A survey on the integrated design of chaotic oscillators

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Multi-scroll
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A B S T R A C T

We present a review on the electronic design of chaotic oscillators. Multi-scroll chaotic oscillators are listed according to their electronic implementations. A 3-scrolls oscillator is analyzed from its mathematical description, and designed with current-feedback operational amplifiers. Finally, we list the integrated realizations, and discuss key points for future research on the design of multi-scroll chaotic oscillators.

1. Introduction

Electronic implementations of chaotic oscillators are relatively new and in some cases there is still mathematical research to be done. Most of the potential applications exploit the deterministic nature of chaotic signals [1,2], while others are related to the random behavior [3]. For example, the synchronization and transmission of encrypted information [4,5], show advantages based on the complexity of chaotic signals, which turns them suitable for message encryption.

It was since nineties, that much research has been oriented to implement chaotic oscillators with electronic devices. A remarkable design of a continuous chaotic oscillator is the case of the two-scroll Chua’s circuit [6–10], which generates a rich variety of chaotic dynamics in a relatively simple electronic implementation. Recently, many new chaotic oscillators have been proposed [11–13]. In all cases a nonlinear part is required to obtain more equilibrium points than the origin and eventually obtain attractive regions. Some approaches use polynomial forms, sinusoidal functions, delay based functions, or piecewise-linear (PWL) functions [3,14]. In particular, the PWL-based implementations show numerous research works because of their capacity to obtain at least partial analytical solutions (for the linear segments), while the obtention of such solutions for other nonlinearities are hard to reach [3]. This motivates the development of new PWL-based chaotic oscillators showing more scrolls (multi-scroll attractors), and more directions [5,11,12,15,16].

Currently, a collection of major developments of chaotic systems can be found in [3], where several researchers summarize key guidelines on modeling, simulation, control synchronization, and applications of chaotic oscillators. Besides, although chaotic oscillators have been intensively studied since a few decades ago, their implementation using electronic devices has been considered only in a very few works. Henceforth, this survey describes the electronic implementation of theoretical chaotic oscillators using discrete active devices and integrated circuit realizations.

2. Electronic implementations of chaotic oscillators

The majority of electronic implementations of chaotic oscillators consist of discrete active devices, mainly operational amplifiers (opamps) [16], whereas other designs are implemented with mixed-mode active devices [17,18], such as the current-feedback opamp (CFOA [8,19], e.g. the Analog Devices AD844), operational transconductance amplifier (OTA) [13].
positive-type second-generation current conveyor (CCII+) [4,20], and unity-gain cells (UGCs) [10–12]. Relevant chaotic continuous-time oscillator designs are summarized in Table 1, where the following issues can be identified:

- The majority of those designs have been obtained by generalizing previous others, for example, the most preferred dynamic system is the Chua’s chaotic oscillator.
- PWL-based oscillators are preferred because of the relatively simple mathematical description, dynamical analysis and circuit synthesis.
- Implementations using integrated circuit technology are few in all cases and multi-scroll designs scarce.

As one can infer, many multi-scroll chaotic oscillators are derived from the double-scroll Chua’s chaotic circuit. Basically, the nonlinear resistor known as Chua’s diode [8] is augmented, as shown in Fig. 1, to have more break points while combining the slopes [19,21], to generate even or odd number of scrolls. Other PWL functions can be added to implement chaotic oscillators in more than one direction (1D) [5,16], e.g. 2D [22], 3D [23], and 4D [24].

The electronic implementations of those chaotic oscillators based on PWL functions use different kinds of active devices; the most used one is the traditional opamp [2]. The CFOA AD844 from Analog Devices is also quite useful, and it can be used as CCII+ [20], as already demonstrated in [3,4]. The OTA has been used to implement double-scroll chaotic oscillators in [6,7]. Those three active devices have been summarized in [13] to show the generic topologies for realizing the PWL saturated function series. From the integrated circuit design point of view, the unity-gain cells (UGCs) can be interconnected or superimposed to derive mixed-mode active devices like CCIIIs and CFOAs [18]. Besides, the UGCs not only can be implemented with the CFOA AD844 [10,19], but also with integrated circuit technology [11,12].

