On planar and non-planar graphs having no chromatic zeros in the interval \((1, 2)\)

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Abstract

We exhibit a family of 2-connected graphs which is closed under certain operations, and show that each graph in the family has no chromatic zeros in the interval \((1,2)\). The family contains not only collections of graphs having no certain minors, but also collections of plane graphs, including near-triangulations found by Birkhoff and Lewis [Chromatic polynomials, Trans. Amer. Math. Soc. 60 (1946) 355–451].

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1. Introduction

Given a graph \(G\), let \(V(G)\), \(E(G)\), \(v(G)\), \(e(G)\), \(c(G)\), \(b(G)\) and \(P(G, \lambda)\) be its vertex set, edge set, order, size, number of components, number of blocks and chromatic polynomial respectively. A zero of \(P(G, \lambda)\) is called a chromatic zero of \(G\).

It is known that the following intervals are the only intervals in which every graph has no chromatic zeros: \((−\infty, 0)\) and \((0, 1)\) (see [4,6], for instance), and \((1, \frac{22}{27})\) (see [5]). Jackson [5] further proved that for any \(\epsilon > 0\), there exists a (plane) graph having a chromatic zero in \((\frac{22}{27}, \frac{22}{27} + \epsilon)\). More generally, Thomassen [8] showed that for any interval \((a, b)\) with \(b > \frac{22}{27}\), there exists a graph having a chromatic zero in \((a, b)\).

There is an old result on certain plane graphs having no chromatic zeros in the interval \((1, 2)\). A near-triangulation is a loopless connected plane graph in which at most one face is not bounded by a cycle of order 3. Birkhoff and Lewis [1] deduced from a more general result on \(P(G, \lambda)\) they established therein that every near-triangulation has no chromatic zeros in \((1, 2)\) (see [10] for a direct proof).

Jackson [5] proposed the following:

**Conjecture 1.1.** Every 3-connected and non-bipartite graph has no chromatic zeros in \((1, 2)\).

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Thus, the conjecture is true for near-triangulations. Although various families of non-bipartite graphs having no chromatic zeros in (1, 2) have also been found in [3], this conjecture is not true in general, as counter-examples have been discovered by Royle [7].

However, the counter-examples in [7] cannot disprove the following conjecture proposed in [3].

**Conjecture 1.2.** For a connected graph $G$, if $c(G - S) \leq |S|$ for every independent set $S$ in $G$, then $G$ has no chromatic zeros in $(1, 2)$.

In this paper, we shall present some results to support Conjecture 1.2. In our main result, namely, Theorem 2.1, we shall introduce a family of graphs which contains $K_2$ and $K_3$ and is closed under certain operations, and show that each graph in the family has no chromatic zeros in $(1, 2)$. When confined to planar graphs, we manage to apply Theorem 2.1 to produce in Theorem 3.2 a number of collections of such plane graphs which contains near-triangulations as a very special case. Theorem 2.1 is further applied to produce in Theorems 4.1 and 4.2, respectively, two new collections of such graphs.

### 2. Main result

We shall establish our main result, namely, Theorem 2.1 in this section. For a subgraph $H$ of a graph $G$, let

$$ N(H) = \{x \in V(G) : x \text{ is adjacent to some vertex in } H\}. \quad (2.1) $$

For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$. For any component $H$ of $G - S$, $G[N(H) \cup V(H)]$ is called an $S$-bridge of $G$. Note that there may exist $S$-bridges $G_i$ of $G$ such that $S \not\subseteq V(G_i)$.

For any edge $uv$ in $G$, let $G - uv$ denote the graph obtained from $G$ by deleting $uv$, and let $G \cdot uv$ denote the graph obtained from $G - uv$ by identifying $u$ and $v$ and deleting all multiple edges but one.

**Theorem 2.1.** Let $\mathcal{U}$ be any family of connected graphs. Suppose that for every $G \in \mathcal{U}$, either $G \in \{K_2, K_3\}$ or $G$ is 2-connected with $v(G) \geq 4$ satisfying one of the following conditions:

(i) $G$ has a cut-set \{x, y\} of $G$ with $xy \in E(G)$ such that all $\{x, y\}$-bridges of $G$ belong to $\mathcal{U}$;

(ii) $G$ has a cut-set \{x, y\} of $G$ with $xy \notin E(G)$ such that $c(G - x - y)$ is even and all $\{x, y\}$-bridges of $G - xy$ and all blocks of $G - xy$ belong to $\mathcal{U}$; and

(iii) there exists $uv \in E(G)$ such that both $G - uv$ and $G \cdot uv$ belong to $\mathcal{U}$.

