A\). As corollaries of their general results, it follows that -
Consider now the case when $A$ coincides with a copula $C \cdot V_{m,C}$ is actually an aggregation function, and special properties of it are analyzed therein.

It is in particular noticed that, when $C$ is the product copula one obtains the Owen extension and, when $C$ is the copula of comonotonicity, namely

$$C(u_1, \ldots, u_n) = \min\{u_1, \ldots, u_n\}, \quad (18)$$

then one obtains the Lovász extension, or the Choquet integral of $y$.

In the present framework, it is useful to give the aggregation function in (16) the form of a Riemann-Stieltjes in integral over $[0, 1]^n$ as follows.

**Theorem 5.** Let $m$ be a capacity over $2^N$ and $C$ an $n$-dimensional copula. For $y \in [0, 1]^n$ one has

$$V_{m,C}(y) = \int_{[0, 1]^n} m[Q(z, y)] \, dC(z) \quad (19)$$

where $Q(z, y)$ is the set defined as in (2).

**Proof.** Let $I \subseteq N$. By definition of $Q$ we have that $Q(z, y) = I$ holds if and only if $z_i \leq y_i$ for $i \in I$ and $z_i > y_i$ for $i \notin I$. Hence

$$m[Q(z, y)] = \sum_{I \subseteq N} m(I) \prod_{j \in I} 1_{z_j \leq y_j} \prod_{j \notin I} 1_{z_j > y_j} = \sum_{I \subseteq N} M_m(I) \prod_{j \in I} 1_{z_j \leq y_j}. \quad (20)$$

By integrating this function over $[0, 1]^n$ w.r.t. the probability measure associated to $C$, one has

$$\int_{[0, 1]^n} m[Q(z, y)] \, dC(z) = \sum_{I \subseteq N} \sum_{j \in I} M_m(I) \prod_{j \notin I} 1_{z_j \leq y_j} \, dC(z)$$

$$= \sum_{I \subseteq N} \sum_{j \in I} 1_{z_j \leq y_j} M_m(I) \, dC(z)$$

$$= \sum_{I \subseteq N} M_m(I) C(y_I). \quad (20)$$

**Remark 6.** Theorem 5 shows in which sense $V_{m,C}$ can be seen as a generalization of the Choquet integral. In fact $V_{m,C}$ reduces to a Choquet integral when $C$ is the copula of comonotonicity.

**Remark 7.** Consider now the case when $C_{\mathcal{X}}$ is the probability distribution function degenerate over $\mathcal{X} \in [0, 1]^n$. In this case, as shown by (19), $V_{m,C_{\mathcal{X}}}$ reduces to

$$V_{m,C_{\mathcal{X}}}(y) = m[Q(\mathcal{X}, y)]. \quad (21)$$

One can notice that, for any copula $C$, $V_{m,C_{\mathcal{X}}}(y) = V_{m,C}(w)$, where $w \in \{0, 1\}^n$ is defined by

$$w_i = \begin{cases} 1 & \text{if } z_i \leq y_i, \\ 0 & \text{if } z_i > y_i. \end{cases}$$

Notice also that Equation (21) is just a different way to read the principle that led us to the Definition (1) of a TB multi-attribute utility function.

As seen in the previous section, aggregation functions of the form (16) emerge in a natural way in the frame of TBA utilities. In such a frame the copula $C$ takes a specific meaning as the copula that describes stochastic dependence properties of random vectors relevant in the decision problem at hand.

Let us consider the expected utility, associated to a multi-attribute prospect $X$, of the target-based utility with target $T$. As shown by expression (15), such expected utility has the form (16), when it is seen as a function of the quantities $\gamma_i = H_i(0), i = 1, \ldots, n,$
introduced in (14). In such a case $C$ has then the meaning of the connecting copula of $D \equiv (T_1 - X_1, \ldots, T_n - X_n)$.

Let furthermore $G_1, \ldots, G_n$, the one-dimensional marginal distributions of $T_1, \ldots, T_n$, be assumed continuous and strictly increasing and let $C$ denote, this time, the connecting copula of $(T_1, \ldots, T_n)$ so that $V_{m,C}(y_1, \ldots, y_n)$ has the meaning of an aggregation function $\hat{U}_{m,C}(y_1, \ldots, y_n)$ of the marginal utilities $y_1, \ldots, y_n$, as shown by (11).