As an example, we describe the electronic implementation of a multi-scroll Chua’s chaotic oscillator using the CFOA AD844. We show the values of the Lyapunov exponents computed by applying the method described in [14,25], and its bifurcation diagram to highlight the operation in chaotic regime.

The design starts by describing the multi-scroll Chua’s chaotic oscillator by (1), where \( \alpha = 10 \), \( \beta = 11.5 \) and the PWL function is represented by (2).

\[
\begin{align*}
\dot{x} &= \alpha(y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y, \\
f(x) &= m_{2n+1}x + \frac{1}{2}\sum_{i=0}^{2n-1}(m_i - m_{i+1})(|x + b_i| - |x - b_i|),
\end{align*}
\]

where \( q = 1 \) to generate \((2n)\) even scrolls, and \( q = 2 \) to generate \((2n - 1)\) odd scrolls. Vector \( m \) represents the slopes of the PWL function and vector \( b \) denotes the break points [19,21]. To generate even scrolls \( m \) is taken as \( [m_0 \ldots m_d] = [-4.416, \ldots] \).

### Table 1

<table>
<thead>
<tr>
<th>System Name</th>
<th>Attractor type</th>
<th>Nonlinear function based on</th>
<th>Proposed circuit</th>
<th>Integrated design</th>
<th>Author Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1D-PWL</td>
<td>Sinusoidal</td>
<td>Opamp</td>
<td></td>
<td>Suykens [27] 2000</td>
</tr>
<tr>
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<td>1D-PWL</td>
<td>Sinusoidal</td>
<td>Trigonometric function generator</td>
<td></td>
<td>Tang [28] 2001</td>
</tr>
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<td>Sinusoidal</td>
<td>Opamp</td>
<td></td>
<td>Zhong [29] 2002</td>
</tr>
<tr>
<td></td>
<td>1D-PWL</td>
<td>Sinusoidal</td>
<td>Opamp</td>
<td></td>
<td>Yu [31] 2007</td>
</tr>
<tr>
<td></td>
<td>2D-PWL</td>
<td>Sinusoidal</td>
<td>FGMOS</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>Multi-scroll</td>
<td>Complex</td>
<td>DSP</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Double scroll</td>
<td>PWL</td>
<td>CFOA,Opamp</td>
<td>OTA, Opamp</td>
<td>Elwakil [34] 2000</td>
</tr>
<tr>
<td>Third order canonical system</td>
<td>double scroll</td>
<td>PWL</td>
<td>CFOA,Opamp</td>
<td>OTA, Opamp</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1D-PWL</td>
<td>PWL</td>
<td>CFOA, Opamp</td>
<td></td>
<td>Yalcin [35] 2002</td>
</tr>
<tr>
<td></td>
<td>1D-PWL</td>
<td>PWL</td>
<td>Opamp</td>
<td></td>
<td>Lü [36] 2004</td>
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<tr>
<td></td>
<td>1D-PWL</td>
<td>PWL</td>
<td>Opamp, diode</td>
<td></td>
<td>Han [37] 2004</td>
</tr>
<tr>
<td></td>
<td>2D-Hysteresis</td>
<td>PWL</td>
<td>Opamp, diode</td>
<td></td>
<td>Ozoguz [38] 2002</td>
</tr>
<tr>
<td></td>
<td>1D-tanh()</td>
<td>PWL</td>
<td>CFOA, TX line</td>
<td></td>
<td>Yalcin [40] 2007</td>
</tr>
<tr>
<td></td>
<td>1D-tanh()</td>
<td>PWL</td>
<td>current mirror, C</td>
<td></td>
<td>Ozoguz [41] 2008</td>
</tr>
</tbody>
</table>

