Nonlinear damped wave equation arising in viscoelastic fluid flows

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Abstract

We consider the nonlinear damped wave equation arising in unsteady Poiseuille flow of shear thinning Maxwell fluid. We show the global existence of the smooth solution to the corresponding nonlinear system. We also give some numerical examples. © 2006 Elsevier Ltd. All rights reserved.

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1. Introduction

The elastic effect in fluids frequently gives rise to striking effects in unsteady flows. Examples are to be seen in the analysis of Rajagopal and Bahatnagar [1], Waters and King [2] and Akyildiz and Jones [3]. In these works, generation of flow by constant pressure gradient was considered. It is seen that the effect of elasticity produces velocity profiles which ‘overshoot’ and oscillate about the terminal velocity. Such velocity overshoot has been observed experimentally as well [4].

In this paper, we consider Maxwell fluid having the constitutive equation [5]

\[
\sigma = -pI + T,
\]

\[
T + \lambda \nabla \cdot T = 2\eta d,
\]

where \( T \) is the stress tensor, \( p \) pressure, \( d \) symmetric part of, and \( \lambda \) is constant relaxation. The nabla denotes the upper-convected time derivative. Shear thinning can be easily added to the Maxwell model by using the Carreau–Bird viscosity law:

\[
\frac{\eta - \eta_\infty}{\eta_o - \eta_\infty} = \left[1 + (\lambda_2 \dot{\gamma})^2\right]^{(n-1)/2},
\]

where \( \dot{\gamma} \) is the rate of strain rate defined in terms of the second invariant in the usual way and \( 0 < n \leq 1 \). In this report, unsteady Poiseuille flow of shear thinning Maxwell fluid through a channel is considered. For this flow, the velocity field and the stress tensor are in the following form

\[
v = u(y, t)i \quad \text{and} \quad T = T(y, t)
\]

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which satisfies the continuity equation identically. The relevant equation of motion is

\[
\frac{\partial u}{\partial t} = -P + \frac{\partial T_{xy}}{\partial y}.
\]  

(1.4)

We first differentiate (1.4) with respect to \( t \) and multiplying by \( \lambda \), we obtain

\[
\rho \lambda \frac{\partial^2 u}{\partial t^2} = \lambda \frac{\partial^2 T_{xy}}{\partial t \partial y}.
\]  

(1.5)

Summing (1.4) with (1.5) and substituting (1.1) into this, after using the nondimensional parameters as in [1], we get

\[
\frac{1}{\lambda} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \frac{8}{\lambda} + \frac{1}{\lambda} \frac{\partial}{\partial y} \left( 1 + \frac{(\eta_0 - \eta_\infty) / \eta_\infty}{1 + \left( \frac{\lambda^2}{\eta_\infty} \frac{\partial u}{\partial y} \right)^{2(1-n)/2}} \right) \frac{\partial u}{\partial y}.
\]  

(1.6)

This equation is to be solved subject to the initial conditions;

\[
u(y, 0) = 0, \quad \nu_t(y, 0) = 8, \quad 0 \leq y \leq 1,
\]  

(1.7)

and the boundary conditions;

\[
u(0, t) = \nu(1, t) = 0, \quad t > 0.
\]  

(1.8)

We note that if we take \( n = 1 \) in (1.6), we obtain the D’Alembert equation and in this case the exact analytical solution is readily available [2]. But, if \( n \neq 1 \), then the exact analytical solution of the equation cannot be found. In this work, we provide global existence of smooth solutions for the nonlinear nonhomogeneous damped wave equations (1.6) with (1.7) and (1.8). A literature survey reveals that shear thinning Oldroyd-B and Maxwell model has also been considered before by Rajagopal et al. [6,7], but they did not consider global existence of the smooth solutions. In [8] Renardy and [9] Guillop´e and Saut used the linear Maxwell model and studied the existence results for the two dimensional flow. This model clearly is the special case of our model. There are also many papers concerning the global existence of the smooth solutions for nonlinear damped wave equations [10–12]. Obviously, our problem has not been covered in these works as well. These give the motivation for the present study.

This paper is organized as follows. In Section 2, we provide global existence of the smooth solutions for Eqs. (1.6)–(1.8). In the last section, we give some numerical examples.

For the remainder of this article, \( H \) denotes the standard Hilbert space \( L^2(\Omega) \), with norm \( \| \cdot \| \) and inner product \( \langle \cdot, \cdot \rangle \).