We then see that both the functions $\hat{U}_{m,C_T}(\cdot)$ and $\hat{U}_{m,C_D}(\cdot)$, defined over $[0,1]^n$, have the same formal expression (16) and are thus two different extensions of the capacity $m$. Starting from a same TBA utility function as in (11), they get different economic meanings. Both of them are definite integrals over $\mathbb{R}^n$, however. In particular, for $\hat{U}_{m,C_T}$ and $\hat{U}_{m,F}$ we can obtain, as a corollary of Theorem 5,

**Proposition 8.** The aggregation functions $\hat{U}_{m,C_T}$ and $\hat{U}_{m,F}$ are respectively given by

$$\hat{U}_{m,C_T}(y) = \int_{[0,1]^n} m(Q(z,y)) \, dC_T(z).$$

$$\hat{U}_{m,F}(\gamma) = \int_{[0,1]^n} m(Q(z,\gamma)) \, dC_D(z).$$

**Remark 9.** In the present frame, the Choquet integral admits the following economic interpretation. The choice of the copula of comonotonicity stands for the choice of a non-dimensional target, where all the random coordinates are just deterministic transformations of one and a same random variable. In this case $\hat{U}_{m,C_T}(y)$ reduces to a Choquet integral.

**4. RELIABILITY-STRUCTURED TBA UTILITIES**

A very special class of capacities $m(\cdot)$ emerges as an immediate generalization of the case in (5) and is of interest in the frame of TBA utilities.

**Definition 10** (See [5], Definition 4). A Target-Based utility function has a reliability structure when the capacity $m(\cdot)$ satisfies the condition

$$m(I) \in \{0,1\} \quad \text{for all } I \in \mathcal{N}.$$

Any such $m(\cdot)$ can then be seen as the structure function of a coherent reliability system $S$ or, more generally, of a semi-coherent one (for further details see [1] and [26]).

We concentrate attention on the case when both the coordinates $(T_1, \ldots, T_n)$ of the target and the coordinates $(X_1, \ldots, X_n)$ of the prospect are non-negative random variables that can then be interpreted as the vectors of the lifetimes of the components of $S$. The above reliability-based interpretation applies in a completely natural way, in this case.

For $\xi \in \mathbb{R}_+^n$, we denote by $\tau(\xi)$ the lifetime of $S$ when $\xi_1, \ldots, \xi_n$ are the values taken by such lifetimes, respectively. Then, as pointed out in [23], $\tau(\cdot)$ is a lattice polynomial function. Then (see [3]) it can be written both in a disjunctive and in a conjunctive form as a combination of the operators $\wedge$ and $\lor$ (see also [23], Proposition 2). When, in particular, the system admitting $m$ as its structure function is coherent, such forms can be based on the path sets and the cut sets of the system (see again [1] and [26]).

The random variable $\tau(T)$ is the lifetime of $S$ when the lifetimes of the components coincide with the coordinates of the DM’s target and $\tau(X)$ is the lifetime of the system when the lifetimes of components coincide with the coordinates of a (random) prospect $X$. Under such positions, the utility function $U_{m,F}(x_1, \ldots, x_n)$ can be read as a probability. More exactly

$$U_{m,F}(x_1, \ldots, x_n) = \mathbb{P}(\tau(T) \leq \tau(x)),$$

and the expected utility in (12) becomes

$$\mathbb{E}(U_{m,F}(X)) = \mathbb{P}(\tau(T) \leq \tau(X)).$$
We can then summarize as follows our conclusions. Consider a reliability-structured multi-attribute Target-Based utility $U_{m,F}(x_1, \ldots, x_n)$ with $F$ the joint distribution function of $n$ non-negative random variables and let $x_i \geq 0, \ i = 1, \ldots, n$. Denote furthermore by

$$G_{\tau(T)}(\xi) := \mathbb{P}(\tau(T) \leq \xi)$$

the marginal distribution function of the lifetime $\tau(T)$. Then we have

**Proposition 11.**

$$U_{m,F}(x_1, \ldots, x_n) = G_{\tau(T)}(\tau(x)). \quad (26)$$

This result shows that, in the reliability-structured case, a multi-attribute Target-Based utility $U_{m,F}$ reduces to a single-attribute Target-Based with a prospect $\tau(X)$ and a target $\tau(T)$. In particular the operator $\tau$ is a *mean* (see e.g. [20]): for $x > 0$, $\tau(x, \ldots, x) = x$. Thus we obtain from (24) that the probability distribution function of $\tau(T)$ is given by

$$G_{\tau(T)}(\xi) = U_{m,F}(\xi, \ldots, \xi). \quad (27)$$

For a different but strictly related expression of $G_{\tau(T)}(\xi)$ see [14].