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decomposed in 5 regions defined by, $E_i / C_0$ given in (2) can now be described by its denoted as:

As a particular case, we will show the analysis for the generation of 3-scrolls. According to Fig. 1, the PWL function can be decomposed in 5 regions defined by,

$$
D_2 = \{(x, y, z)|x \geq E_2\},
D_1 = \{(x, y, z)|x \geq E_1\},
D_0 = \{(x, y, z)|E_1 \leq x \leq E_1\},
D_{-1} = \{(x, y, z)|x \leq -E_1\},
D_{-2} = \{(x, y, z)|x \leq -E_2\}.
$$

where $E_i$ denotes a break point. In this case: $E_1$ and $E_2$ being $b_2 = 0.8$ and $b_3 = 1.4$, respectively. The nonlinear function $f(x)$ given in (2) can now be described by its $m_i$ slopes and $b_i$ breakpoints as:

$$
f(x) =
\begin{cases}
  m_2 x + (m_5 - m_6)b_2 + (m_6 - m_7)b_3 & x \geq b_3, \\
  m_5 x + (m_6 - m_7)b_2 & x > b_2, \\
  m_6 x & -b_2 \leq x \leq b_2, \\
  m_7 x + (m_4 - m_5)b_2 & x < -b_2, \\
  m_6 x + (m_4 - m_5)b_2 + (m_3 - m_4)b_3 & x \leq -b_3.
\end{cases}
$$

From (1), one can compute the equilibrium points (EPs), where the static equations are given as,

$$
\begin{align}
  f(x) &= 0, \\
y &= 0, \\
z &= -x.
\end{align}
$$

In each PWL region, $D_2, \ldots, D_{-2}$, the system has a unique EP. The 5 EPs are given as,

$$
\begin{align}
  EP_{D_2}^+ &= (k_1, 0, -k_1) \in D_2, \\
  EP_{D_1}^+ &= (k, 0, -k) \in D_1, \\
  EP_0 &= (0, 0, 0) \in D_0, \\
  EP_{D_{-1}}^- &= (-k, 0, k) \in D_{-1}, \\
  EP_{D_{-2}}^- &= (-k_1, 0, k_1) \in D_{-2}.
\end{align}
$$

where $k = (g_2 - g_1)E_1/(g_2 + 1)$ and $k_1 = ((g_2 - g_1)E_1 + (g_1 - g_2)E_2)/(g_1 + 1)$, with $g_1 = m_3 = m_5 = m_7$ and $g_2 = m_4 = m_6$, according to (2).

The mathematical representation from (1) derives a linear expression of the form $\dot{x} = Ax + b$ [19]. The 5 regions can be denoted as:

$$
\dot{x} =
\begin{cases}
  A(x, \beta, m_3)(x - k_1) & x \in D_2, \\
  A(x, \beta, m_4)(x - k) & x \in D_1, \\
  A(x, \beta, m_5)x & x \in D_0, \\
  A(x, \beta, m_6)(x + k) & x \in D_{-1}, \\
  A(x, \beta, m_7)(x + k_1) & x \in D_{-2}.
\end{cases}
$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{PWL description of Chua's diode to generate multi-scroll attractors.}
\end{figure}
where \( x = [x, y, z]^T \), \( k = [k_0, -k_0]^T \), \( k_1 = [k_1, 0, -k_1]^T \) and the Jacobian matrix can be expressed by (8). The characteristic equation of the linearization around the equilibrium points is given by (9), where \( m_i \) is the corresponding slope for the 5 regions defined by (7).

\[
A(x, \beta, m_i) = \begin{bmatrix}
-x(m_i + 1) & x & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{bmatrix},
\]

(8)

\[
\lambda = \lambda^3 + [1 + x(m_i + 1)]\lambda^2 + [\beta + x \lambda m_i] \lambda + \alpha \beta (1 + m_i).
\]

