Reinterpreting a fuzzy subset by means of a Sincov’s functional equation

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Abstract. Throughout the paper it is shown that the classical definition of a fuzzy subset carries additional structures. The concept of a fuzzy subset is regarded from an alternative point of view: namely, the characteristic function of a fuzzy subset may be reinterpreted in terms of a Sincov’s functional equation in two variables. Since the solutions of a Sincov’s functional equation are also closely related to the existence of representable total preorders, an special attention is paid to the relationship between fuzzy subsets and total preorders defined on a universe. Some possible applications of this approach are pointed out in the final section of the manuscript.

Keywords: Fuzzy subsets, Sincov’s functional equation, representable total preorders

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1. Introduction

A new look at the classical definition of a fuzzy subset, introduced by L.A. Zadeh in his seminal work [30] (see also [6]), reveals other structures that were hidden in the original setting.

Starting from the standard definition (see [6, 30]) of a fuzzy subset \( X \) (of a set \( U \), called the universe) by means of a membership function \( \mu_X : U \to [0, 1] \), the degree of membership of two elements \( x, y \in U \) could be compared by directly reckoning the difference \( \mu_X(y) - \mu_X(x) \). The bivariate function \( F : U \times U \to [-1,1] \), given by \( F(x, y) = \mu_X(y) - \mu_X(x) \) for every \( x, y \in U \), satisfies the so-called Sincov’s functional equation (see e.g. [17, 20])

\[
F(x, y) + F(y, z) = F(x, z) (x, y, z \in U)
\]

Therefore, the mere definition of a fuzzy subset immediately induces the solution of a Sincov’s functional equation on the universe.

Conversely, provided that \( U \) is a universe and \( F : U \times U \to [-1,1] \) is a bivariate map that satisfies the Sincov’s functional equation, one may wonder if there exists a fuzzy subset \( X \) of \( U \), such that the characteristic function \( \mu_X \) accomplishes that \( F(x, y) = \mu_X(y) - \mu_X(x) \) for every \( x, y \in U \).

The answer is yes, but the fuzzy subset \( X \) is not unique, in general. However, if it is a priori known, for instance, that \( X \) is a normal fuzzy subset of the universe \( U \), then \( F \) completely determines \( X \) in a unique way.

Incidentally, a crucial fact related to the Sincov’s functional equation is that of defining a representable total preorder on a given nonempty crisp set (see e.g. [7]). Having this in mind, one may wonder whether or not those orderings can directly be defined by means of suitable fuzzy subsets.

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The structure of the paper goes as follows:

After the Introduction and a Section 2 of previous concepts, notations and background, in Section 3 the relationship between fuzzy subsets and Sincov’s functional equations is analyzed. In Section 4, representable total preorderings coming from a fuzzy subset or generated by a Sincov’s functional equation are studied. The paper is closed by pointing out some possible generalizations, applications and suggestions for further research.

2. Preliminaries

2.1. The classical concept of a fuzzy subset

Throughout the paper, a set in the standard sense is said to be crisp (or conventional), in opposition to the term “fuzzy” introduced below in Definition 2.1.

The standard definition of a fuzzy subset, already pointed out in the Introduction, is established now in a formal way.

Definition 2.1. ([30]) Let $U$ be a nonempty set, usually called universe. A fuzzy subset $X$ of $U$ is defined by means of a map $\mu_X : U \rightarrow [0, 1]$. The map $\mu_X$ is said to be the membership function (or indicator) of $X$. The support of $X$ is the crisp subset $\text{Supp}(X) = \{ t \in U : \mu_X(t) \neq 0 \} \subseteq U$, whereas the kernel of $X$ is the crisp subset $\text{Ker}(X) = \{ t \in U : \mu_X(t) = 1 \} \subseteq U$. The fuzzy subset $X$ is said to be normal if $\text{Supp}(X) = U$.

Given $a \in [0, 1]$, the crisp subset of $U$ defined by $U_a = \{ t \in U : \mu_X(t) \geq a \}$ is said to be the $a$-cut of the fuzzy subset $X$.

