Fourier–Galerkin domain truncation method for Stokes’ first problem with Oldroyd four-constant liquid

F. Talay Akyildiz\textsuperscript{a}, K. Vajravelu\textsuperscript{b,\ast}, H. Ozekes\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Ondokuz Mayis University, 55139, Kurupelit Samsun, Turkey
\textsuperscript{b}Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
\textsuperscript{c}Department of Mathematics, Okan University, 34959, Akfirat-Tuzla, Istanbul, Turkey

Received 3 August 2007; accepted 13 August 2007

Abstract

Using the Fourier–Galerkin method with domain truncation strategy, Stokes’ first problem for Oldroyd four-constant liquid on a semi-infinite interval is studied. It is shown that the Fourier–Galerkin approximations are convergent on the bounded interval. Moreover, an efficient and accurate algorithm based on the Fourier–Galerkin approximations is developed and implemented in solving the differential equations related to the present problem. Also, the effects of non-Newtonian parameters on the flow characteristics are obtained and analyzed. The method developed here is so general that it can be used to study the mathematical models that involve the flow of viscous fluids with shear rate-dependent properties: For example, models dealing with polymer processing, tribology & lubrication, and food processing.

\textcopyright{} 2007 Elsevier Ltd. All rights reserved.

Keywords: Fourier–Galerkin method; Stokes’ first problem; Oldroyd four-constant model; Discontinuous boundary condition; Quasilinear parabolic equation; Regularized boundary layer function

1. Introduction

Stokes’ first problem for a channel, also known as Rayleigh’s problem, was first studied in 1851 by Stokes \cite{1}. This unsteady flow problem examines the diffusion of vorticity in a half-space filled with a viscous, incompressible fluid that is set in motion when an infinite flat plate (i.e., the bounding plane) suddenly moves with a constant velocity parallel to itself from rest. While the original problem solved by Stokes dealt with a viscous Newtonian fluid, many researchers \cite{2–6} have studied the problem for a variety of non-Newtonian fluids.

However, as discussed by Dunn and Rajagopal \cite{7}, the inadequacy of the classical Navier–Stokes theory to describe rheological complex fluids such as polymer solution (for example, blood, paints, certain oils and greases) has led to the development of several theories for non-Newtonian fluids. Amongst the many models used to describe the non-Newtonian behavior, the fluids of Oldroyd type have received special attention, and the liquids of Oldroyd have been...
studied successfully in various types of flow situations. (We may mention here some of the studies made by Fetecau and Kannan [8] and Hayat et al. [9].)

To the best of our knowledge, the approximate solution for the Stokes’ first problem with an Oldroyd four-constant liquid has received very little attention. Here we study the effects of non-Newtonian parameters of Oldroyd four-constant liquid on that problem. Since our problem is defined on the semi-infinite interval, the domain truncation method is chosen so that this new problem will be amenable to numerical computation and this method can be applied to any problem on a bounded interval.

After the domain truncation, we apply two types of spectral methods. The first type imposes artificial boundary conditions and uses Fourier–Galerkin/pseudospectral approximation. (Please see Refs. [10,11].) The other type uses Chebyshev or Legendre–Galerkin/pseudospectral approximation. (For details, see [12,13].)

Besides Fourier–Galerkin, we did some numerical experiments with Chebyshev–Galerkin method, for Stokes’ first problem with Newtonian fluid where exact analytical solutions (see Ref. [14]) are possible. We found that the Fourier–Galerkin method is more efficient than the Chebyshev–Galerkin method, and in a follow-up paper, we would like to compare the results of the above two methods.

2. Mathematical formulation of the problem and the Fourier–Galerkin method

Taking the positive y-axis of a Cartesian coordinate system in the upward direction, let the Oldroyd four-constant liquid fill the half-space $y > 0$ above and be in contact with a flat plate occupying the $xz$-plane. Initially, both the fluid and the plate are at rest. At time $t = 0+$, the fluid is set in motion by the sudden acceleration of the plate along the $x$-axis to a constant velocity $U_0$ ($\neq 0$); that is, the velocity of the plate is given by $(U_0\theta(t), 0, 0)$, where $\theta(t)$ denotes the Heaviside unit step function. Under these conditions, the fluid velocity at a given point in the half-space depends only on its $y$-coordinate and the time $t$. That is

$$v = (u(t, y), 0, 0),$$

and the equation of motion reduces to

$$\frac{\partial p_{xy}}{\partial y} = \frac{\partial u}{\partial t},$$

where $p_{xy}$ is the $xy$-component of the stress tensor.

