Abstract—Our aim is to propose tests for (non-)existence of nonlinear relationships between signals, which, after passing a test, can be interpreted as input and output signals of a certain system, if its characteristic is sufficiently smooth. The proposed tests are based on the theoretical results on equality of fractal dimensions of these signals as well as on estimation of fractal dimensions from observations. They are applicable when at least one of these signals has the fractal dimensions strictly larger than one, i.e., it is rough enough. The tests are then verified on simulated data. Their applicability is illustrated by two sets of real data, namely, observations of two financial time series and samples of displacement-force signals in a magneto-hydrological damper.

Index Terms—fractal dimension, pre-identification, quality monitoring

I. INTRODUCTION

Most of the papers on system identification starts with the statement: “Let \( u(t) \) and \( y(t) \) be the input and output of a system ...”. In many cases, especially when a system is man-made, or if there is a theory based on the laws of physics and/or chemistry, we can indeed be sure that a certain input-output relationship between the signals exists. If we are faced with possibly related signals, but existence of a system for which they are input and output is not certain, then only correlation-like methods are widely used. It is, however, well known that the correlation coefficient and the cross-correlation functions fail to detect wide classes of nonlinearities.

Our aim in this paper is to propose a new approach to testing existence of nonlinear input-output (I/O) relationships. The idea is based on estimating and comparing fractal dimensions from samples of input and output (or increments of output) and it is inspired by the example of jet engine measurements, which is shortly described in [1] (page 195).

We refer the reader to [2], [3] and [1] for definitions of fractal dimensions and for the classical methods of their estimation and to [4], [5], [6], [7], [8], [9] for more recent contributions, relevant to this paper.

The paper is organized as follows. In the next section the theoretical background of the proposed method is presented together with the study of its robustness. Then, in Section III, we summarize the well known correlation method and more recent quadratic variation method of estimating fractal dimension of signals. The results of comprehensive simulation studies on testing the accuracy of the correlation method are also presented forming foundations for the empirical version of the test for memoryless systems, which is then verified by simulating a wide variety of memoryless systems. In Section IV the approach is extended to dynamical systems and tested by simulating second and third order systems. Finally, in Section V we describe two applications of the method to real-life data.

II. TESTING INPUT-OUTPUT RELATIONSHIPS FOR MEMORYLESS SYSTEMS

Let us suppose that two signals, \( u(t) \) and \( y(t) \) say, were observed for \( t \in (0, T) \), where \( T > 0 \) is the observation horizon. The Hausdorff dimension of signals, \( u(t), y(t), t \in (0, T) \) is denoted by \( F_{\text{dim}}(u(\cdot)), F_{\text{dim}}(y(\cdot)) \), respectively (see [3] for the definition of the Hausdorff and other related dimensions). An important assumption is that \( F_{\text{dim}}(u(\cdot)) > 1 \). We need this assumption in order to reduce the probability that two unrelated signals have the same fractal dimension (see Remark 7 for more detailed discussion). The example of a signal having \( F_{\text{dim}}(u(\cdot)) > 1 \) is presented in Fig. 3. As one can see such a signal is “rough”.

Remark 1: We shall use the Hausdorff dimension in theoretical considerations. In simulations an estimator of the correlation dimension is used in the version discussed in details in [6] (see [10] Chap. 3.7 for the discussion on using the correlation dimension in estimation of dynamical systems), but we stress that any other reasonably accurate estimator can be used in all the tests proposed below.

Remark 2: It is important for possible practical applications to indicate that fractal signals having fractal dimension greater than 1 must not be “totally fractal” in the sense that not all range of their amplitudes must look as wildly as in the above mentioned figure. In other words, one can obtain a signal with \( F_{\text{dim}}(u(\cdot)) > 1 \) by superimposing a fractal signal with a small amplitude on slowly varying signal (see left panel in Fig. 7).

Our aim is to propose the method for testing the existence of a smooth function, \( G : R \to R \) such that \( y(t) = G(u(t)), t \in (0, T) \). By “smooth” function we mean a function which
is bi-Lipschitz in the set \( \mathcal{U}_T = \{ u(t), t \in [0, T] \} \), i.e.,
\[
c_1|x_1 - x_2| \leq |G(x_1) - G(x_2)| \leq c_2|x_1 - x_2|, \quad x_1, x_2 \in \mathcal{U}_T
\]
for certain constants \( 0 < c_1 \leq c_2 < \infty \). The class of all such functions will be denoted by \( \mathcal{B} \mathcal{L} (\mathcal{U}_T) \).

**Theoretical test for memoryless systems.** According to the general methodology of science, observations of any kind are not able to confirm a theoretical model. For this reason we can only propose the method of its falsification.

**Test 1 - theoretical version**

Step 1 State the hypothesis \( H \): there is no \( G \in \mathcal{B} \mathcal{L} (\mathcal{U}_T) \) for which \( y(t) = G(u(t)), \ t \in (0, T) \).