Remark 2.2. Notice that through this Definition 2.1, the fuzzy subset $X$ is indeed identified to its membership function $\mu_X$. It is quite common to use these two notions, namely $X$ and $\mu_X$, interchangeably.

2.2. The Sincov’s functional equation

The concept of a Sincov’s functional equation is introduced now.

Definition 2.3. Let $U$ be a nonempty crisp set (universe). A bivariate function $F : U \times U \rightarrow \mathbb{R}$ is said to satisfy Sincov’s functional equation if $F(x, y) + F(y, z) = F(x, z)$ holds for every $x, y, z \in U$.

Remark 2.4. Although the Sincov’s functional equation is so-named after the Russian mathematician Dmitri Matveevich Sincov (1867–1946), who furnished an elegant solution for it in a couple of papers dated 1903 (see [28, 29]), it had already been considered and analyzed by the German historian of mathematics Moritz Cantor in 1896 (see [16, 20]).

The following result is well-known (see e.g. [16, 17, 20, 28, 29] or [1], pp. 122 and ff.).

Proposition 2.5. Let $U$ be a universe. A bivariate function $G : U \times U \rightarrow \mathbb{R}$, satisfies the Sincov’s functional equation (if and only if $G(x, y) = G(x) + G(y)$ (for $x, y \in U$), for some function $G : U \rightarrow \mathbb{R}$, that only depends of one single variable.

Proof. The elegant Sincov’s proof [28] goes as follows: since $F(x, y) = F(x, z) - F(y, z)$ is true for every $x, y, z \in U$, one may fix any $z$ in the right side of the above equality. Thus, after fixing an element $a \in U$, define the function $G : U \rightarrow \mathbb{R}$ by declaring that $G(t) = -F(a, t)$ for every $t \in U$. With this definition, $F(x, y) = G(y) - G(x)$ holds for all $x, y \in U$. □

Definition 2.6. Let $U$ be a universe. Let $F : U \times U \rightarrow \mathbb{R}$ be a function that satisfies the Sincov’s functional equation. Having Proposition 2.5. in mind, a function $G : U \rightarrow \mathbb{R}$ is said to generate $F$ if $F(x, y) = G(y) - G(x)$, for every $x, y \in U$.

Proposition 2.7. Let $U$ be a universe. Let $F : U \times U \rightarrow \mathbb{R}$ be a function that satisfies the Sincov’s functional equation. Let $G : U \rightarrow \mathbb{R}$ be a function that generates $F$. Then a function $G_k : U \rightarrow \mathbb{R}$ also generates $F$ if and only if there exists $k \in \mathbb{R}$, such that $G_k(x) = G(x) + k$ for every $x \in U$. (In other words: a function that generates $F$ is unique up to a constant).

Proof. To prove the direct implication, notice that if $G_1$ generates $F$, then $F(x, y) = G_1(y) - G_1(x) = G_1(y) - G(x)$ holds for every $x, y \in U$. Thus we have that $G_1(y) - G(x)$ is a constant function. The converse implication is immediate. □

The following corollary is an immediate consequence of Proposition 2.7.

Corollary 2.8. Let $U$ be a universe. Let $F : U \times U \rightarrow \mathbb{R}$ denote a function that satisfies the Sincov’s functional equation. Then a function $G : U \rightarrow \mathbb{R}$ is said to generate $F$ if and only if the function $G_k$ is a constant function.
functional equation. If an element \( a \in U \) and a real number \( k \in \mathbb{R} \) have been fixed, then there exists a unique function \( G : U \to \mathbb{R} \) such that \( G \) generates \( F \) and \( G(a) = k \).

2.3. Total preorders

Let us recall now the notion of a total preorder, as well as some related concepts.

**Definition 2.9.** Let \( U \) denote a universe. A preorder \( \preceq \) on \( U \) is a binary relation on \( U \) which is reflexive and transitive.

An antisymmetric preorder is said to be an order. A total preorder \( \preceq \) on a set \( U \) is a preorder such that if \( a, b \in U \) then \( [a \preceq b] \) or \([b \preceq a]\) holds true. A total order is also called a linear order.