Then $(-1)^{v(G)} P(G, \lambda) > 0$ holds for all $G \in \mathcal{U}$ and all real $\lambda \in (1, 2)$.

**Proof.** Suppose that there exists $H \in \mathcal{U}$ and $\alpha \in (1, 2)$ such that

$$ (-1)^{v(H)} P(H, \alpha) \leq 0. \quad (2.2) $$

Fix $\alpha$ and assume that $H$ is such a graph with minimum $v(H) + e(H)$. Thus, for each $H' \in \mathcal{U}$, if $v(H') + e(H') < v(H) + e(H)$, then

$$ (-1)^{v(H')} P(H', \alpha) > 0. \quad (2.3) $$

Since

$$ (-1)^{v(K_j)} P(K_j, \alpha) > 0, \quad (2.4) $$

for each $j \geq 2$, we have $v(H) \geq 4$.

**Claim 1.** $H$ contains no cut-set \{x, y\} with $xy \in E(H)$ such that all $\{x, y\}$-bridges of $H$ belong to $\mathcal{U}$.

Suppose that $H$ contains such a cut-set \{x, y\}. Let $H_1, H_2, \ldots, H_k$ be the $\{x, y\}$-bridges of $H$. For any $1 \leq i \leq k$, since $H_i \in \mathcal{U}$ and $v(H_i) + e(H_i) < v(H) + e(H)$, we have $(-1)^{v(H_i)} P(H_i, \alpha) > 0$. Observe that

$$ v(H_1) + v(H_2) + \cdots + v(H_k) = v(H) + 2(k - 1). \quad (2.5) $$
Thus
\[
(-1)^{v(H)} P(H, z) = \frac{(-1)^{v(H)} \prod_{1 \leq i \leq k} P(H_i, z)}{x^k-1 (x-1)^{k-1}} = \frac{\prod_{1 \leq i \leq k} (-1)^{v(H_i)} P(H_i, z)}{x^k-1 (x-1)^{k-1}} > 0,
\]
(2.6)
a contradiction.

**Claim 2.** \(H\) contains no cut-set \(\{x, y\}\) with \(xy \notin E(H)\) such that \(c(H - x - y)\) is even and all \(\{x, y\}\)-bridges of \(H + xy\) and all blocks of \(H \cdot xy\) belong to \(\mathcal{U}\).

Suppose that \(H\) contains such a cut-set \(\{x, y\}\). Let \(H_1, H_1, \ldots, H_k\) be \(\{x, y\}\)-bridges of \(H\). Then \(k\) is even and both \(H_i + xy\) and \(H_i \cdot xy\) belong to \(\mathcal{U}\) for each \(i\). Thus, for \(i = 1, 2, \ldots, k\),
\[
(-1)^{v(H_i)} P(H_i + xy, z) > 0,
\]
(2.7)
and
\[
(-1)^{v(H_i \cdot xy)} P(H_i \cdot xy, z) > 0.
\]
(2.8)
By the method used in proving Claim 1, we can show that
\[
(-1)^{v(H + xy)} P(H + xy, z) > 0.
\]
(2.9)
Observe that
\[
v(H) = 2 - k + \sum_{i=1}^{k} v(H_i \cdot xy).
\]
(2.10)
Thus
\[
(-1)^{v(H)} P(H \cdot xy, z) = \frac{(-1)^{v(H)} \prod_{1 \leq i \leq k} P(H_i \cdot xy, z)}{x^k-1} = \frac{1}{x^k-1} \prod_{1 \leq i \leq k} (-1)^{v(H_i \cdot xy)} P(H_i \cdot xy, z)
\]
\[
> 0.
\]
(2.11)
Hence
\[
(-1)^{v(H)} P(H, z) = (-1)^{v(H)} P(H + xy, z) + (-1)^{v(H)} P(H \cdot xy, z) > 0,
\]
(2.12)
a contradiction.