Step 2 Select (or observe) input signal \( u(.) \) with \( 1 < F_{dim}(u(.)) \leq 2 \). Verify this hypothesis by calculating the fractal dimensions of \( u(.) \) and \( y(.) \) and
\[
\text{reject} \ H \text{ if } F_{dim}(u(.)) = F_{dim}(y(.)) \tag{1}
\]

**Remark 3:** For commonly encountered signals \( u(.) \) and \( y(.) \) with fractal dimensions \( 1 \) it can happen that \( F_{dim}(u(.)) = F_{dim}(y(.)) \), even if \( u(.) \) and \( y(.) \) are not related. It is, however, highly improbable to have \( F_{dim}(u(.)) = F_{dim}(y(.)) \) when \( u(.) \) and \( y(.) \) are random signals with high fractal dimensions and there is no relationship between them.

The next theorem justifies this test (see [3] page 30).

**Theorem 1:** If \( G \in \mathcal{B} \mathcal{L} (\mathcal{U}_T) \) and
\[
y(.) = G(u(.)), \text{ then } F_{dim}(u(.)) = F_{dim}(y(.)).
\]

Note that the roles of \( u(.) \) and \( y(.) \) can be interchanged, since if \( G \) is bi-Lipschitz then it is necessarily one-to-one, hence also bi-Lipschitz with the constants \( c_2^{-1} \leq c_1^{-1} \), respectively.

Assume that the input and/or output of system are measured with additive errors, which are samples of signals \( z_y(.) \) and \( z_u(.) \), say. We also assume that fractal dimensions to these signals exist. Then, we have the following result.

**Theorem 2 (Robustness against measurement errors):** Assume that \( F_{dim}(u(.)) > F_{dim}(z_u(.)) \) and \( F_{dim}(z_y(.)) > F_{dim}(z_y(.)) \). Let for \( t \in (0, T) \) the observable input signal \( U(.) \) and the observable output signal \( Y(.) \) be related by
\[
Y(t) = G(U(t)) + z_y(t), \quad U(t) = u(t) + z_u(t). \tag{2}
\]

If \( G \in \mathcal{B} \mathcal{L} (\mathcal{U}_T) \), then \( F_{dim}(U(.)) = F_{dim}(Y(.)) \).

**Proof.** We start by invoking the following result (see [1], [3] and [11] page 149): for two fractal signals \( x_1(.) \) and \( x_2(.) \), if \( F_{dim}(x_1(.)) \neq F_{dim}(x_2(.)) \) then
\[
F_{dim}(x_1(.) + x_2(.)) = \max \{ F_{dim}(x_1(.)), F_{dim}(x_2(.)) \}, \tag{3}
\]
(see [11] page 150 for the discussion on other sufficient conditions for (3) to hold). This result and the assumption \( F_{dim}(u(.)) > F_{dim}(z_u(.)) \) imply \( F_{dim}(u(.)) = F_{dim}(U(.)) \). Now, it follows from Thm. 1 \( F_{dim}(U(.)) = F_{dim}(G(U(.))). \)

Invoking (3) again we infer from this equality \( F_{dim}(U(.)) = F_{dim}(G(U(.)) + z_y(.)) \), which concludes the proof by recalling the definition of \( Y(.) \) (see (2)).

Thus, we can use Test 1 also in the cases when input and output signals are corrupted by errors, provided that \( F_{dim} \) of errors is smaller than that of input signal.

**Remark 4:** The consequence of this statement is the following. If we expect that \( \max \{ F_{dim}(z_u(.)), F_{dim}(z_y(.)) \} \leq \epsilon_{\text{max}} < 2 \), then we should use input signals with \( F_{dim}(u(.)) > \epsilon_{\text{max}} \). The largest possible \( F_{dim} \) of any univariate signal equals 2. Thus, if we are able to apply an input signal having \( F_{dim}(u(.)) = 2 \), we are guarded against any error signal with \( F_{dim}(z_u(.)) < 2 \) and \( F_{dim}(z_y(.)) < 2 \).

**III. Empirical test and accuracy of estimating fractal dimensions by the correlation method**

A large number of methods for estimating the fractal dimension from samples were developed, the box-counting being the most popular. Although the proposed approach does not rely on a particular method of estimating the fractal dimension, it is clearly desirable to use an accurate, reliable and easily implementable one. For these reasons the correlation method (CM) in the chord version (see [2]) was selected here as a tool for estimating fractal dimensions in the whole range of \( 1 \leq F_{dim} \leq 2 \). If it occurs that \( 1 \leq F_{dim} \leq 1.25 \), then it is desirable to repeat the estimation process using the quadratic variation method (QVM) ([8], [9]). QVM has smaller mean squared error (MSE) than CM for \( 1 \leq F_{dim} \leq 1.25 \) (see [8]).

One can also use the Grassberg-Proccacia version of estimating the correlation dimension (see, e.g., [10]). We refer the reader also to [12] for interesting results on the local correlation dimension in the multiparameter case, its relationships to the Hausdorff dimension and on estimating local correlation dimension of fractional Brownian motions.

Below we briefly describe CM and QVM in the form, which were used in our simulation experiments.