The formula (26) can be used in the two different directions: one can analyze questions about systems’ reliability by using tools in the theory of aggregation operators and of extensions of capacities or, viceversa, different aspects of aggregation operators can be interpreted in terms of reliability practice, when the capacities are $\{0,1\}$-valued. In particular, the aggregation function $\hat{U}_{m,C}$ in (11) can be given special interpretations in the present setting. From a technical point of view, in a reliability-structured frame, $G_1, \ldots, G_n$ are the one-dimensional marginal distributions of the components’ lifetimes $T_1, \ldots, T_n$ of a system and $C$ denotes the connecting copula of $T$. By taking into account the equation (26) we obtain, for $y \in [0,1]^n$,

$$\hat{U}_{m,C}(y) = G_{\tau(T)}(\tau(G_1^{-1}(y_1), \ldots, G_n^{-1}(y_n))). \quad (28)$$

Notice that the operator $\tau(x)$ appearing in (24) and (28) is only determined by the capacity $m$, whereas the probability law $G_{\tau(T)}$ also depends on the copula $C$ of $F$. In any case $\hat{U}_{m,C}(y)$ is an integral, w.r.t. the capacity $m$, and the function to be integrated depends on $C$.

We also notice that, from a purely mathematical viewpoint, $m$ can be paired with any copula $C$. Any capacity $m$, for instance, can be paired with the comonotonic copula to obtain that also $\hat{U}_{m,C}$ is a lattice polynomial, given by a Choquet Integral. From the economic point of view, on the contrary, imposing conditions describing the attitudes towards risk by part of a Decision Maker, creates some constraints on the choice of the pair $(m,C)$. See also the brief discussion in the next section.

### 4.1. Symmetric Reliability-Structured Cases.

Here we consider special conditions of invariance with respect to permutations. First we look at the very restrictive, but important, case of *symmetric* reliability-structured capacities. The reliability systems admitting permutation-invariant structure functions $\phi$ are those of the type $k$-out-of-$n$. More precisely, a system is $k$-out-of-$n$ when, for $x \in \{0,1\}^n$, its structure function has the form

$$\phi_{k,n}(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } \sum_i x_i \geq k, \\ 0 & \text{if } \sum_i x_i < k. \end{cases} \quad (29)$$

This is then the case of a system which is able to work as far as at least $k$ of its components are working or, in other words, which fail at the instant of the $(n-k+1)$-th failure among its components. In (29), the structure function is seen as a function over $N$. Equivalently, when $\phi_{k,n}$ is seen as a set function, the value $\phi_{k,n}(I)$ is 0 or 1, only depending on the cardinality of $I$ being larger or smaller than $k$. 

Proposition 12. In the case of a $k$-out-of-$n$ capacity $m = \phi_{k:n}$, we have

$$U_{m,F}(x) = \sum_{I \subseteq N, |I| \geq k} (-1)^{|I|-k} \binom{|I|-1}{|I|-k} \mathbb{P}(T \leq x_I).$$

Proof. Recall Equation (10) and notice that, for $m = \phi_{k:n}$, the coefficients of the Möbius transform are given by

$$M_m(I) = \begin{cases} (-1)^{|I|-k} \binom{|I|-1}{|I|-k} & |I| \geq k, \\ 0 & \text{otherwise}. \end{cases}$$

It is clear that, in the case of a $k$-out-of-$n$ systems, the function $\tau(\xi_1, \ldots, \xi_n)$ is

$$\tau(\xi) = \xi_{(n-k+1)},$$

where $\xi_{(1)}, \ldots, \xi_{(n)}$ denote the order statistics of $\xi_1, \ldots, \xi_n$ and the formula (26) takes the special form

$$U_{\phi_{k:n},F}(x) = G_{T_{(n-k+1)}}(x_{(n-k+1)}).$$

From (27), we in particular obtain the probability law of $T_{(n-k+1)}$:

$$G_{T_{(n-k+1)}}(\xi) := \mathbb{P}(T_{(n-k+1)} \leq \xi) = U_{\phi_{k:n},F}(\xi, \ldots, \xi). \quad (30)$$

A different remarkable case of reliability-structured TBA utilities is obtained by imposing the condition of permutation-invariance over the joint distribution $F$, rather than over the capacity $m$. This is the case when $T_1, \ldots, T_n$, the coordinates of the target $T$, are assumed to be non-negative exchangeable random variables, namely the joint distribution $F(x_1, \ldots, x_n)$ is assumed invariant under permutations of its arguments $x_1, \ldots, x_n$. In this case the concept of signature of the system enters in the expression of the utility function $U_{m,F}$.