2. Global existence for the nonlinear system

**Theorem 2.1.** There is \( T > 0 \) such that (1.6)–(1.8) has a unique local solution

\[
u \in \bigcap_{k=1}^3 C^k([0, T], H^{4-k}(0, 1)) \cap H^1_0((0, 1)) \cap C^4([0, T], L^2(0, 1)).
\]

As explained in [13], it is useful to rewrite (1.6)–(1.8) as a first-order system as

\[
V = (u_t, u_y)^T.
\]

(2.1)

Then it is easy to see that (1.6)–(1.8) can be written as

\[
V_t + \begin{pmatrix} \frac{1}{\lambda} & -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & 0 \end{pmatrix} V = \begin{pmatrix} 8 + B(u_y)u_{yy} \\ 0 \end{pmatrix} = \begin{pmatrix} 8 + M(V, V_y) \end{pmatrix}
\]

(2.2)

\[
V_0 = (8, 0)^T
\]
Lemma 2.2. There are constant \( c_1, c_2 > 0 \) which are independent from \( V_0 \) and \( T \), such that the local solution given in Theorem 2.1 satisfies for \( t \in [0, T] \):

\[
\| V(t) \|_{H^3}^2 \leq c_1 \| V_0 \|_{H^3}^2 e^{2\alpha T} e^{c_2 \int_0^T (\| V(\tau) \|_{H^2} + \| V(\tau) \|_{H^3})^2 d\tau}.
\]  

(2.7)

Proof. The proof can be found in [14] in detail. \( \square \)

Lemma 2.3. Suppose that \( V \in H^3 \) and \( c > 0 \). Then \( \| M(V, V_y) \| \leq c_6 \| V \|_{H^2}^2 \| V \|_{H^3} \).

Proof. First, from the Young inequality and Sobolev (Gagliardo–Nirenberg) imbedding, we have

\[
\| B(u_y)u_{yy} \|_{H^2} \leq \| B(u_y) \|_{H^2} \| u_{yy} \|_{H^2} \\
\leq c \| B(u_y) \|_{H^2}^{1/2} \| B(u_y) \|_{H^2}^{1/2} \| u_{yy} \|_{H^2}^{1/2} \| u_{yy} \|_{L^\infty}^{1/2} \\
\leq c \left( \| B(u_y) \|_{H^2} \| u_{yy} \|_{L^\infty} \right)^{1/2} \left( \| B(u_y) \|_{H^2} \| u_{yy} \|_{H^2} \right)^{1/2} \\
\leq c \left( \| B(u_y) \|_{H^2} \| u_{yy} \|_{L^\infty} \right) \left( \| B(u_y) \|_{L^\infty} \| u_{yy} \|_{H^2} \right).
\]  

(2.8)

Since \( B(u_y) = (\eta_0 - \eta_x) / \eta_x \frac{n^2 \lambda \beta u_{yy}^2}{(1 + \lambda^2 u_y^2)^{1/2}} \) then we have \( \| B(u_y) \|_{L^\infty} \leq c_5 \sup |u_y^2| \), similarly, \( \| B(u_y) \|_{H^2} \leq \| u_y \|_{L^\infty} \| u_y \|_{H^3} \). Therefore, using the Sobolev imbedding, we obtain

\[
\| B(u_y)u_{yy} \|_{H^2} \leq c_6 \| V \|_{H^2}^2 \| V \|_{H^3}.
\]  

(2.9)

Since from (2.6), we have \( H^2 \) estimates as

\[
\| V(t) \|_{H^2} \leq ce^{-\alpha t} \| V_0 \|_{H^2} + c \int_0^T e^{-\alpha (t-\tau)} \| 8 + M(V, V_y) \|_{H^2} d\tau,
\]  

(2.10)

and substituting (2.9) into this, we get

\[
\| V(t) \|_{H^2} \leq ce^{-\alpha t} \| V_0 \|_{H^2} + c_7 \int_0^T e^{-\alpha (t-\tau)} \left( \| 8 \|_{H^3} + \| V(\tau) \|_{H^2}^2 \| V(\tau) \|_{H^3} \right) d\tau
\]  

(2.11)

which can be used as the starting point for the following lemma. \( \square \)
**Lemma 2.4.** For $0 \leq t \leq T$, if

$$B_1(t) = \sup_{0 \leq r \leq t} \left( e^{\tau_0 \alpha} \| V(t) \|_{H^2} \right) \quad (0 < \tau_0 \leq \alpha)$$

Then there are $B_0 > 0$ and $\delta > 0$ such that if $\| V_0 \|_{H^2}$, for all $0 \leq t \leq T$, we have

$$B_2(t) \leq B_0 < \infty.$$  \hspace{1cm} (2.13)

where $B_0$ independent of $T$ and $V_0$.