**Correlation method of estimating the fractal dimension.** Let \( s(t) \) denote a stationary stochastic process with a finite variance and write \( \gamma(t) = \text{cov}(s(t), s(0)) \) for its covariance function at lag \( t \). As in [6] let us assume a relatively simple model for the covariance function
\[
\gamma(t) = \gamma(0) - c|t|^{2H} + o(|t|), \quad t \to 0 \tag{4}
\]
for a certain \( c > 0 \). In the above \( 0 < H \leq 1 \) characterizes smoothness of trajectories of \( s(t) \) (for a Gaussian process \( H = 1 \) if \( s(.) \) is differentiable). For a large class of processes \( H \) is related to the fractal dimension of \( s \) as follows
\[
F_{dim}(s) = 2 - H. \tag{5}
\]

**Remark 5:** Equality (5) holds for a wide subset of second-order stochastic processes, but it is not valid for every second-order process. We refer the reader to the classical monograph [13] (Chapter 8) and to [9], [14] and [11] for conditions, which imply (5). The above comment is relevant also to QVM. Following [6] we shall describe the correlation method (CM). Let \( s_i \) denote equidistant samples of process \( s(\tau) \), \( \tau > 0 \), \( i = 1, 2, \ldots, n \). Select the number of lags, \( 1 < M < n \) say, which should be a fraction of \( n \). Define the variogram
\[
g_j = (n-j)^{-1} \sum_{i=1}^{n-j} (s_{i+j} - s_i)^2, \tag{6}
\]
\( j = 1, 2, \ldots, M \), which estimates \( 2(\gamma(0) - \gamma(j \tau)) \). According to (4), for \( |t| \) small enough, \( \log(\gamma(0) - \gamma(t)) = 2H \log(|t|) + \)
estimation accuracy was repeated: proposed in [17], which simultaneously tests for normality density of errors in estimating the fractal dimension (the reader runs. The problem of precisely estimating the density of errors

1

\[ \text{EMPIRICAL DISPERSION OF ESTIMATING} \]

\[ \text{F}_{\text{dim}}(s) \]

\[ \text{QVM outperforms CM in the MSE sense also for a moderate sample size, the same simulated samples with known} \]

\[ \text{QVM estimated by CM is in (or close to) the interval (1,1.25). To check whether} \]

\[ \text{QVM outperforms CM in the MSE sense also for a moderate sample size, provided that estimated} \]

\[ \text{QVM} \]

\[ \text{test lead to the following.} \]

\[ \text{Test step 1} \]

\[ \text{ES} \]

\[ \text{EMPIRICAL DISPERSION OF ESTIMATING} \]

\[ \text{F}_{\text{dim}} \]

<table>
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<th>1.1</th>
<th>1.05</th>
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<td>45</td>
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\[ \text{EMPIRICAL DISPERSION OF ESTIMATING} \]

\[ \text{F}_{\text{dim}} \]

\[ \text{QVM (LOWER TABLE) V.S.} \]

\[ \text{F}_{\text{dim}} \]

\[ \text{EMPIRICAL DISPERSION OF ESTIMATING} \]

\[ \text{F}_{\text{dim}} \]

\[ \text{AND BY QVM (LOWER TABLE) V.S.} \]

\[ \text{F}_{\text{dim}} \]

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</tr>
</tbody>
</table>
is selected as 0.001, 0.01, 0.05). In the tables of the standard normal distribution find the critical level \( N_{cr}(\alpha) \) such that \( P\{|N| > N_{cr}(\alpha)\} < \alpha \), where \( N \) is the standardized Gaussian random variable. Estimate fractal dimensions of \( u(.) \) and \( y(.) \) by QVM or the correlation method and

\[
\text{reject } G \text{ if } \frac{|\hat{F}_{dim}(u(.)) - \hat{F}_{dim}(y(.))|}{\sqrt{2\sigma_d}} < N_{cr}(\alpha). \quad (9)
\]

In practice, we accept existence of I/O relationship if \( G \) is rejected. In (9) it was assumed that errors in estimating \( \hat{F}_{dim}(u(.)) \) and \( \hat{F}_{dim}(y(.)) \) are independent. Selecting \( \alpha \approx 0.003 \) in Step 3 we find \( N_{cr} = 3 \), which was used in the simulations reported below.

Remark 6: Test (9) is adopted to our purposes, version of the classical procedure for testing the equality of two means of normal distributions with the same dispersion \( \sigma_d \), which is assumed to be known. This test is uniformly most powerful in the class of all unbiased tests (see, e.g., [19]). Its power function equals to \( \Phi \left( \theta/(\sqrt{2\sigma_d}) + N_{cr}(\alpha) \right) - \Phi \left( \theta/(\sqrt{2\sigma_d}) - N_{cr}(\alpha) \right) \), where \( \Phi(.) \) is the cumulative distribution function of the standard normal distribution, while \( \theta = \hat{F}_{dim}(u(.)) - \hat{F}_{dim}(y(.)) \). Setting \( \theta = 0 \) in the above expression we obtain the size of this test.

Tab. I and the simulations reported below indicate that \( \sigma_d \) in the interval between 0.04 and 0.05 is a good approximation of the real dispersion in estimating \( \hat{F}_{dim} \)'s. This justifies treating \( \sigma_d \) as known in Test 2. One may wish to estimate \( \sigma_d \) from one sufficiently long sample, by dividing it into subsamples. In such a case it is easy to modify Test 2 by replacing \( \sigma_d \) in (9) by its estimate \( \hat{\sigma}_d \) and to establish \( N_{cr}(\alpha) \) from the tables of t-Student distribution.