**Claim 3.** There is no \(uv \in E(H)\) such that both \(H - uv\) and \(H \cdot uv\) belong to \(\mathcal{U}\).

Suppose that \(H\) contains such an edge \(uv\). Then both \(H - uv\) and \(H \cdot uv\) belong to \(\mathcal{U}\). Since
\[
v(H - uv) + e(H - uv) < v(H) + e(H),
\]
(2.13)
we have
\[
(-1)^{v(H-uv)} P(H - uv, z) > 0.
\]
(2.14)
Similarly, we have
\[
(-1)^{v(H \cdot uv)} P(H \cdot uv, z) > 0.
\]
(2.15)
Thus
\[ (-1)^{v(H)} P(H, x) = (-1)^{v(H)} (P(H - uv, x) - P(H : uv, x)) \]
\[ = (-1)^{v(H - uv)} P(H - uv, x) + (-1)^{v(H - uv)} P(H : uv, x) \]
\[ > 0, \]
(2.16)
a contradiction.

Notice that Claims 1–3 contradict the given conditions. Thus there is no such graph \( H \) in \( \mathcal{V} \). Therefore \( (-1)^{v(G)} P(G, \lambda) > 0 \) for all \( G \in \mathcal{V} \) and all real \( \lambda \in (1, 2) \). \( \square \)

3. Plane graphs

For a loopless connected plane graph \( G \), let \( \rho(G) \) be the number of faces of \( G \) which are not bounded by cycles of order 3. Thus, \( G \) is a near-triangulation if and only if \( \rho(G) \leq 1 \).

Birkhoff and Lewis [1] showed that every near-triangulation has no chromatic zeros in \( (1, 2) \). In this paper we shall apply Theorem 2.1 to extend their result.

Jackson [5] pointed out that every 2-connected bipartite graph \( G \) of odd order has a chromatic zero in \( (1, 2) \) as stated below (a proof is given in [3]).

**Theorem 3.1.** For any connected bipartite graph \( G \) with \( v(G) \geq 2 \), if \( v(G) + \rho(G) \) is even, then \( G \) has a chromatic zero in \( (1, 2) \).

For any integer \( k \geq 1 \), let \( \Phi_k \) be the family of connected graphs \( G \) such that \( c(G - S) > |S| \) for some independent set \( S \) of \( G \) with \( |S| = k \). Since every bipartite graph \( G \) of odd order contains an independent set \( S \) such that \( c(G - S) > |S| \), Theorem 3.1 thus says that for every \( k \geq 2 \), there exists a graph in \( \Phi_k \) which has chromatic zeros in \( (1, 2) \).

We shall show in this section that Conjecture 1.2 holds for plane graphs \( G \) with \( \rho(G) \leq 4 \). To this end, we shall first prove a result that if \( G \) is a 2-connected plane graph in \( \Phi_s \cup \Phi_{s-1} \), then \( \rho(G) \geq s + 1 \). We begin with the following:

**Lemma 3.1.** Let \( G \) be a 2-connected graph with a cut-set \( \{x, y\} \), where \( xy \notin E(G) \). If one block of \( G \cdot xy \) belongs to \( \Phi_k \) for some \( k \geq 2 \), then \( G \in \Phi_k \cup \Phi_{k+1} \).

**Proof.** Let \( B \) be a block of \( G \cdot xy \) with \( B \in \Phi_k \). Then \( c(B - S) > |S| = k \) for some independent set \( S \) of \( B \).

Let \( w \) be the new vertex in \( G \cdot xy \) after identifying \( x \) and \( y \). If \( w \notin S \), then \( S \) is an independent set of \( G \) and \( c(G - S) > k = |S| \); if \( w \in S \), then \( S' = (S \setminus \{w\}) \cup \{x, y\} \) is an independent set of \( G \) and \( c(G - S') > k + 1 = |S'| \). Hence \( G \in \Phi_k \cup \Phi_{k+1} \). \( \square \)

**Lemma 3.2.** Let \( G \) be a bipartite graph with bipartition \((A, B)\) such that \( |A| > |B| \). If
\[ c(G - B') \leq |B'| \]
holds for every \( B' \subset B \) with \( |B'| = |B| - 1 \), then
\[ e(G) \geq |A| + 2|B| - 1. \]
(3.2)

**Proof.** Let \( k = |A| - |B| \). We first show that \( d(x) \geq k + 2 \) for every \( x \in B \). Suppose that \( d(x_0) \leq k + 1 \) for some \( x_0 \in B \). Then for \( B' = B \setminus \{x_0\} \),
\[ c(G - B') \geq |A| - (k + 1) + 1 = |A| - k = |B| > |B'|, \]
(3.3)
contradicting the given condition.