If \( \preceq \) is a preorder on \( U \), then the associated asymmetric relation is denoted by \( < \), whereas \( \sim \) will stand for the associated equivalence relation. These relations are respectively defined by \( [a < b \iff (a \preceq b) \land \lnot(b \preceq a)] \) and \([a \sim b \iff (a \preceq b) \land (b \preceq a)]\). Moreover, the binary relation \( \preceq \) is defined by \( a \preceq b \iff b \preceq a \) for every \( a, b \in U \), which is also a preorder on \( U \), is said to be the dual preorder associated to \( \preceq \).

The asymmetric part of a linear order is said to be a strict linear order.

**Definition 2.10.** Let \( U \) be a universe. Let \( \preceq \) be a total preorder defined on \( U \). The preorder \( \preceq \) is said to be representable if there exists a function \( u : U \to \mathbb{R} \) such that \( a \preceq b \iff u(a) \leq u(b) \), for every \( a, b \in U \). The order-preserving function \( u \) involved is said to be a utility function (also known as an isotomy or an order isomorphism) for the preorder \( \preceq \) on \( U \).

**Remark 2.11.** For further information concerning characterizations of the representability of total preorders through utility functions, see, e.g., the first three chapters of [9], or else [4]. There are also other alternative numerical representations for total preorders as well as for other particular kinds of orderings defined on a universe (see Definition 4.2., below, and [2, 8, 11]). As a matter of fact, some of those alternative representations lean on different kinds of fuzzy numbers (see [10, 12]).

**Example 2.12.** A well-known example of a linear order which is not representable is the lexicographic plane, namely \( \mathbb{R}^2 \) endowed with the strict linear order \( \prec \) given by \((x, y) \prec (z, t) \iff (x < z) \lor (x = z \land y < t)\), for any \( x, y, z, t \in \mathbb{R} \) (see e.g. [9, 13]).

**Remark 2.13.** Notice that a utility function (if there is any) \( u : U \to \mathbb{R} \) completely determines a representable total preorder on \( U \) as \( x \preceq y \iff u(x) \leq u(y) \). However, a representable total preorder \( \preceq \) cannot fix a (unique) utility function \( u \) that represents it. Indeed, if \( u : U \to \mathbb{R} \) is a utility representation of \( \preceq \), and \( h : \mathbb{R} \to \mathbb{R} \) is a strictly increasing function, the composition \( h \circ u : U \to \mathbb{R} \) also represents \( \preceq \) since \( x \preceq y \iff u(x) \leq u(y) \iff h(u(x)) \leq h(u(y)) \) for every \( x, y \in U \).

3. Sincov’s functional equations vs. fuzzy subsets

Let \( U \) be a universe. Given a fuzzy subset \( X \) of \( U \), defined by means of a membership function \( \mu_X : U \to [0, 1] \), a bivariate function \( F : U \times U \to [-1, 1] \) is immediately generated by setting \( F(x, y) = \mu_X(y) - \mu_X(x) \), for every \( x \in U \). Therefore, given a fuzzy subset of a universe \( U \), it has immediately associated a solution of the Sincov’s functional equation, since obviously \( F(x, y) + F(y, z) = (\mu_X(y) - \mu_X(x)) + (\mu_X(z) - \mu_X(y)) = (\mu_X(z) - \mu_X(x)) = F(x, z) \), for every \( x, y, z \in U \).

Let us analyze the converse: assume that \( F : U \times U \to [-1, 1] \) is a function that satisfies the Sincov’s functional equation. One may wonder if there is a fuzzy subset \( Y \) of \( U \) such that \( F(x, y) = \mu_Y(y) - \mu_Y(x) \) for every \( x, y \in U \), where \( \mu_Y \) stands for the indicator of \( Y \) as a fuzzy subset of \( U \).

The answer is affirmative, as next Theorem 3.1. shows.

**Theorem 3.1.** Let \( U \) be a universe. Let \( F : U \times U \to [-1, 1] \) be a function that satisfies the Sincov’s functional equation. Then there exists a fuzzy subset \( X \) of \( U \), such that \( F(x, y) = \mu_X(y) - \mu_X(x) \), for all \( x, y \in U \), where \( \mu_X \) denotes the membership function of \( X \).