Remark 7: An important question concerning Test 1 and Test 2 is: how unlikely is that the input and output have the same fractal dimension but they are not related? Below we provide partial answers

1) It is highly probable to meet two unrelated smooth signals having \( F_{dim} = 1 \). Our assumption that \( F_{dim}(u(.)) > 1 \) reduces this probability, since such signals change wildly. However, having a finite resolution of distinguishing \( F_{dim}(u(.)) \) and \( F_{dim}(y(.)) \) we can not be sure that inequality (9) holds for unrelated signals. The situation is similar to that in the correlation analysis. Namely, quite unrelated observations can have a high correlation coefficient. Applying Test 2 we are in somewhat better position, namely, if we are able to apply input signals with changing \( F_{dim}(u(.)) \) and estimated \( F_{dim}(y(.)) \) follows these changes then we can be more convinced that there is a system for which \( u(.) \) and \( y(.) \) are input and output, respectively.

2) Let us assume that the covariance functions of \( u(.) \) and \( y(.) \) are of the form (4) each with parameter \( H \) drawn at random from the uniform distribution in \([0,1]\). Then \( F_{dim}(u(.)) \neq F_{dim}(y(.)) \) almost surely and Test 1 rejects hypothesis \( G \). Nevertheless, if \( F_{dim}(u(.)) \) and \( F_{dim}(y(.)) \) are estimated from observations, then there is nonzero probability that Test 2 fails to reject \( G \), suggesting that \( u(.) \) and \( y(.) \) are related by a smooth transformation. This probability can be decreased by increasing significance level \( 0 < \alpha < 1 \).

In Fig. 3 the input and output signal of the following static nonlinearity \( y(t) = \sqrt{\cos(5 \cdot u(t)) + 2} \) is shown. By looking at these plots it is hard to decide whether these signals are related or not. However, their fractal dimensions are 1.52 for the input signal and 1.64 for the output one, indicating (according to Test 2) that we have to reject \( G \), suggesting that \( u(.) \) and \( y(.) \) are related by a smooth transformation. Simulations were performed according to the following scheme:

**Scheme of simulations**

1) The input signal with a specified fractal dimension (usually 1.5 or 1.75) was generated using the algorithm Fractional Brownian Motion described in [15]. Then, its fractal dimension was estimated.

2) This input signal is fed to the system \( y(t) = G(u(t)) \) with a chosen static characteristic \( G(.) \) and output signal is sampled.

3) The samples of output are then used for evaluating its fractal dimension, which is compared with the fractal dimension of the input signal.

4) For comparison, the empirical correlation coefficient \( \hat{\rho} \) between input and output samples is calculated in the classical way, i.e., \( \hat{\rho} = \frac{n^{-1} \sum_{i=1}^{n} (y_i - \bar{y})(u_i - \bar{u})}{\sqrt{n^{-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \cdot n^{-1} \sum_{i=1}^{n} (u_i - \bar{u})^2}} \), where \( \hat{\sigma}_n^2 = n^{-1} \sum_{i=1}^{n} (u_i - \bar{u})^2 \), \( \bar{u} = n^{-1} \sum_{i=1}^{n} u_i \) and analogously for \( \hat{\sigma}_n^2 \) and \( \bar{y} \).

In Tab. II the results of applying the above scheme are summarized for different functions \( G(.) \). As one can see from this table, in all the cases the differences \( |F_{dim}(u(.)) - F_{dim}(y(.))| \) are less than \( 3\sqrt{2}\sigma_d \approx 0.2 \). Thus, according to Test 2, in all the cases displayed in Tab. II we reject the hypothesis on

![Fig. 2. MSE of estimating \( \hat{F}_{dim} \) versus "true"value of \( F_{dim} \). Each point was obtained by averaging 500 simulation runs of the length \( n = 1000 \) each. Solid line – \( F_{dim} \) estimated by QVM. Dashed line – \( F_{dim} \) obtained by CM.](image1)

![Fig. 3. Left panel – samples of input signal, right panel – samples of output signal of the system \( y(t) = \sqrt{\cos(5 \cdot u(t)) + 2} \).](image2)
nonexistence of a smooth function relating \( u(\cdot) \) and \( y(\cdot) \).

On the other hand, the last column indicates that Test 2 is able to detect an input-output dependence both in the cases of low correlation and when the correlation is large. Tab. II contains estimates of \( F_{\dim} \) obtained from one sample, since in the practical use of Test 2 the one sample behavior is of importance. However, one may wish to know what is the variability of the estimates. Tab. III presents the results of averaging of 100 repetitions of estimates \( F_{\dim}(u(\cdot)) \) and \( F_{\dim}(y(\cdot)) \), where \( y(t) = \sin(2\pi u(t)) \), \( n = 1000 \). \( u(t) \) was normalized to \([-1, 1]\) before applying it to the simulated system. QVM was applied in the first two rows of Tab. III, while in the rest CM was used. As one can notice, the empirical means (denoted as \( \mathcal{A}(\cdot) \)) of \( F_{\dim}(u(\cdot)) \) and \( F_{\dim}(y(\cdot)) \) are fairly close to the expected theoretical values for a wide range of \( F_{\dim}(u(\cdot)) \) applied to the system. Also the empirical dispersions (denoted as \( \mathcal{V}(\cdot) \)) of \( F_{\dim}(u(\cdot)) \) and \( F_{\dim}(y(\cdot)) \) are in a good agreement with the above discussion on the accuracy of \( F_{\dim}(\cdot) \) estimates. During the reported 100 simulations Test 2 was applied each time and each time Test 2 provided the correct decision (in each row of Tab. III).