Thus, \( d(x) \geq k + 2 \) for every \( x \in B \), and we have
\[ e(G) = \sum_{x \in B} d(x) \geq (k + 2)|B| = (k - 1)(|B| - 1) + |A| + 2|B| - 1. \]
(3.4)
As \( k \geq 1 \), the result holds. \( \square \)
Lemma 3.3. Let $G$ be a 2-connected plane graph. Then

$$\rho(G) \geq 2 - |S| - r + \sum_{i=1}^{r} |V(G_i) \cap S|$$

holds for any independent set $S$ of $G$, where $r = c(G - S)$ and $G_1, G_2, \ldots, G_r$ are the $S$-bridges of $G$.

Proof. Since $S$ is an independent set in $G$, we have

$$e(G) = \sum_{i=1}^{r} e(G_i).$$

Note that $|V(G_i) \cap S| \geq 2$ and $G_i + xy$ is still a plane graph for some distinct $x, y \in V(G_i) \cap S$. Thus, $\rho(G_i) \geq 1$ for $i = 1, 2, \ldots, r$. By Euler’s formula for plane graphs, $G_i$ has exactly $e(G_i) - v(G_i) + 2$ faces. Thus, each $G_i$ has at most $e(G_i) - v(G_i) + 1$ faces bounded by cycles of order 3. Again, since $S$ is an independent set of $G$, the number of faces of $G$ which are bounded by cycles of order 3 is at most

$$\sum_{i=1}^{r} (e(G_i) - v(G_i) + 1).$$

By Euler’s formula again, the number of faces in $G$ is

$$e(G) - v(G) + 2 = 2 - v(G) + \sum_{i=1}^{r} e(G_i).$$

Thus

$$\rho(G) \geq 2 - v(G) + \sum_{i=1}^{r} e(G_i) - \sum_{i=1}^{r} (e(G_i) - v(G_i) + 1) = 2 - v(G) - r + \sum_{i=1}^{r} v(G_i).$$

Since

$$\sum_{i=1}^{r} v(G_i) = v(G) - |S| + \sum_{i=1}^{r} |V(G_i) \cap S|,$$

the result then follows. □

We shall now apply Lemmas 3.2 and 3.3 to prove the following:

Lemma 3.4. Let $G$ be a 2-connected plane graph. If $G \in \Phi_s \setminus \Phi_{s-1}$, where $s \geq 2$, then $\rho(G) \geq s + 1$.

Proof. Since $G \in \Phi_s$, $G$ has an independent set $S$ with $|S| = s$ such that $c(G - S) > s$. Let $G_1, G_2, \ldots, G_r$ be the $S$-bridges of $G$, where $r = c(G - S) > s$. By Lemma 3.3,

$$\rho(G) \geq 2 - s - r + \sum_{i=1}^{r} |V(G_i) \cap S|.$$  

(3.9)

It remains to show that

$$\sum_{i=1}^{r} |V(G_i) \cap S| \geq r + 2s - 1.$$  

(3.10)

Let $H$ be the graph obtained from $G$ by contracting $G_i - (S \cap V(G_i))$ into a vertex $x_i$ for $i = 1, 2, \ldots, r$. Then $H$ is a bipartite graph with bipartition $(A, S)$, where $A = \{x_1, x_2, \ldots, x_r\}$. It is clear that $|A| = r > s = |S|$. Since $G \notin \Phi_{s-1}$, we have

$$c(H - S') = c(G - S') \leq |S'|$$

(3.11)
for every \( S' \subset S \) with \( |S'| = |S| - 1 \). By Lemma 3.2,
\[
e(H) \geq |A| + 2|S| - 1 = r + 2s - 1.
\]
Hence
\[
\sum_{i=1}^{r} |V(G_i) \cap S| = \sum_{i=1}^{r} d_H(x_i) = e(H) \geq r + 2s - 1. \tag{3.13}
\]
A graph \( H \) is called a minor of a graph \( G \), denoted by \( H \preceq G \), if