Given the structure function $\phi : \{0,1\}^n \to \{0,1\}$ of a semi-coherent system, the signature $s(\phi) = s = (s_1, \ldots, s_n)$ is a probability distribution over $N = \{1, \ldots, n\}$ (as a basic reference, see e.g. [28]). For $j = 1, \ldots, n$, consider the events

$$E_j := \{\tau(T) = T_{(j)}\},$$

with $T_1, \ldots, T_n$ denoting again the lifetimes of the components and $\tau(T)$ the lifetime of the system. When $T_1, \ldots, T_n$ are such that

$$\mathbb{P}(T_{i'} = T_{i''}, \text{ for some } i', i'') = 0, \quad (31)$$

$E_1, \ldots, E_n$ are exhaustive and pair-wise disjoint, and we have

$$\sum_{j=1}^n \mathbb{P}(E_j) = 1.$$

The components $s_1, \ldots, s_n$ of the signature are defined as $s_j = \mathbb{P}(E_j)$. It is easy to prove that, when $T_1, \ldots, T_n$ are exchangeable, the following properties hold:

a): $s(\phi)$ does not depend on the joint probability distribution of $T_1, \ldots, T_n$;

b): For $\xi > 0$ and $j = 1, \ldots, n$, the event $(T_{(j)} \leq \xi)$ is stochastically independent of $E_1, \ldots, E_n$. 

By the formula of total probabilities we then can write, for any \( \xi > 0 \),
\[
\mathbb{P}(\tau(T) \leq \xi) = \sum_{j=1}^{n} \mathbb{P}(E_j) \mathbb{P}(\tau(T) \leq \xi | E_j)
\]
\[
= \sum_{j=1}^{n} \mathbb{P}(\tau(T) = T_{(j)}) \mathbb{P}(\tau(T) \leq \xi | \tau(T) = T_{(j)})
\]
\[
= \sum_{j=1}^{n} s_j^{(s)} \mathbb{P}(T_{(j)} \leq \xi). \tag{32}
\]

By the property a) the signature \( s^{(s)} \) is a combinatorial invariant of the system. See in particular [24] for the relations between the signature \( s^{(s)} \) and the “reliability function” of the system in case of i.i.d. components. For a discussion about the relations between \( s^{(s)} \) and symmetry properties see also [31]).

In view of (32) the signature \( s^{(s)} \) has a role in the representation of the utility function \( U_{\phi, F} \) when \( F \) is exchangeable. By (27) we obtain

**Proposition 13.** Let \( F \) be an exchangeable joint distribution function over \( \mathbb{R}_+^n \), satisfying the condition (31). For any reliability-structured capacity \( m : 2^N \to \{0, 1\} \) and for \( x \in \mathbb{R}_+^n \), one has

\[
U_{m,F}(x) = \sum_{j=1}^{n} s_j^{(m)} \mathbb{P}(T_{(j)} \leq \tau(x)). \tag{33}
\]

The terms \( s_j^{(m)} \) and \( \tau(x) \) in (33) are determined by \( m \), whereas \( F \) determines the probability law of \( T_{(j)} \), for \( j = 1, \ldots, n \). The formula (32) is a special case of (33): for \( x = (\xi, \ldots, \xi) \) we again obtain

\[
G_{\tau(T)}(\xi) = U_{m,F}(\xi, \ldots, \xi) = \sum_{j=1}^{n} s_j^{(m)} \mathbb{P}(T_{(j)} \leq \xi) = \sum_{j=1}^{n} s_j^{(m)} U_{\phi_{\infty, m}}(\xi, \ldots, \xi).
\]

5. TBA UTILITIES AND ATTITUDES TOWARD GOODS AND RISK

Here we think of a Decision Maker who describes her/his attitudes towards \( n \) goods \( G_1, \ldots, G_n \) through a capacity \( m \) and defines her/his utility by choosing a target \( T \) with joint distribution function \( F \). Thus \( U_{m,F}(x) \) evaluates the satisfaction of the DM in receiving the quantity \( x_1 \) for the good \( G_1 \), \( x_2 \) for the good \( G_2 \) and so on. Different properties with economic meaning of a multi-attribute utility function can take a special form in the TBA case and in the reliability-structured TBA case, more in particular. One should analyze how can different properties be influenced by the choice of the parameters \( m, F \) or, in other terms, which constraints on the pair \((m, F)\) are induced by fixing DM’s attitudes toward risk. In this Section, we concentrate our attention on the basic concepts of supermodularity and submodularity (see [32, 33]) and present some related comments.