**Proof.** The proof can be found in [9]. □

Now, we can give the main theorem on global existence and exponential decay.

**Theorem 2.5.** Let $\delta > 0$ be given if $\| V_0 \|_{H^2} < \delta$, there is a unique global solution $u$ to (1.6)–(1.8) satisfying

$$u \in \bigcap_{l=1}^{3} C^l ([0, T], H^{4-k} (0, 1)) \cap \mathcal{H}_0^1 ((0, 1)) \cap C^4 ([0, T], L^2 (0, 1)).$$

Moreover, there are constants $c_0 = c_0 (V_0) > 0$ and $c_1 > 0$ such that

$$\| V(t) \|_{H^2} \leq c_0 e^{-\alpha t},$$

and

$$\| V(t) \|_{H^3} \leq c_8 \| V_0 \|_{H^2} e^{\frac{1}{2} \alpha t}. \hspace{1cm} (2.14)$$

**Proof.** From Lemmas 2.2 and 2.4, we get for the local solution as

$$\| V(t) \|_{H^3} \leq c_1 \| V_0 \|_{H^2} e^{\frac{1}{2} \alpha t} \left( c_2 \int_0^t \left( \| V_0 \|_{H^2} + \| V(t) \|_{H^2} \right) \right) dt$$

$$\leq c \| V_0 \|_{H^2} e^{\frac{1}{2} \alpha t} (B_0 + B_0^2 + B_0^3)$$

$$\leq c_8 \| V_0 \|_{H^2} e^{\frac{1}{2} \alpha t}. \hspace{1cm} (2.15)$$

From where, the global existence follows by the usual continuation argument. □

**3. Numerical method and discussion of the method**

Since the partial differential equation (1.6) with (1.7) and (1.8) is non-linear, we cannot solve this initial boundary value problem by the direct finite-difference method. In solving such non-linear equations, iterative methods are usually employed.

We now construct an iterative procedure in the form:

$$\frac{1}{\lambda} \frac{\partial^2 u^{k+1}}{\partial t^2} + \frac{\partial^2 u^{k+1}}{\partial y^2} = \frac{8}{\lambda} + \frac{1}{\lambda} \frac{\partial^2 u^{k+1}}{\partial y^2} \left( 1 + \alpha - \frac{n \lambda_2^2 \left( \frac{\partial u^k}{\partial y} \right)^2}{\left( 1 + \left( \lambda_2 \frac{\partial u^k}{\partial y} \right) \left( \frac{1-n}{\lambda_2^2} \right) \right)^2} \right),$$

\hspace{1cm} (3.1)

where the index $(k)$ indicates the iterative step and $\alpha = (\eta_0 - \eta_\infty) / \eta_\infty$. If the indices $(k)$ are withdrawn (3.1) is consistent with (1.6). Eq. (3.1) is subject to initial conditions

$$u^{k+1} (y, 0) = 0, \quad u^{k+1}_t (y, 0) = 8, \quad 0 \leq y \leq 1,$$

\hspace{1cm} (3.2)

and the boundary conditions

$$u^{k+1} (0, t) = u^{k+1} (1, t) = 0, \quad t > 0.$$ \hspace{1cm} (3.3)

The finite difference method is used to solve a linear partial differential equation for which a mesh imposed over $(y, t)$ plane, such that $y = i \Delta y, (i = 0, 1, \ldots, m)$, and $t = j \Delta t, (j = 0, 1, \ldots)$ and the above equations are discretized
to facilitate the use of a numerical time-marching technique. Manipulating the discrete equations yields the following three term recurrence relation

$$A_i^k (u_{i+1}^{j+1})^{k+1} + B_i^k (u_i^{j+1})^{k+1} + C_i^k (u_{i-1}^{j+1})^{k+1} = D_i^k$$

(3.4)

where \( (u_{i+1}^{j+1})^{k+1}, (u_i^{j+1})^{k+1} \) and \( (u_{i-1}^{j+1})^{k+1} \) are unknown values of the discrete velocity field and \( A_i^k, B_i^k, C_i^k \) and \( D_i^k \) are some complicated functions of quantities at time \( j \Delta t \) and therefore, as explained earlier, known. The results are shown in Fig. 1 which show the effect of both shear thinning (\( n \)) and elasticity (\( \lambda \)) on the velocity field.

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**References**