Tab. IV contains the results of applying Test 2 to the system \( y(t) = \mathrm{sign}(u(t)) \), where \( \mathrm{sign}(u) = 1 \) if \( u > 0 \) and \( \mathrm{sign}(u) = -1 \) otherwise. This system is not bi-Lipschitz and our aim was to check whether Test 2 does not reject hypothesis \( \mathcal{G} \) sufficiently often when there is no a smooth function relating \( u(t) \) and \( y(t) \). In the last column of Tab. IV No \( \mathcal{G} \) denotes the number of cases when Test 2 did not reject hypothesis \( \mathcal{G} \) in 100 trials with \( \alpha = 0.003 \). The test seems to be sufficiently sensitive to nonexistence of a smooth input-output transformation.

In order to verify robustness of Test 2 a fractal error signal was added to the output of the system. \( n = 5000 \) samples of the input and output were taken. The results are summarized in Tab. V. In all the cases reported in this table Test 2 rejects the hypothesis on nonexistence of a smooth function, which links input and output signals.

### IV. Testing existence of I/O relationship for dynamical systems

If the fractal dimensions of \( u(\cdot) \) and \( y(\cdot) \) are essentially different, then - according to Test 2 - we claim that a smooth, i.e., bi-Lipschitz nontrivial function linking them does not exist. This, however, does not mean that there are no other relationships between these signals, but we should look for them between transformations which are not \( F_{\dim} \) preserving. A large class of such transformations is generated by stable systems described by ordinary differential equations. Transformations of this type change the fractal dimension of input signals since the integration increases smoothness of signals. Thus, selecting a smoother signal as a possible output of a dynamical system one can try to test whether it is related to the less smooth signal, which is treated as the input.

Theoretical background and empirical version of the test. Consider a system described by

\[
x'(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad y(t) = g(x(t)),
\]

where \( x(t) \in \mathbb{R}^s \) is the state system at time \( t \) and \( x_0 \) denotes an unknown initial condition. \( y(t) \in \mathbb{R} \) is the output and \( u(t) \in \mathbb{R} \) is the input signal. In (10) \( g : \mathbb{R}^s \to \mathbb{R} \) denotes the output function of the system. Both \( f \) and \( g \) are unknown, but we have to impose certain regularity conditions on them. \( D1) \) \( f : \mathbb{R}^s \times \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous with respect to the first \( s \) arguments and \( f(x, .) \in \mathcal{BL} (\mathcal{U}^s) \) for every \( x \in \mathbb{R}^s \). \( D2) \) \( g \) is assumed to have continuous and non-vanishing derivatives with respect to each state variable and

\[
(\text{grad}_x g(x))^T \cdot f(x, u) \neq 0, \quad \text{a.e. in} \quad \mathbb{R}^s \times \mathbb{R}.
\]

Clearly, if \( g \) and \( f \) are unknown it is not possible to verify condition (11). On the other hand, if this condition does not hold, then there exists a set \( \Lambda \subset \mathbb{R}^s \times \mathbb{R} \) with nonzero Lebesgue measure and such that \( dy(t)/dt = (\text{grad}_x g(x))^T \cdot f(x(t), u(t)) = 0 \) for all \((x(t), u(t)) \in \Lambda \) and \( t \) denotes the transposition. In other words, violation of (11) implies existence of a “fat” set \( \Lambda \) such that output is constant (does not respond) if input and state variables are in this set.

Theorem 3: If \( u(\cdot) \) and \( y(\cdot) \) are related by (10) and the above assumptions imposed on \( f \) and \( g \) hold, then

\[
F_{\dim}(y'(\cdot)) = F_{\dim}(u(\cdot)),
\]

where fractral dimensions of \( u(t) \) and \( y'(t) \) defined \( dy(t)/dt \) are measured on \((0, T)\).

### TABLE II

**Test 2 applied to memoryless systems.** \( F_{\dim}(u(\cdot)) = 1.5 \).

<table>
<thead>
<tr>
<th>System ( y(t) = )</th>
<th>( F_{\dim}(u(\cdot)) )</th>
<th>( F_{\dim}(y(\cdot)) )</th>
<th>( \rho )</th>
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</thead>
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<tr>
<td>( u^2(t) )</td>
<td>1.494</td>
<td>1.55</td>
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</tr>
<tr>
<td>( u^3(t) )</td>
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<td>1.56</td>
<td>0.86</td>
</tr>
<tr>
<td>( \sqrt{</td>
<td>u(t)</td>
<td>} )</td>
<td>1.498</td>
</tr>
<tr>
<td>( \arcsin(u^2(t)) )</td>
<td>1.48</td>
<td>1.538</td>
<td>0.21</td>
</tr>
<tr>
<td>( \sin(2u(t)) )</td>
<td>1.51</td>
<td>1.61</td>
<td>0.09</td>
</tr>
<tr>
<td>( \sqrt{\cos(5u(t)) + 2} )</td>
<td>1.48</td>
<td>1.57</td>
<td>0.03</td>
</tr>
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</table>