For a function \( U : \mathbb{R}^n \to \mathbb{R} \) and for \( x', x'' \in \mathbb{R}^n \), set

\[
\nu^U(x', x'') := U(x' \lor x'') + U(x' \land x'') - U(x') - U(x''). \tag{34}
\]

**Definition 14.** The function \( U \) is supermodular when \( \nu^U(x', x'') \geq 0 \) for all \( x', x'' \in \mathbb{R}^n \), and submodular when \( \nu^U(x', x'') \leq 0 \). If \( U \) is both supermodular and submodular, then it is called modular.

Under the condition that the function \( U \) is twice differentiable, an equivalent formulation in terms of the second order derivatives of \( U \) can be given. In particular the condition of supermodularity is given by

\[
\partial^2 U(x)/\partial x_i \partial x_j \geq 0 \tag{35}
\]
For all $x \in \mathbb{R}^n$ and $i \neq j, i, j = 1, \ldots, n$.

For a utility function, it is well-known that supermodularity describes the case of complementary goods (see [15, 29, 33]), while submodularity is associated to substitutable goods. Two or more goods are called complementary if “they have little or no value by themselves, but they are more valuable when combined together”, while they are called substitutable when “each of them satisfies the same need of the DM that the other good fulfills”. In these settings we can say that a collection of goods are complements (and each pair is said to be complementary) if they have a real-valued supermodular utility function (Bulow et al. [6] use the term strategic complements to describe any two activities $i$ and $j$ for which formula (35) holds).

As a related interpretation, the properties of supermodularity, submodularity, and modularity of a multi-attribute utility $U$ respectively describe, in an analytic language, the properties of correlation affinity, correlation aversion, and correlation neutrality (see e.g. [12], [32] and [33]). In particular the concept of submodularity gives rise to a specific definition of greater correlation between two joint probability distributions (see Definition 4 in [17]).

Let us now come to TB utilities and to related problems of prospects choosing. We are essentially interested in decision problems where the following objects are considered to be fixed: the capacity $m$, the marginal distributions $G_1, \ldots, G_n$ of the targets’ components $T_1, \ldots, T_n$, and the marginal distributions $G_{X_1}, \ldots, G_{X_n}$ for the components of the prospect. Since we have assumed stochastic independence between $X$ and $T$, the marginal probability distribution function $H_i(\cdot)$ of $D_i = T_i - X_i$ is suitably obtained by convolution from $G_i$ and $G_{X_i}$. Then, at least in principle, the vector $\gamma = (\gamma_1, \ldots, \gamma_m)$ is known, where $\gamma_i = H_i(0)$. The DM is supposed to declare the copula $C$ of the target vector $T$ and, on this basis, to select a copula for the random prospect $X$. The choice of a prospect then amounts to the choice of a copula $C_D$ for the vector $D = T - X$.

For a TB utility function $U$, the expression in (34) becomes

$$
\nu^U(x', x'') = \sum_{I \subseteq N} M_m(I) \nu^F(x'_I, x''_I) \tag{36}
$$

for any pair of vectors $x', x'' \in \mathbb{R}^n$. The notation $x'_I, x''_I$ is as used in (7). Then the conditions of supermodularity, or submodularity, become

$$
\sum_{I \subseteq N} M_m(I) \nu^F(x'_I, x''_I) \geq 0. \tag{37}
$$

Let the DM manifest correlation aversion or correlation affinity. Namely she/he wants to use a submodular, or supermodular, utility function. Of course correlation aversion/affinity concerns attitudes toward dependence among the coordinates of the prospect. On the other hand, for fixed $m$, the properties of supermodularity and submodularity are expressed through the choice of the connecting copula $C$ for the target $T$. Such properties are generally determined by the interplay between $m$ and $F$. In conclusion, we are interested in conditions on the pair $(m, F)$ for which condition (37) holds. In this direction we now discuss some special cases.