The constitutive equations for the Oldroyd four-constant model (for details see Ref. [15]) are

$$p_{ik} = -p\delta_{ik} + s_{ik},$$

$$s_{ik} + \lambda_1 \frac{\Delta s_{ik}}{\Delta t} + \mu_0 s_{ik} e^{(1)}_{ik} = s \left( e^{(1)}_{ik} + \lambda_2 \frac{\Delta e^{(1)}_{ik}}{\Delta t} \right),$$

where $p$ is the isotropic pressure, $\delta_{ik}$ is the identity tensor, $s_{ik}$ is the deviatoric stress tensor, and $\lambda_1$, $\lambda_2$ and $\mu_0$ are dimensionless material parameters. In Eqs. (2.1)–(2.4) we used the following non-dimensional quantities: $u = \frac{u'}{U_0}$, $y = y'(U_0/v)$, $s_{ik} = \frac{s_{ik}}{\mu_0 U_0^2}$, $\lambda_1 = \lambda'_1 U_0^2/v$ and $\lambda_2 = \lambda'_2 U_0^2/v$ where $v = \mu/\rho$. The first rate of strain tensor defined by $e^{(1)}_{ik} = (\partial u_i/\partial x_k + \partial u_k/\partial x_i)$, and $\frac{\Delta}{\Delta t}$ is the contravariant convected derivative defined by Oldroyd [15] as

$$\frac{\Delta A^{ij}}{\Delta t} = \frac{\partial A^{ij}}{\partial t} + u^r \frac{\partial A^{ij}}{\partial x^r} - \frac{\partial u^i}{\partial x^j} A^{ij} - \frac{\partial u^j}{\partial x^i} A^{kj}.$$ (2.5)

Using the velocity distribution in Eq. (2.1), the only non-zero components of the first rate of strain tensor reduce to $e^{(1)}_{xy} = e^{(1)}_{yx} = (\partial u/\partial y)/2$. The equations of state given by Eq. (2.5) reduce to

$$s_{xy} + \lambda_1 \frac{\partial s_{xy}}{\partial t} + \frac{1}{2} \mu_0 s_{xx} \frac{\partial u}{\partial y} = \left( \frac{\partial u}{\partial y} + \lambda_2 \frac{\partial^2 u}{\partial y \partial t} \right),$$

$$s_{xx} + \lambda_1 \frac{\partial s_{xx}}{\partial t} - 2 \lambda_1 s_{xy} \frac{\partial u}{\partial y} = -2 \lambda_2 \left( \frac{\partial u}{\partial y} \right)^2.$$ (2.7)
The aim of the present work is to find the solution for Eqs. (2.2), (2.6) and (2.7) subject to initial boundary conditions
\[
\begin{align*}
\frac{\partial u}{\partial t} + \theta_1^N(t) e^{-\gamma} &= 0, & t \in [0, T], \\
\left( \begin{array}{c}
\partial_t \theta_1^N(t) e^{-\gamma} - \partial_3 s_{xy}^{N, 2} \\
\partial_t s_{xy}^{N, 3} + \frac{1}{2} \mu_0 s_{xx}^N (\partial_3 v_N + \theta_1^N(t) e^{-\gamma}) - \left( \frac{\partial_3 v_N + \theta_1^N(t) e^{-\gamma}}{\sqrt{L}} + \lambda_2 \left( \frac{\partial_3 v_N + \theta_1^N(t) e^{-\gamma}}{\sqrt{L}} \right)^2, \omega \right)
\end{array} \right), & (2.19)
\end{align*}
\]
for all \( \varphi \in S_N \) and \( \omega \in R_N \), where for each \( t, v_N(.,.) \), \( s_{xy}^{N, 3}(.,.) \) and \( s_{xy}^{N, 2}(.,.) \) have the form

The well-posedness of (2.2), (2.6) and (2.7) in the classical sense can be found in Ladyzhenskaya et al. [16]. Here the problem has discontinuous boundary condition. Since spectral methods are very sensitive to the smoothness of the solutions, it is crucial to design a sensible treatment for the singular boundary condition. We emphasize that the singular boundary conditions are usually a mathematical idealization of the physical situation. The singularity can never be realized in experiments nor in numerical computations. Therefore, it is appropriate to use a regularized boundary layer function to approximate the actual physical situation as in [17, 18]). In our case the singular boundary condition can be approximated by
\[
\theta_1^N = 1 - e^{-t/\epsilon},
\]
to within any prescribed accuracy, by choosing an appropriate \( \epsilon \). Such an approach proved successful in [17, 18]. We define now a new independent variable \( v(y, t) \) by
\[
v(y, t) = u(y, t) - \theta_1^N(t) e^{-\gamma}.
\]