(i) \( H \in \{G \cup \{G - x : x \in V(G)\} \cup \{G - e, G \cdot e : e \in E(G)\} \) or

(ii) \( H \preceq G' \) and \( G' \preceq G \) for some graph \( G' \).

Thus, \( H \not\preceq G \) means that \( H \) is not a minor of \( G \).

The next three results reveal the fact that the family \( \Phi_2 \cup \Phi_3 \) includes neither \( G - e \) nor \( G \cdot e \) for any 3-connected plane graph \( G \) with \( \rho(G) \leq 4 \) and any edge \( e \in E(G) \).

**Lemma 3.5.** Let \( G \) be a 3-connected graph such that \( K_{3,3} \not\preceq G \). Then \( G - e \notin \Phi_2 \cup \Phi_3 \) for each \( e \in E(G) \).

**Proof.** Let \( H = G - e \), where \( e \in E(G) \). Suppose that \( H \in \Phi_2 \). Then \( c(H - S) \geq 3 \) for some independent set \( S \) of \( H \) with \( |S| = 2 \). So \( S \) is a cut-set of \( G = H + e \), implying that \( G \) is not 3-connected, a contradiction.

Suppose that \( H \in \Phi_3 \). Then \( c(H - S) \geq 4 \) for some independent set \( S \) of \( H \) with \( |S| = 3 \). Let \( H_1, H_2, \ldots, H_r \) be the components of \( H - S \), where \( r \geq 4 \). Since \( H \) is 2-connected, we have \( 2 \leq |N(H_i) \cap S| \leq |S| = 3 \) for each \( i = 1, 2, \ldots, r \). Since \( G = H + e \) is 3-connected, \( |N(H_i) \cap S| = 2 \) holds for at most two \( i \)'s. As \( K_{3,3} \not\preceq G \), \( |N(H_i) \cap S| = 3 \) holds for at most two \( i \)'s. Thus \( r = 4 \) and \( H \) has exactly two components, say \( H_1 \) and \( H_2 \), such that \( |N(H_i) \cap S| = 2 \). This implies that \( |N(H_i) \cap S| = 3 \) for \( i = 3, 4 \).

Let \( S = \{x_1, x_2, x_3\} \). If \( N(H_1) \cap S = N(H_2) \cap S \), then \( G = H + e \) is not 3-connected, a contradiction. We may assume that \( N(H_1) \cap S = \{x_1, x_2\} \) and \( N(H_2) \cap S = \{x_2, x_3\} \). Since \( N(H_i) \cap S = \{x_1, x_2, x_3\} \) for \( i = 3, 4 \) and \( G = H + e \) is 3-connected, the two ends of \( e \) must be in \( H_1 \) and \( H_2 \), respectively, which implies that \( K_{3,3} \preceq G \), a contradiction. Therefore, \( G - e \notin \Phi_3 \). \( \square \)

**Lemma 3.6.** Let \( G \) be 3-connected such that \( K_{3,3} \not\preceq G \). Then \( G \cdot e \notin \Phi_2 \) for every \( e \in E(G) \).

**Proof.** Let \( e \in E(G) \) and \( H = G \cdot e \). Suppose that \( H \in \Phi_2 \). Then \( c(G - \{u, v\}) \geq 3 \) for some non-adjacent vertices \( u, v \) in \( G \). Since \( G \) is 3-connected, either \( u \) or \( v \), say \( u \), is obtained after identifying the two ends \( x \) and \( y \) of \( e \) in \( G \). Thus, \( G - \{x, y, v\} \not\preceq H - \{u, v\} \). Again, since \( G \) is 3-connected, we have \( \{x, y, v\} \subseteq N(G_i) \) for each component \( G_i \) of \( G - \{x, y, v\} \). Since \( c(G - \{x, y, v\}) \geq 3 \), \( G \) contains a subdivision of \( K_{3,3} \), a contradiction. Hence \( G \notin \Phi_2 \). \( \square \)

**Lemma 3.7.** Let \( G \) be a 3-connected plane graph. For any \( e \in E(G) \), if \( \rho(G \cdot e) \leq 4 \), then \( G \cdot e \notin \Phi_3 \).