### TABLE III

**Test 2 applied to the system** \( y(t) = \sin(2\pi u(t)) \).

<table>
<thead>
<tr>
<th>( F_{\dim}(u(\cdot)) )</th>
<th>( \mathcal{A}(u) )</th>
<th>( \mathcal{V}(u) )</th>
<th>( \mathcal{A}(\mathcal{y}) )</th>
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<td>1.57</td>
<td>0.037</td>
<td>1.59</td>
<td>0.041</td>
</tr>
<tr>
<td>1.7</td>
<td>1.67</td>
<td>0.044</td>
<td>1.69</td>
<td>0.044</td>
</tr>
</tbody>
</table>

### TABLE IV

**Test 2 applied to the system** \( y(t) = \mathrm{sign}(u(t)) \).

<table>
<thead>
<tr>
<th>( F_{\dim}(u(\cdot)) )</th>
<th>( \mathcal{A}(u) )</th>
<th>( \mathcal{A}(F_{\dim}(\mathcal{y})) )</th>
<th>No ( \mathcal{G} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>1.34</td>
<td>1.63</td>
<td>96</td>
</tr>
<tr>
<td>1.4</td>
<td>1.39</td>
<td>1.68</td>
<td>98</td>
</tr>
<tr>
<td>1.5</td>
<td>1.48</td>
<td>1.73</td>
<td>93</td>
</tr>
<tr>
<td>1.6</td>
<td>1.56</td>
<td>1.76</td>
<td>91</td>
</tr>
</tbody>
</table>
and to invoke Thm. 1. In comparing the fractal dimensions of the left and the right hand sides of (13) one should observe that $F_{dim}(x_i) < F_{dim}(u_i)$. This follows from expressing (10) as $x(t) = x(0) + \int_0^t f(x(\tau), u(\tau)) d\tau$, since the integration reduces the fractal dimension of signals (see [1]).

Remark 8: Note that we do not need to specify $dim(x(t))$ in the above theorem. On the other hand, this theorem does not provide direct hints on $dim(x(t))$. However, if we have samples from signals $x_1(t), x_2(t), \ldots x_m(t)$, say, which are “suspected” to be state variables of a hypothetical system, then we can compare $F_{dim}(u_i)$ with $F_{dim}(x_1(i)), F_{dim}(x_2(i)), \ldots F_{dim}(x_m(i))$ and select those, which are close to each other as a subset of state variables. Validity of this approach follows from Thm. 3, applying it for $g(x(t)) = x_j(t)$, $j = 1, 2, \ldots, m$. Following the reasoning used in Theorem 2 we can be sure that also additive fractal errors $z(t)$ in $\frac{dx(t)}{dt} = f(x(t), u(t)) + z(t)$ do not change our main equality (12), provided that $F_{dim}(u_i) > F_{dim}(z(t))$.

Now, we are prepared to present the empirical test for testing Hypothesis $D$: there does not exist initial conditions $x(0)$ and a dynamical system described by (10) with $D1), D2)$ for which given signals $u(.)$ and $y(.)$ are the input and the output, respectively.

**Test 3 – Empirical test for dynamical systems**

Step 1 State the hypothesis $D$. Step 2 Select or observe input $u(.)$ with $1 < F_{dim}(u(.)).$ Collect equidistant samples $u_i$ and $y_i, i = 1, 2, \ldots n$. Step 3 Form the differences $d_i = y_{i+1} - y_i, i = 1, \ldots, (n-1)$. Step 4 Estimate their fractal dimensions by CM or QVM and reject $D$ if $|\hat{F}_{dim}(u.) - \hat{F}_{dim}(d.)|/\sqrt{2\sigma_d} < N_{cr}(\alpha), (14)$

where $\alpha$ and $N_{cr}(\alpha)$ are selected exactly in the same way as in Test 2. In (14) the operation $\hat{F}_{dim}(d.)$ means calculating the empirical fractal dimension from $d_i, i = 1, 2, \ldots (n-1)$.

If $D$ is rejected, then in practice we accept existence of a dynamical system with input $u(.)$ and output $y(.)$. In the simulations reported below we take $N_{cr} = 3$ for the r.h.s. of (14). The intuitive justification of this test is the same as in Remark 3. The simulations studies reported in this section were performed according to the scheme similar to that from Section III with obvious modifications. The differential equations were solved by the fourth order Runge-Kutta method.

**TABLE VI**

<table>
<thead>
<tr>
<th>System</th>
<th>$F_{dim}(u(1))$</th>
<th>$F_{dim}(y(1))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(t) = u(t)$</td>
<td>1.73</td>
<td>1.84</td>
</tr>
<tr>
<td>$y(t) = \sqrt{</td>
<td>u(t)</td>
<td>}$</td>
</tr>
<tr>
<td>$y(t) = \sin(2\pi u(t))$</td>
<td>1.69</td>
<td>1.79</td>
</tr>
</tbody>
</table>

One sample of the state trajectory is shown in Fig. 4 (left panel). The observations were made in the interval $[0, 100]$ with the step size 0.005. The empirical mean from 100 repetitions of $F_{dim}(u(.)$) is 1.48, while the corresponding empirical mean calculated from $F_{dim}$ of differenced output equals 1.41. This provides arguments on trying to construct a model of dynamical system linking $u(.)$ and $y(.)$. Let us note that we have no indications concerning the order of the system. Test 3 applied to these 100 repeated samples correctly rejected hypothesis $D$ 97 times.