Now, the governing equations and the boundary conditions can be written as follows:
\[
\frac{\partial p_{xy}}{\partial y} = \frac{\partial v}{\partial t} + \partial_t \theta_1^N(t) e^{-\gamma},
\]
\[
as_{xy} + \lambda_1 \frac{\partial s_{xy}}{\partial t} + \frac{1}{2} \mu_0 s_{xx}^N \left( \frac{\partial v}{\partial y} + \theta_1^N(t) e^{-\gamma} \right) = \left( \frac{\partial v}{\partial y} + \theta_1^N(t) e^{-\gamma} + \lambda_2 \left( \frac{\partial^2 v}{\partial y^2} + \theta_1^N(t) e^{-\gamma} \right) \right),
\]
\[
s_{xx} + \lambda_1 \frac{\partial s_{xx}}{\partial t} - 2 \lambda_1 s_{xy} \left( \frac{\partial v}{\partial y} + \theta_1^N(t) e^{-\gamma} \right) = -2 \lambda_2 \left( \frac{\partial v}{\partial y} + \theta_1^N(t) e^{-\gamma} \right)^2,
\]
\[
u(0, t) = 0, \quad u(L, t) = 0,
\]
\[
u(0, t) = 0, \quad (y > 0),
\]
\[
s_{xx}(y, 0) = s_{xy}(y, 0) = 0, \quad s_{xx}(L, t) = s_{xy}(L, t) = 0,
\]
where \( L \) is an arbitrary but fixed positive number.

Next, we chose \( \{e_k(y), k \in \mathbb{N}\} = \left\{ \frac{\sqrt{2} \sin(k\pi y/L)}{\sqrt{L}}, k \in \mathbb{N} \right\} \) and \( \{f_k(y), k \in \mathbb{N}\} = \left\{ \frac{\sqrt{2} \cos(k\pi y/L)}{\sqrt{L}}, k \in \mathbb{N} \right\} \) to be an orthonormal basis of the Hilbert spaces \( L^2[0, L] \) and \( \mathbb{L}^2[0, L] \) respectively. Then, the subspaces of these Hilbert spaces are spanned by \( S_N = \left\{ \frac{\sqrt{2} \sin(k\pi y/L)}{\sqrt{L}}, 1 \leq k \leq N \right\} \) and \( R_N = \left\{ \frac{\sqrt{2} \cos(k\pi y/L)}{\sqrt{L}}, 0 \leq k \leq N \right\} \).

Fourier–Galerkin approximations of (2.13)–(2.18) are used to find functions \( \{v_N, s_{xx}^N, s_{xy}^N\} \) from \( [0, T] \) to \( \{S_N, R_N\} \) which satisfy
\[
\frac{\partial_3 v_N + \theta_1^N(t) e^{-\gamma}}{\sqrt{L}} + \lambda_2 \left( \frac{\partial_3 v_N + \theta_1^N(t) e^{-\gamma}}{\sqrt{L}} \right)^2, \omega, 
\]
for all \( \varphi \in S_N \) and \( \omega \in R_N \), where for each \( t, v_N(.,.), s_{xy}^{N, 3}(.,.) \) and \( s_{xy}^{N, 2}(.,.) \) have the form
\[ v_N(y, t) = \sum_{k=1}^{\infty} \hat{v}_N(k, t) \frac{\sqrt{2} \sin(k\pi y/L)}{\sqrt{L}}, \]

\[ s_{xx}^N(y, t) = \sum_{k=0}^{\infty} \hat{s}_{xx}^N(k, t) \frac{\sqrt{2} \cos(k\pi y/L)}{\sqrt{L}}, \]

and

\[ s_{xy}^N(y, t) = \sum_{k=0}^{\infty} \hat{s}_{xy}^N(k, t) \frac{\sqrt{2} \cos(k\pi y/L)}{\sqrt{L}}. \]

(2.20)

Using \( \varphi = \sin(k\pi y/L) \) for \( 1 \leq k \leq N \) and \( \omega = \cos(k\pi y/L) \) for \( 0 \leq k \leq N \) in (2.19), we obtain the following system of equation for the Fourier coefficient of \( v_N, s_{xx}^N \) and \( s_{xy}^N \):

\[ \frac{d}{dt} \hat{v}_N(k, t) + \frac{k\pi}{L} \hat{s}_{xy}^N(k, t) - \frac{2}{L} \int_0^L \frac{\partial}{\partial t} \theta_1^y(t) e^{-y} \sin(k\pi y/L) dy = 0, \]