First of all we consider the case in which the capacity $m$ is totally monotone. We remind that a capacity $m$ is said totally monotone if its M"obius transform $M(I)$ is positive for all $I \subseteq N$ (see [19]). Since all the multivariate distribution functions are supermodular, we immediately see from (37) that if $m$ is totally monotone, the utility function $U_{m,F}$ is supermodular whatever the distribution function $F$ of the target is. So, in this special case, the condition of supermodularity is completely determined by the capacity $m$.

A further interesting case is met when the capacity $m$ is additive: in this situation the interplay among variables has no effect on the overall amount of the utility $U_{m,F}$. In fact,
the formula for $U_{m,F}$ reduces to

$$U_{m,F}(\mathbf{x}) = \sum_{i=1}^{n} m_i \mathbb{P}(T_i \leq x_i),$$

with $m_1 + \ldots + m_n = 1$. The expression in the r.h.s. represents an Ordered Weighted Average (see [19]) of the marginal distributions of the targets $T_i$. It is immediate to notice that $U_{m,F}(\mathbf{x})$ is modular for any choice of $F$. Furthermore it does not depend on the copula $C$ of $F$. We notice that, in this case, the expected value of the utility $\mathbb{E}[U_{m,F}(\mathbf{X})]$ (see formulas (12) and (15)) for a fixed prospect $\mathbf{X}$ becomes $\tilde{U}_{m,F}(\gamma) = \sum_{i=1}^{n} m_i \gamma_i$.

Another likely situation is that in which the DM only considers interactions among small groups of goods, say $k$ at most. In other words the DM is not interested in how they behave when considered in groups of cardinality larger than $k$. This condition leads to the choice of a $k$-additive capacity (see e.g. [18]). More in details

**Definition 15.** A capacity $m$ is said $k$-additive if the coefficients of its Mobius transform $M_m$ satisfy the condition $M_m(I) = 0$ for all $I$ such that $|I| > k$, and $M_m(I) \neq 0$ for at least one element $I$ with $|I| = k$.

The assumption of $k$-additivity generally simplifies the study of the utility function. Under this hypothesis condition (37) reduces to

$$\nu^F(\mathbf{x}', \mathbf{x}'') = \sum_{|I| = 2, \ldots, k} M_m(I) \nu^F(\mathbf{x}_I', \mathbf{x}_I'') \geq 0.$$  

We notice, in any case, that the possible validity of the conditions of submodularity and supermodularity generally depends on both the capacity $m$ and the distribution $F$. In particular, in the case $k = 2$, a sufficient condition for supermodularity (submodularity) reads $M_m(\{i,j\}) \geq 0 (\leq 0)$, for all $i \neq j$.

Also of interest is the special case of reliability-structured utility functions, that we have considered in the previous section. First we notice that $m$ being $\{0,1\}$-valued has a direct economic interpretation: like a binary system, that can be up or down according to the current state (up or down) of each of its $n$ components, so the DM is completely satisfied or completely unhappy according to which is the subset of targets that have been achieved. Cases where such utilities can be of economic relevance are discussed in [5]. Also, the special forms of TB utilities with $m$ describing series systems or parallel systems are discussed there: these are the cases when $m$ is the minimal or the maximal capacity, respectively, and correspond to the two extreme cases of perfect complementarity and perfect substitutability. In such cases we encounter supermodularity and submodularity, respectively, independently of the form of $F$. In all the other cases the condition of supermodularity, or submodularity respectively, reads

$$G_{\tau(\mathbf{T})}(\tau(\mathbf{x}' \lor \mathbf{x}'')) + G_{\tau(\mathbf{T})}(\tau(\mathbf{x}' \land \mathbf{x}'')) - G_{\tau(\mathbf{T})}(\tau(\mathbf{x}')) - G_{\tau(\mathbf{T})}(\tau(\mathbf{x}'')) \geq 0. \quad (38)$$

The validity of such a condition depends on the behavior of both the capacity $m$ and the distribution function $F$ of the targets. Notice that, when $\mathbf{T}$ is exchangeable, $G_{\tau(\mathbf{T})}$ is of the form (32), then condition (38) becomes

$$\sum_{j=1}^{n} s_j^{(\mathbf{0})} \cdot [G_{(j)}(\tau(\mathbf{x}' \lor \mathbf{x}'')) + G_{(j)}(\tau(\mathbf{x}' \land \mathbf{x}'')) - G_{(j)}(\tau(\mathbf{x}')) - G_{(j)}(\tau(\mathbf{x}''))] \geq 0,$$

where $G_{(j)}(x) = \mathbb{P}(T_{i(j)} \leq x)$.