**Proof.** Let \( H = G \cdot e \), where \( e \in E(G) \). Suppose that \( S \) is an independent set of \( H \) such that \( |S| = 3 \) and \( H \) has \( r \) \( S \)-bridges \( H_1, H_2, \ldots, H_r \), where \( r \geq 4 \). It is clear that \( |S \cap V(H_i)| \geq 2 \) for \( i = 1, 2, \ldots, r \).

If \( H \) has at least two \( S \)-bridges \( H_i \) such that \( S \subseteq V(H_i) \), then, by Lemma 3.4,
\[
\rho(H) \geq 2 - |S| - r + \sum_{i=1}^{r} |N(H_i) \cap S|
\geq 2 - 3 - r + 2 \times 3 + (r - 2) \times 2
\geq r + 1 \geq 5,
\]
a contradiction. So \( H \) has at least three \( S \)-bridges \( H_i \), say \( H_1, H_2 \) and \( H_3 \), such that \( |V(H_i) \cap S| = 2 \). Since \( H \notin \Phi_2 \) by Lemma 3.6, we have
\[
V(H_i) \cap S \neq V(H_j) \cap S \quad \tag{3.15}
\]
for all \(i, j\) with \(1 \leq i < j \leq 3\). Let \(S = \{x_1, x_2, x_3\}\). We may assume that \(V(H) \cap S = \{x_i, x_{i+1}\}\) for \(i = 1, 2, 3\), where \(x_4 = x_1\). Since \(G\) is 3-connected, one vertex in \(S\), say \(x_1\), must be obtained by identifying the two ends \(x\) and \(y\) of \(e\) in \(G\). But then \(\{x_2, x_3\}\) is still a cut-set of \(G\), a contradiction. Hence \(H \not\in \Phi_3\). \(\Box\)

We are now in a position to prove the main result of this section.

**Theorem 3.2.** Let \(G\) be a 2-connected plane graph without loops and multiedges. If either \(\rho(G) \leq 2\) or \(3 \leq \rho(G) \leq 4\) and \(G \notin \bigcup_{2 \leq t \leq \rho(G) - 1} \Phi_t\), then

\[
(-1)^{v(G)} P(G, \lambda) > 0
\]

for all \(\lambda \in (1, 2)\).

**Proof.** We shall apply Theorem 2.1 to prove this result. By Lemma 3.4, if \(G\) is a 2-connected plane graph and \(G \notin \Phi_t\) for each \(t\) with \(2 \leq t < \rho(G) \leq 4\), then \(G \notin \Phi_2 \cup \Phi_3\).

Let \(\mathcal{U}\) be the family of the following graphs:

(i) \(K_2\) and

(ii) 2-connected plane graphs \(G\) such that \(\rho(G) \leq 4\) and \(G \notin \Phi_2 \cup \Phi_3\).

It is clear that \(K_3 \in \mathcal{U}\). We shall show that \(\mathcal{U}\) satisfies the three conditions stated in Theorem 2.1. Let \(G \in \mathcal{U}\) with \(v(G) \geq 4\).

Assume that \(\{x, y\}\) is a cut-set of \(G\) and \(xy \in E(G)\). Let \(G_i\) be any \(\{x, y\}\)-bridge. It is clear that \(G_i\) is a 2-connected plane graph and \(\rho(G_i) \leq \rho(G) \leq 4\). Also observe that \(G_i \notin \Phi_2 \cup \Phi_3\). Hence \(G_i \notin \mathcal{U}\).

Assume that \(\{x, y\}\) is a cut-set of \(G\) and \(xy \notin E(G)\). Since \(G \notin \Phi_2\), \(G\) has only two \(\{x, y\}\)-bridges. Let \(G_i\) be any \(\{x, y\}\)-bridge. It is clear that \(G_i + xy\) is 2-connected, \(\rho(G_i + xy) \leq \rho(G) \leq 4\) and \(G_i + xy \notin \Phi_2 \cup \Phi_3\). Thus, \(G_i + xy \in \mathcal{U}\).