(9)

The analysis of each linear region \( m_i \) is done using \( \alpha = 10, \beta = 11.5, g_1 = -0.276 \) and \( g_2 = -3.036 \), according to (2). Therefore, the computed EPs are given by

\[
EP_2^+ = (2.2873, 0, -2.2873) \in D_2,
\]

\[
EP_1^+ = (1.0845, 0, -1.0845) \in D_0,
\]

\[
EP_0 = (0.0, 0) \in D_0,
\]

\[
EP_1^- = (-1.0845, 0, 1.0845) \in D_{-1},
\]

\[
EP_2^- = (-2.2873, 0, 2.2873) \in D_{-2}.
\]

For \( m_i = g_1 \) (regions \( D_2, D_0, D_{-2} \)) the Jacobian matrix from \( (1) \) at the equilibrium points \( (EP_2^+, 0, EP_2^-) \) is given as:

\[
J(EP_2^+, 0) = \left[ \frac{df}{dx} \right] = \begin{bmatrix}
-7.2400 & 10.0000 & 0 \\
1.0000 & -1.0000 & 1.0000 \\
0 & -11.5000 & 0
\end{bmatrix},
\]

(10)

which has the following eigenvalues,

\[
i_{EP2}^{1,2,0} \approx -8.3823,
\]

\[
i_{EP2}^{1,2,0} \approx 0.0712 + 3.1508i,
\]

\[
i_{EP2}^{1,2,0} \approx 0.0712 - 3.1508i.
\]

For \( m_i = g_2 \) (regions \( D_1, D_{-1} \)) the Jacobian matrix from \( (1) \) at the equilibrium points \( (EP_1^+, EP_1^-) \) becomes:

\[
J(EP_1^+, EP_1^-) = \left[ \frac{df}{dx} \right] = \begin{bmatrix}
20.3600 & 10.0000 & 0 \\
1.0000 & -1.0000 & 1.0000 \\
0 & -11.5000 & 0
\end{bmatrix},
\]

(11)

which has the following eigenvalues,

\[
i_{EP1}^{1,2,1} \approx 20.8072,
\]

\[
i_{EP1}^{1,2,1} \approx -0.7236 + 3.2755i,
\]

\[
i_{EP1}^{1,2,1} \approx -0.7236 - 3.2755i.
\]

From the computed eigenvalues, those can be classified as saddle point of index 2 and saddle point of index 1, respectively.

2.1. Numerical simulation

The simulation of (1), leads us to compute the PWL function shown in Fig. 2(a), two state variables are shown in Fig. 2(b), and the 3-scrolls attractor computed using \( X = [0.1, 0, 0] \) as initial conditions, is shown in Fig. 2(c).

2.2. Lyapunov exponents

The Lyapunov exponents are computed by applying the method given in [14,25]. Those exponents are related to the expansion or contraction nature of the different directions in the phase diagram. In fact, in a dissipative continuous dynamical system, the possible spectrums for the Lyapunov exponents in a third order system are: \((+, 0, -)\), strange attractor; \((0, 0, \cdot)\), double torus; \((0, -, \cdot)\), limit cycle; and \((-,-,-)\), fixed point [42]. In our case, the Lyapunov exponents are 0.37522, 0.0044 and \(-7.7984\), confirming chaotic regime.

2.3. Bifurcation diagram

The bifurcation diagram is generated on the plane \( X-Y \), as shown in Fig. 3 by varying \( \alpha \) from 5 to 15 in steps of 0.01. The integration time was set to \( T = 400 \) for each increment in \( \alpha \) [14]. As one sees, the chaotic behavior is present for \( \alpha \) between 7.8 and 10.7.
Fig. 2. 3-Scrolls generated from a modified Chua’s circuit [19,21].
2.4. Electronic implementation using CFOAs

The multi-scroll chaotic oscillator described in this section was implemented using the CFOA AD844 [19], as shown in Fig. 4.