3) Linear second order system observed with fractal noise: The same system (15), (16) was simulated, but the fractal noise with $F_{dim}$ equal to 1.25 was added to the first equation and the input signal had $F_{dim}(u(.)$) = 1.75. The state trajectory is shown in Fig. 4 (right panel). For the observations in the interval $[0, 25]$ with the step size 0.01 the empirical $F_{dim}$ of input signal, averaged from 100 repetitions, was equal to 1.72. The output was sampled with the step size 0.001, corrupted by fractal noise with the dimension 1.25 and then differenced. The empirical mean from 100 samples of $F_{dim}$ of this differenced signal was equal to 1.68. Even in the presence of noise $D$ was correctly rejected 95 times in 100 repetitions of the test.

4) The Lorentz nonlinear system perturbed by fractal input: Consider the Lorentz system (see, e.g., [2]), but modified by

\begin{align*}
  x'(t) &= \begin{bmatrix} -3 & 0 \\ 5 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 10 \\ 0 \end{bmatrix} u(t) \\
  y(t) &= [0, 1] \cdot x(t), \quad x(0) = [10, 1]^T
\end{align*}

TABLE V

**Test 2 applied to output signals corrupted by noise $z(t)$ with $F_{dim}(z(1)) = 1.25, F_{dim}(u(1)) = 1.75$.

<table>
<thead>
<tr>
<th>System</th>
<th>$F_{dim}(u(1))$</th>
<th>$F_{dim}(y(1))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(t) = u(t)$</td>
<td>1.73</td>
<td>1.84</td>
</tr>
<tr>
<td>$y(t) = \sqrt{</td>
<td>u(t)</td>
<td>}$</td>
</tr>
<tr>
<td>$y(t) = \sin(2\pi u(t))$</td>
<td>1.69</td>
<td>1.79</td>
</tr>
</tbody>
</table>

Fig. 4. Trajectory of state of system (15), (16) driven by input signal with $F_{dim}(u(.) = 1.5$ (left panel) and $F_{dim}(u(.) = 1.75$ (right panel).
The null hypothesis. Thus, we suspect a certain relationships between them. This is confirmed by comparing the fractal dimensions of our data. Namely, for Shares 1 we obtain $\hat{F}_{\text{dim}} = 1.5327$ and for Shares 2 $\hat{F}_{\text{dim}} = 1.5439$. The coincidence is evident, but it is not evident what is an appropriate mathematical model for the dependence between prices of Shares 1 and Shares 2. We know, however, that it is not hopeless to look for a smooth nonlinear model between prices of these shares.

Application in quality monitoring. Our aim in this subsection is to illustrate how empirical measurements of fractal dimensions can be used for monitoring quality characteristics of industrial processes and devices.

Firstly, note that CM can easily be modified in such a way that last $n$ observations in a sliding window are taken into account for estimating $F_{\text{dim}}$. Additionally, the least squares fit in Step 3 can be realized in the on-line mode.

Suppose that an input signal with a fixed fractal dimension is fed to a process (or device). If this process runs properly, then the fractal dimension of its output signal also has a fixed fractal dimension. If the process runs out of control, e.g., as a result of failure of sensors or actuators or in the process itself, then one can expect changes in the fractal dimension of the output signal. Thus, tracking the fractal dimensions of the input and output signals one can detect changes in quality characteristics.

The above idea is illustrated by the following experiment performed on a magnetorheological (MR) damper, which can be used in cars to suppress shocks. A MR damper is a highly nonlinear device and for its precise mathematical description it is necessary to use a set of nonlinear differential equations (see [20]). Such a damper uses MR fluid to produce damping. MR fluids can change from a free-flowing, viscous fluid to a semi–solid when exposed to magnetic field, which is generated by suitably formed changes of a current. Namely, a pulsating current is supplied to the electromagnetic in the MR damper that is proportional to a displacement current (DC) input voltage in the range from 0 to 3 V.

The experiment was performed as follows: A hydraulic actuator generated a sinusoidal displacement (input signal), which was superimposed on a small, frequently changing signal, having a fractal structure (see left panel in Fig. 7). Its $\hat{F}_{\text{dim}}$ equals to 1.59. The input signal has the form

\begin{align}
\dot{x}(t) &= -5(x(t) - y(t)) + 100u(t) \\
\dot{y}(t) &= -x(t)z(t) + 26.5x(t) - y(t) \\
\dot{z}(t) &= x(t)y(t) - z(t).
\end{align}

Adding the input signal to the first equation

\[ x'(t) = -5(x(t) - y(t)) + 100u(t) \]  \hspace{1cm} (17)

\[ y'(t) = -x(t)z(t) + 26.5x(t) - y(t) \]  \hspace{1cm} (18)

\[ z'(t) = x(t)y(t) - z(t). \]  \hspace{1cm} (19)

\[ x(0) = z(0) = 0, \ y(0) = 1. \]  \hspace{1cm} (17)-(19)

In (17)-(19) we changed the notation in order to keep the traditional symbols used in describing the Lorentz system. In our simulations we took $y(t)$ to be the system output, while the theoretical fractal dimension of the input signals was 1.75. In Fig. 5 differenced output samples are plotted. One sample estimates of fractal dimensions of input and differenced output are 1.72 and 1.799, respectively. Thus, we reject the hypothesis on nonexistence of a dynamical system linking $u(.)$ and $y(.)$.