Observe that \(G_i \cdot xy\) is a 2-connected plane graph and \(\rho(G_i \cdot xy) \leq \rho(G) \leq 4\). Now suppose \(G_i \cdot xy \in \Phi_k\), where \(2 \leq k \leq 3\). Then, by Lemma 3.1, \(G \in \Phi_k \cup \Phi_{k+1}\). Since \(G \notin \Phi_2 \cup \Phi_3\), we have \(k = 3\) and \(G \in \Phi_4\). But, by Lemma 3.4, \(\rho(G) \geq 5\), a contradiction. Thus, \(G_i \cdot xy \notin \Phi_2 \cup \Phi_3\).

Finally, we assume that \(G\) is 3-connected. We may assume that the external face of \(G\) is not bordered by three vertices and 3-edges if \(\rho(G) \geq 1\). Let \(e\) be any edge on the border of the external face of \(G\). It is clear that both \(G - e\) and \(G \cdot e\) are 2-connected. If \(\rho(G) = 0\), then \(\rho(G - e) \leq 1\); otherwise, \(\rho(G - e) \leq \rho(G) \leq 4\). We also have \(\rho(G \cdot e) \leq \rho(G) \leq 4\). Since both \(G - e\) and \(G \cdot e\) are plane graphs, we have \(K_{3,3} \notin G - e\) and \(K_{3,3} \notin G \cdot e\). By Lemmas 3.5–3.7, \(G - e \notin \Phi_2 \cup \Phi_3\) and \(G \cdot e \notin \Phi_2 \cup \Phi_3\). Hence \(G - e \notin \mathcal{U}\) and \(G \cdot e \notin \mathcal{U}\).

Therefore, the family \(\mathcal{U}\) satisfies the conditions in Theorem 2.1. The result thus follows from Theorem 2.1. \(\Box\)

By Lemma 3.4, Theorem 3.2 shows that Conjecture 1.2 holds for plane graphs \(G\) with \(\rho(G) \leq 4\). We guess Conjecture 1.2 holds for all plane graphs. When confined to plane graphs, by Lemma 3.4, Conjecture 1.2 can be stated as follows.

**Conjecture 3.1.** Let \(G\) be a 2-connected plane graph without loops and multiedges. If \(G \notin \bigcup_{2 \leq t \leq \rho(G) - 1} \Phi_t\), then \((-1)^{v(G)} P(G, \lambda) > 0\) for all \(\lambda \in (1, 2)\).

**4. On graphs without some special minors**

It is known that every outerplanar graph \(G\) has no chromatic roots in \((1, 2)\). Actually, Wakelin and Woodall [9] showed that any chromatic root of an outerplanar graph is of the form \(1 + u\), where \(u\) is a root of the equation \(u^k = 1\) for some positive integer \(k\).

Note that \(K_{2,3}\) is not a minor of any outerplanar graph. Applying Theorem 2.1, we shall show in what follows that every 2-connected graph without a minor \(K_{2,3}\) has no chromatic zeros in \((1, 2)\).

**Theorem 4.1.** If \(G\) is 2-connected and \(K_{2,3} \not\subseteq G\), then \((-1)^{v(G)} P(G, \lambda) > 0\) for all \(\lambda \in (1, 2)\).

**Proof.** Let \(\mathcal{S}\) be the family of 2-connected graphs \(G\) such that \(K_{2,3} \not\subseteq G\). We may assume that \(K_2\) is a special member in \(\mathcal{S}\). It is clear that \(K_3 \in \mathcal{S}\). It suffices to show that \(\mathcal{S}\) satisfies the conditions in Theorem 2.1.
Let $G \in \mathcal{F}$ with $v(G) \geq 4$. If $G$ is 3-connected, then both $G - e$ and $G \cdot e$ belong to $\mathcal{F}$.