The values of $a$ and $b$ correspond to the coefficients $\frac{R_{73}}{R_{72}}$ and $\frac{R_{32}}{R_{31}}$, respectively. The resistances $R_{61} = R_{62}$ are tuned to adjust the gains of the PWL function, which is built by using the finite gain model of the CFOA. That way, several saturated-cells, as the one shown in Fig. 5, can be connected in parallel to synthesize the PWL function [13]. For instance, the behavior of the saturated cell shown in Fig. 5, can be described by (12).

$$V_{\text{out}}(V_{\text{in}}) = \begin{cases} 
\frac{R_{\text{out}}(V_{\text{in}} - E)}{R_x} & V_{\text{in}} < V^- , \\
\frac{R_{\text{out}}(V^- - E)}{R_x} & V^- < V_{\text{in}} < V^+ , \\
\frac{R_{\text{out}}(V^+ - E)}{R_x} & V_{\text{in}} > V^+.
\end{cases} \quad (12)$$

To generate 3-scrolls, three saturated cells are connected in parallel as shown in Fig. 6. The values are assigned according to the procedure introduced in [19]. Fig. 7 shows the 3-scrolls attractor simulated in HSPICE using the macromodel of the CFOA AD844. Experimental results using the CFOA AD844 are shown in [3,19,43,44].

3. Integrated design of chaotic oscillators

As mentioned in the previous sections, the majority of chaotic oscillator designs have been implemented mainly by using discrete active devices, e.g. opamps in the majority of cases. Moreover, the use of practically recent mixed-mode active devices [18], with IC technology allows us to implement integrated systems, as recently demonstrated in [11,12,45]. However, still many open problems in the development of integrated chaotic oscillators, are good opportunities for future research. For instance, a brief discussion on the integrated implementations of chaotic oscillators is given below.

Cruz and Chua design: One of the first monolithic implementations was the Chua’s chaotic oscillator [26]. The fabricated IC required external resistors connected between terminals 1 and 2 and between 5 and 6 of the developed chip shown by Fig. 3 in [26], with the purpose of adjusting the nonlinear function gain. That nonlinear function was generated by using two OTAs, each one with different transconductance values and labeled as A and B, and also each one with different current bias levels. The inductor was designed from the gyrator concept also using OTAs.

Rodríguez and Delgado design: A similar Chua’s implementation was realized based on the state variables perspective [7] (the normalized system). Arranging identical OTAs in parallel used gm-C integrators. The nonlinear function was realized by two OTAs connected in parallel, but each one with independent bias control.

Elwakil et al. Design: The two-scroll version of the original canonical system was also realized by selecting the capacitance or resistances of the loads, as described in [34]. Thus allowing having positive and negative integrals. All the system parameters were available since all the loads were externally connected. Using an inverter as comparator made the nonlinear function.

Fujiwara et al. design: These authors proposed a circuit similar to that introduced by Cruz and Chua [26]. A simulated inductor was also used, and transistors were biased to operate in the linear region to implement the resistor [30]. A particular change was introduced in the design of the nonlinear function, which was designed using floating-gate MOS transistors.
The FGMOS transistors were exploited as switches in the current path. That way, the nonlinear function approached by a PWL one, allows increasing the number of linear segments by connecting several cells in parallel. The disadvantage of that design is the requirement of external controls to adjust the slope and breakpoints of the voltage-to-current (V-I) characteristic of the PWL function. The authors in [30] reported a 3-scroll experimental attractor, obtained from the introduction of the first multi-scroll chaotic oscillator, implemented with IC technology. However, the poor common-mode rejection ratio (CMRR) did not allow proving the generation of more scrolls.

Table 2 summarizes some details of these integrated chaotic oscillators, and the new one introduced by Trejo-Guerra and colleagues in [11,12]. The center frequency listed in the last row of Table 2, of 3.5 MHz was observed by using the internal capacitances, not the external ones already shown in [11,12]. Most important is that from this Table one can infer the necessity of developing novel IC designs for chaotic oscillators generating not only multi-scrolls, but also multi-scrolls in multiple directions.