(2.21)

\[ \lambda_1 \frac{d}{dt} \hat{s}_{xy}^N(k, t) + \hat{s}_{xy}^N(k, t) + \frac{1}{2} \mu_0 \sum_{i,j=0}^{N} j \frac{k\pi}{L} c_{ijk} \hat{s}_{xx}^N(i, t) \hat{v}_N(j, t) \]

\[ + \frac{1}{\mu_0} \int_0^L \hat{s}_{xx}^N(k, t) \theta_1^y(t) e^{-y} \cos(k\pi x/L) \]

\[ - \frac{k\pi}{L} \hat{v}_N(k, t) - (\lambda_2 + 1) \frac{2}{L} \int_0^L \frac{\partial}{\partial t} \theta_1^y(t) e^{-y} \cos(k\pi y/L) dy - \lambda_2 \frac{k\pi}{L} \frac{d}{dt} \hat{v}_N(k, t), \]

(2.22)

\[ \lambda_1 \frac{d}{dt} \hat{s}_{xx}^N(k, t) + \hat{s}_{xx}^N(k, t) - 2\lambda_1 \frac{k\pi}{L} \sum_{i,j=0}^{N} c_{ijk} \hat{s}_{xy}^N(i, t) \hat{v}_N(j, t) \]

\[ + \frac{4}{L} \lambda_1 \int_0^L \hat{s}_{xy}^N(k, t) \theta_1^y(t) e^{-y} \]

\[ \cos(k\pi y/L) dy + \int_0^L \left( \frac{2k\pi}{L} \hat{v}_N(k, t) \theta_1^y(t) e^{-y} \right) \]

\[ \cos(k\pi y/L) dy, \]

(2.23)

\[ \hat{v}_N(k, 0) = 0, \quad \hat{s}_{xy}^N(k, 0) = 0 \quad \text{and} \quad \hat{s}_{xx}^N(k, 0) = 0, \]

(2.24)

where \( c_{ijk} = \int_0^L \cos(i\pi y/L) \cos(j\pi y/L) : \cos(k\pi y/L) dy. \)

(2.25)

### 3. Results and discussion

The system of nonlinear differential equations is solved by a standard Runge–Kutta method. To illustrate the spectral accuracy, the time step is chosen to be sufficiently small so that the error is dominated by the spatial discretization. We first consider Stokes’ first problem for Newtonian fluid; i.e. \( \lambda_1 = \lambda_2 = \mu_0 = 0. \) In this case the exact analytical solution is possible and can be written as

\[ u(y, t) = 1 - \frac{2}{\pi} \int_0^\infty e^{-s^2 t} \frac{\sin(ys)}{s} ds. \]

(3.1)

In Fig. 1, we compare our approximate solution for \( L = 6\pi \) with the exact analytical solution (3.1), with number of nodes \( N = 25. \) It has been observed that there is no significant difference between the result obtained for \( N = 25 \) and \( 40. \) From Fig. 1, we see that the error around \( t = 0 \) is 6.5% (this is expected, because of the discontinuous boundary condition) and this error decreases when the value of \( t \) increases. Fig. 2 shows the effects of the non-Newtonian parameters \( (\lambda_1, \lambda_2, \mu_0) \) on the velocity profiles for fixed \( t. \) In Fig. 2, we see that the Oldroyd four-constant liquid velocity is always less than the Newtonian fluid velocity. This behavior is true for all values of the non-Newtonian parameters \( (\lambda_1, \lambda_2, \mu_0). \) Fig. 3 shows the variations of normal stress \( s_{xx}(y, 1) \) with \( y. \) Here, \( s_{xx}(y, 1) \) is not equal to zero. It is well-known that \( s_{xx}(y, 1) = 0 \) for Newtonian fluid.
Fig. 1. Velocity profiles at $y = 2$: the circles indicate the exact analytical solution and the continuous line is for the solution of the present study.

Fig. 2. Velocity profiles at $t = 1$: the circles for the Newtonian fluid and the continuous line for the Oldroyd four-constant fluid, with $\lambda_1 = 3$, $\lambda_2 = 1$ and $\mu_0 = 1$.

Fig. 3. Variations in normal stress $s_{xx}(y, 1)$ profile at $t = 1$ for the Oldroyd four-constant fluid with $\lambda_1 = 3$, $\lambda_2 = 1$ and $\mu_0 = 1$.
In conclusion, we have shown that the domain truncation can be applied to a system of nonlinear partial differential equations arising in Stokes’ first problem for Oldroyd four-constant fluid. We have also presented the numerical results. We would like to present the results of error and convergence analyzes for the above problem in a follow-up paper.

References