Still concerning the properties of supermodularity/submodularity, a very clear situation is met in the special case $n = 2$. We first notice that, in this case, formula (10) becomes

$$U_{m,F}(x_1, x_2) = M_1 \mathbb{P}(T_1 \leq x_1) + M_2 \mathbb{P}(T_2 \leq x_2) + M_{1,2} \mathbb{P}(T_1 \leq x_1, T_2 \leq x_2), \quad (39)$$
where we have used, for \( m \) and \( M \), the shorter notation \( m_1 = m(\{1\}) \), \( M_1 = M_m(\{1\}) \), and so on. As a strongly simplifying feature of the present case, the utility function \( U_{m,F} \) in (39) is, in any case, supermodular or submodular. In fact condition (36) reads
\[
\nu^U(x', x'') = M_{1,2} \nu^F(x', x'').
\]
Hence, since any joint distribution function \( F \) is supermodular, submodularity and supermodularity are respectively equivalent to the conditions
\[
M_{1,2} \leq 0 \quad \text{and} \quad M_{1,2} \geq 0,
\]
or, in terms of \( m \),
\[
m_1 + m_2 \geq 1 \quad \text{and} \quad m_1 + m_2 \leq 1.
\]
Focus now attention, in particular, to the cases of perfect complementarity and perfect substitutability. The first one is equivalent to the condition \( m_1 = m_2 = 0 \) or, equivalently, \( M_{1,2} = 1 \), and describes the maximal possible affinity to correlation for the DM. Here the expression of the utility \( U_{m,F} \) reduces to
\[
U_{m,F}(x_1, x_2) = F(x_1, x_2),
\]
which is exactly the joint distribution function of the two-dimensional target. In the opposite case, the utility reduces to \( U_{m,F}(x_1, x_2) = G_1(x_1) + G_2(x_2) - F(x_1, x_2) \) or, analogously, \( \hat{U}_{m,C}(y_1, y_2) = C^*(y_1, y_2) \), where \( C^* \) stands for the dual of the copula \( C \) (for further details see [25]). All other cases can be grouped mainly into two sets, the strictly supermodular ones, with \( m_1 + m_2 < 1 \), and the submodular ones, with \( m_1 + m_2 > 1 \). Finally we notice a region of neutrality, along the diagonal corresponding to \( m_1 + m_2 = 1 \): this is the case of additivity of the capacity \( m \), already discussed above. All these cases are summarized in Figure 2.

![Figure 2](image-url)

We already noticed that, w.r.t. the capacity \( m \), the aggregation function \( \hat{U}_{m,C} \) is an integral of \( m \), depending on \( C \), the connecting copula of \( F \). For a fixed \( m \), there is no restriction in the choice of \( C \), from a purely mathematical point of view. We can see, on the contrary, that certain constraints on the pair \( (m, C) \) can arise from an economic point of view, depending on the attitudes of our Decision Maker. In other words, the type of integral of \( m \), that the DM is led to consider as an aggregation \( \hat{U}_{m,C} \), depends on \( m \) itself once the attitudes of the DM have been fixed. As a simple example, let us consider the case of perfect complementarity in (42). In such case \( \hat{U}_{m,C} \) becomes \( \hat{U}_{m,C}(y_1, y_2) = C(y_1, y_2) \). Thus the aggregation function \( \hat{U}_{m,C} \) will grow with the growth of the copula \( C \). This
entails that a DM, who will manifest risk aversion besides correlation affinity, will choose the target which exhibits the greatest possible copula. Thus the most profitable choice is the maximal copula, \( C(u, v) = u \land v \), namely the one of comonotonicity. Similar arguments can be developed for the study of the extreme opposite case, \( m_1 = m_2 = 1 \) \((M_{1,2} = -1)\).