Reference [11] summarizes integrated realizations of multi-scroll chaotic oscillators, and introduces a modified Chua’s circuit whose PWL function is a saw-tooth one. That work shows the analysis of the dynamical system to obtain the circuit design requirements. The article proposes an innovative design using FGMOS transistors to implement a V-I PWL function to generate multi-scrolls. The experimental confirmation of that integrated design is given in [12], where the authors introduce for the first time: parameter, technology and process variation analysis. The article shows the experimental observation of 3- and 5-scrolls from the fabricated IC in technology of 0.5 μm.

Discussion on the design of chaotic oscillators using IC technology is given in [11,12]. For instance, as seen in Table 1, PWL functions allow obtaining several multi-scroll oscillator schemes. To design those functions with electronic devices, a
Fig. 6. PWL function implemented with saturated cells to generate 3-scrolls.

Fig. 7. Hspice simulation result.

Table 2
Chaotic oscillators implemented with CMOS IC technology.

<table>
<thead>
<tr>
<th>Author</th>
<th>Technology</th>
<th>Bias Levels</th>
<th>Center Freq.</th>
<th>PWL function strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cruz &amp; Chua [26]</td>
<td>2 μm</td>
<td>-</td>
<td>160 kHz</td>
<td>Distinct bias currents and saturation</td>
</tr>
<tr>
<td>Rodríguez &amp; Delgado [7,46]</td>
<td>2.4 μm</td>
<td>±2.5 V</td>
<td>-</td>
<td>Distinct bias currents and saturation</td>
</tr>
<tr>
<td>Elwakil et al. [34]</td>
<td>1.2 μm</td>
<td>±2.5 V</td>
<td>118 kHz</td>
<td>Simple inverter</td>
</tr>
<tr>
<td>Fujiwara et al. [30]</td>
<td>0.35 μm</td>
<td>±1.65 V</td>
<td>7 MHz</td>
<td>Switching currents FGMOS transistors</td>
</tr>
<tr>
<td>Trejo-Guerra et al. [11,12]</td>
<td>0.5 μm</td>
<td>±2.5 V</td>
<td>3.5 MHz</td>
<td>FGMOS inverter</td>
</tr>
</tbody>
</table>
switching component may be used. Direct applications will achieve stair-like or saw-tooth functions; also, by controlling the biasing of some linear active cell, other PWL functions can be generated. But the implementation of the nonlinear or PWL function is a challenge design issue because the behavior must be accurate to keep the relations of the dynamical system and its equilibrium points, simultaneously. It means that the design of nonlinear functions has several dimensions of difficulty compared to a simple linear design. The first problem is how to mitigate input and output offset points on PWL segments, and active function blocks performing integration, addition or subtraction operations. The slow rate affects the bandwidth capacity in different manners (according to the specific PWL function), the hysteresis effects may produce more than two different signal paths, and so on.

4. Conclusions

This article summarized recent developments on the electronic implementation of chaotic oscillators. It was focused on multi-scroll chaotic oscillators, and specifically those based on PWL functions. Several PWL-function based multi-scroll chaotic oscillators were listed and we highlighted that the majority of them are modifications of the well-known double-scroll Chua’s circuit.

Key references for using active devices such as opamps, CFOAs, OTAs, CCIIs, and unity-gain cells were listed for the discrete implementation of chaotic oscillators.

A table summarizing the very few works on the integrated design of chaotic oscillators was given in the last section. As a final conclusion, we appeal performing research on the electronic implementation of multi-scroll chaotic oscillators using not only discrete active devices, but also using modern IC technologies. Interested designers can find guidelines on the IC implementation of multi-scroll chaotic oscillators based on PWL function approaches in [11,12].

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