Suppose that $G$ has a cut-set $\{x, y\}$. If $xy \in E(G)$, it is clear that every $\{x, y\}$-bridge of $G$ is 2-connected and does not include $K_{2,3}$ as a minor, implying that each $\{x, y\}$-bridge of $G$ belongs to $\mathcal{F}$. If $xy \notin E(G)$, then $c(G - x - y) = 2$; otherwise, $K_{2,3} \cong G$. Observe that every $\{x, y\}$-bridge of $G + xy$ is 2-connected and does not include $K_{2,3}$ as a minor, and each block of $G \cdot xy$ is either $K_2$ or 2-connected and does not include $K_{2,3}$ as a minor.

Hence $\mathcal{F}$ satisfies the conditions in Theorem 2.1 and the result follows. \hfill \Box

Let $G_0$ be the graph shown in Fig. 1.

**Lemma 4.1.** Let $G$ be a 2-connected graph. If $G \notin \Phi_3 \setminus \Phi_2$, then $G_0 \cong G$.

**Proof.** Let $T = \{x_1, x_2, x_3\}$ be an independent set of $G$ such that $c(G - T) \geq 4$. Since $G$ is 2-connected, $|V(G_i) \cap T| \geq 2$ for every $T$-bridge $G_i$ of $G$.

Since $G \notin \Phi_2$, there are no $T$-bridges $G_1'$ and $G_2'$ of $G$ such that

$$V(G_1') \cap T = V(G_2') \cap T \neq T.$$ (4.1)

Since $c(G - T) \geq 4$, by (4.1), $G$ has a $T$-bridge $G_i$ such that $T \subseteq V(G_i)$, say $i = 1$. Again, by (4.1), there exist another three $T$-bridges $G_2, G_3$ and $G_4$ such that $\{x_i, x_{i+1}\} \subseteq V(G_{i+1})$ for $i = 1, 2, 3$, where $x_4 = x_1$. Thus, $G_0 \cong G$, as required. \hfill \Box

**Theorem 4.2.** Let $G$ be a 2-connected graph. If $G \notin \Phi_2$ and $G_0 \cong G$, where $G_0$ is the graph of Fig. 1, then $(-1)^{v(G)} P(G, \lambda) > 0$ for all $\lambda \in (1, 2)$.

**Proof.** Let $\mathcal{F}$ be the family of 2-connected graphs $G$ such that $G \notin \Phi_2$ and $G_0 \cong G$. We may assume that $K_2$ is a special member in $\mathcal{F}$. It is clear that $K_3 \notin \mathcal{F}$. We shall show that $\mathcal{F}$ satisfies the conditions in Theorem 2.1.

Let $G \in \mathcal{F}$ with $v(G) \geq 4$. Assume that $G$ is 3-connected. If $v(G) = 4$, then both $G - e$ and $G \cdot e$ belong to $\mathcal{F}$ for any edge $e$ in $G$. Assume now that $v(G) \geq 5$. Since $G$ is 3-connected, it is known that there exists an edge $e$ such that $G \cdot e$ is 3-connected (see, for instance, [2, Lemma 3.2.1]). This implies that $G \cdot e \notin \Phi_2$. It is also clear that $G - e \notin \Phi_2$, $G_0 \cong G \cdot e$ and $G_0 \cong G - e$. Hence both $G \cdot e$ and $G - e$ belong to $\mathcal{F}$.

Now assume that $G$ has a cut-set $\{x, y\}$. If $xy \in E(G)$, then each $\{x, y\}$-bridge of $G$ is 2-connected, does not belong to $\Phi_2$ and does not include $G_0$ as a minor. Thus, each $\{x, y\}$-bridge of $G$ belongs to $\mathcal{F}$. Suppose $xy \notin E(G)$. Then $c(G - x - y) = 2$, as $G \notin \Phi_2$. Each block of $G \cdot xy$ is either $K_2$ or 2-connected and does not include $G_0$ as a minor. If some block belongs to $\Phi_2$, then by Lemma 3.1, $G \in \Phi_2 \cup \Phi_3$. Since $G \notin \Phi_2$, we have $G \in \Phi_3$. But, then by Lemma 4.1, $G_0 \cong G$, a contradiction. Thus, each block of $G \cdot xy$ belongs to $\mathcal{F}$.

Hence $\mathcal{F}$ satisfies the conditions in Theorem 2.1, and the result follows. \hfill \Box

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