V. APPLICATIONS

In this section we discuss two applications of the proposed approach. The first one illustrates its applicability in the step, which precedes a system identification. The second application elucidates the use of the above tests in quality monitoring.

Testing existence of a relationship between prices of shares

Our aim in this section is to apply the proposed approach to real data selected from the Polish stock market. Prices of shares of two companies were collected during the consecutive $n = 531$ sessions (see Fig. 6). Later we call them “Shares 1” and “Shares 2”, since the name of these companies is irrelevant to our considerations. What is important is that they represent quite different branches of industry, which are, in a broad sense, unrelated. We are interested in the following question: is it possible to predict prices of Shares 1 knowing prices of Shares 2 (or conversely). Firstly, we apply the classical approach based on the empirical correlation coefficient $r$. In this case we have $r = 0.104$. Our null hypothesis is $\rho = 0$.

The standard test statistic for this hypothesis has the form:

\[ t_s = \frac{r}{\sqrt{1-r^2}} \cdot \sqrt{n-2} = 2.408 \]

and t-Student PValue = 0.106, so for the significance level 0.01 we do not reject the null hypothesis. Thus, we suspect that there is no strong linear dependence between prices of Shares 1 and Shares 2. On the other hand, it is well known that in the Polish stock market, which was not fully developed in the middle nineties, most of the shares prices were usually simultaneously growing or falling. Hence, one can expect...
A \cdot \sin(2\pi \omega t) + \text{fractalsignal}, \text{ where } A \text{ is the displacement amplitude (in mm) and } \omega \text{ is the frequency (in Hz). In all the experiments } \omega \text{ was fixed to 1 Hz. The second possible input is the DC voltage, but here it was kept fixed with only one exception, when its change was used to evoke the change of the output signal fractal dimension, which simulates its undesirable behavior. The response of the MR damper is the measured force which is also periodic. Two periods of a typical MR damper response are shown in Fig. 7 (right panel). Magnification of a part of damper's output (see the box in Fig. 7) reveals its fractal nature (see Fig. 8 left panel). Its empirical fractal dimension, estimated from several periods, equals to 1.54. Comparing it with the fractal dimension of input signal 1.59 we conclude that } D \text{ is rejected, suggesting existence an input-output relationship.}

Then, malfunctioning of the damper was evoked by changing DC voltage from 1V to 3V, which simulated freezing of an oil in a damper and almost broke the connection between its input and output. The aim of the subsequent measurements was to check whether this malfunctioning can be detected by comparing fractal dimensions. After collecting more samples from the output signal its fractal dimension was evaluated again, providing } F_{\text{dim}} = 1.33. \text{ The difference between this dimension and fractal dimension of input signal is larger than } 3\sqrt{2} \Delta F \approx 0.2. \text{ Thus, } D \text{ is not rejected, what indicates an essential change of the mode of functioning in the monitored process. The comparison between } F_{\text{dim}} = 1.33 \text{ and the fractal dimension of output signal in the normal operating conditions (equal to 1.54) confirms this observation. Such a change usually means that a certain kind of tuning or repairing is necessary.}

VI. CONCLUDING REMARKS

The tests for verifying the hypothesis on (non-)existence of a nonlinear relationship between two signals are proposed. These tests are based on estimating the fractal dimensions of these signals and they are robust if errors have smaller fractal dimensions than that of the input signal.

The proposed approach can be a starting point for various generalizations, e.g., to distributed-parameter systems. Here we mention only one extension, namely to multiple-input, multiple-output systems. The generalization of the tests to single-input, multiple-output systems is immediate, since it suffices to estimate the fractal dimension of the input signal and then to compare it with the fractal dimension of each output signal. The generalization to multiple input systems is more complex. To explain the reason consider two input signals } u_1(t), u_2(t) \text{ and a smooth function } y(t) = G(u_1(t), u_2(t)), t \in (0, T). \text{ Equation (3) suggests that also in a general case } F_{\text{dim}}(y(t)) = \max(F_{\text{dim}}(u_1(.)), F_{\text{dim}}(u_2(\cdot))) \text{ and the influence of an input with smaller fractal dimension is suppressed. This difficulty can be avoided by first selecting } u_1 \text{ with } F_{\text{dim}}(u_1(.)) > F_{\text{dim}}(u_2(.)) \text{ and comparing } F_{\text{dim}}(u_1(.)) \text{ with } F_{\text{dim}}(y(\cdot)), \text{ then switching to } F_{\text{dim}}(u_1(.)) < F_{\text{dim}}(u_2(.)) \text{ and performing the test for the second parts of the signals. The test for dynamical systems can be generalized analogously.}

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REFERENCES

Ewaryst Rafajłowicz received the M.Sc., Ph.D. and D.Sc. degrees in control theory from the Wrocław University of Technology, Poland, in 1977, 1979 and 1987, respectively. He published about 100 papers on system identification, experiment design and nonparametric regression estimation.