6. SUMMARY AND CONCLUDING REMARKS

By introducing the target-based approach, Bordley and Li Calzi in [4] and Castagnoli and Li Calzi in [8] had in particular developed a new way to look at utility functions, and related extensions, in the field of decision problems under risk. In those papers, emphasis was given to the single-attribute case where, practically, there is no loss of generality in considering target-based utilities. As to the multi-attribute case, a treatment proposed a few years later (in [5, 34, 35]) had further added some new ideas to the field. In fact, the proposed extension is something different from the single-attribute definition. Actually, a direct generalization of the latter would lead one to consider much too special and restrictive forms of utilities, as we have remarked in the Introduction.

A principle of individual choice, clearly enucleated in [5], is at the basis of the given definition of multi-attribute target-based utilities. This principle is indeed quite natural and is related to the evaluation, by part of a Decision Maker, about the relative importance attributed to any possible subset of achieved targets. It emerges then that such an evaluation depends on the individual propensity toward the possible “coalitions” of attributes and that it is related with the concept of capacity.

Starting from the latter observation, we have formally considered a multi-attribute target-based utility (Definition 1) as a pair \((m, F)\), where \(m\) is a capacity over \(2^N = \{0, 1\}^n\) and \(F\) is a probability distribution function over \(\mathbb{R}^n\). On this basis, we have pointed out that the theory of multi-attribute target-based utilities can hinge on a formal apparatus, provided by the field of fuzzy measures, extensions of fuzzy measures, and fuzzy, or universal, integrals. On the other hand, multi-attribute target-based utilities give rise to applications of the concepts and of results in this field. In particular, under special conditions, the arguments and results presented in [21] can have an interpretation useful to an heuristic view of the differences among various fuzzy integrals. As we have briefly recalled in Section 3, operators of the form

\[
V_{m,A}(y) = \sum_{I \subseteq N} M_m(I) A(y_I)
\]  

have been analyzed in [21] as extensions of capacities \(m\) over \(2^N\). Generally speaking, the function \(A\) appearing in (43) is an aggregation function. In our frame, interest is concentrated on the special case when \(A\) is replaced by an \(n\)-dimensional copula \(C\). The effect of such a particular condition is two-fold: on the one hand, it makes \(V_{m,A} = V_{m,C}\) to have, itself, the properties of an aggregation function. On the other hand, it gives \(V_{m,C}\) the meaning of an aggregation of marginal utilities; the special form of aggregation depends on the special type of stochastic dependence that is assumed among the coordinates of the target. An extreme condition of dependence with a special decisional meaning of its own, namely positive comonotonicity, lets such an aggregation coincide with a Choquet integral. We thus see the aggregation functions \(\hat{U}_{m,C}\) as a natural class of operators generalizing such integrals.

Concerning Choquet integrals, it is well known that they have been very widely studied and discussed in the past literature concerning utilities and decision under risk. In particular, in [10] and [30], it is shown how this concept allows one to build a quite general model of decision making under uncertainty, generalizing the Expected Utility model, in the frame of single-attribute decisions. We point out that its role in the present study appears under a rather different form: it is not used in fact to explain a general principle for decisions under uncertainty. It emerges as an extremely special case, just in the
frame of Expected Utility. However its meaning in the TB Approach is peculiar of the multi-attribute case.

In multi-attribute decision problems under risk, the profile of a Decision Maker can be specified by taking into account different types of attitudes and forms of behavior, such as risk-aversion (or risk-affinity), correlation-aversion (or correlation-affinity), cross-prudence, etc. Generally these conditions are described in terms of qualitative properties of the utility functions (see e.g. [12, 13, 16]).

Let us come to the specific case of multi-attribute utility functions, that we had identified with the pairs \((m, F)\). As a challenging program for future research, one should detail how the mentioned qualitative properties of utility functions determine (or are determined by) the form of \(m\) and \(F\) and reciprocal relations between them. For a DM with given attitudes toward risk, the choice of \(F\) - and then, in particular, of the copula \(C\) - is not completely free, but is influenced by the form of \(m\) itself. In the above Section 5, we have considered some significant special cases and sketched some conclusion in this direction. A more general analysis may result from future achievements about qualitative descriptions of target-based utilities.

Further research suggested by our work also concerns specific aspects of multivariate copulas. As shown by formula (15), the analysis of the present approach would benefit from new results concerning the connecting copula of the vector \(D\) obtained as the difference between the vectors \(T\) and \(X\). Here we have assumed stochastic independence.

More complex arguments would be involved in the cases when the possibility of some correlation between the vectors \(T\) and \(X\) is admitted. Some specific aspects, related with stochastic dependence between target and prospect for the special case \(n = 1\), have been dealt with in [11].

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