**Proof.** Define \( \mu_X : U \to \mathbb{R} \) by declaring that \( \mu_X(x) = \sup\{F(t, x) : t \in U\} \), for every \( x \in U \). Hence, by definition, \( 0 \leq F(x, x) \leq \mu_X(x) \) holds for all \( x \in U \). Moreover, \( \mu_X(x) \leq 1 \) for every \( x \in U \), because, by hypothesis, \( F(t, x) \leq 1 \) for every \( t \in U \). Thus \( \mu_X \) is indeed the indicator of a fuzzy subset \( X \) of \( U \). Furthermore, given \( x, y \in X \), it follows that \( \mu_X(y) - \mu_X(x) = \sup\{F(t, y) : x \in \)
by a Sincov’s functional equation

Remark 3.2. Despite the membership function \( \mu_X \) of the fuzzy subset \( X \) being defined from the bivariate function \( F \) in a quite natural way, there may exist other fuzzy subsets of the universe \( U \) that satisfy the same property, namely, their corresponding indicators also generate \( F \). These fuzzy subsets are characterized through the following Proposition 3.3.

Proposition 3.3. Let \( U \) be a universe. Let \( F : U \times U \to [-1, 1] \) be a function that satisfies the Sincov’s functional equation. Define \( \mu_F(x) = \sup\{F(t, x) : t \in U\} \), for every \( x \in U \). Let \( a = \inf\{\mu_F(t) : t \in U\} \) and \( b = \sup\{\mu_F(t) : t \in U\} \). Then \( Y \) is another fuzzy subset of \( U \), whose indicator \( \mu_Y \) also generates \( F \), if and only if there exists a constant \( k \in [-a, 1 - b] \) such that \( \mu_Y(x) = \mu_F(x) + k \) for every \( x \in U \).

Proof. To prove the direct implication, notice that by Proposition 2.7, if \( F(x, y) = \mu_F(y) - \mu_F(x) \) for every \( x, y \in U \), there exists a constant \( k \in \mathbb{R} \) such that \( \mu_Y(x) - \mu_F(x) = k \) for all \( x \in U \). Therefore, since \( 0 \leq \mu_Y(x) \leq 1 \) must hold for every \( x \in U \), the constant \( k \) belongs to the interval \([-a, 1 - b]\).

To prove the converse implication, take \( k \in [-a, 1 - b] \) and define \( G : U \to \mathbb{R} \) as \( G(x) = \mu_Y(x) - k \) for all \( x \in U \). It is clear that \( F(x, y) = G(y) - G(x) \) and \( 0 \leq G(x) \leq 1 \) hold for every \( x, y \in U \). Thus, the function \( G \) generates \( F \) and is the membership function of a fuzzy subset \( Y \) of \( U \).

To force the uniqueness of the fuzzy subset induced by a Sincov’s functional equation \( F : U \times U \to [-1, 1] \) on a universe \( U \), some additional condition is compulsory (see Corollary 3.6 below).

Definition 3.4. Let \( U \) be a universe. Let \( X \) denote a fuzzy subset of \( U \), defined through the membership function \( \mu_X : U \to [0, 1] \). The fuzzy subset \( X \) is called quasi-normal if it holds that \( \sup\{\mu_X(t) : t \in U\} = 1 \).

Remark 3.5. A quasi-normal fuzzy subset \( X \) of a universe \( U \) may fail to be normal. In other words, despite \( 1 \) being the supremum of the set \( \sup\{\mu_X(t) : t \in U\} \), it could still happen that this supremum is not attained at any point of the universe \( U \). As a clear example, consider \( U = \{0, 1\} \) and the fuzzy subset \( X \) defined by \( \mu_X(t) = 1 \) for every \( t \in U \).

Corollary 3.6. Let \( U \) be a universe. Let \( F : U \times U \to [-1, 1] \) be a function that satisfies the Sincov’s functional equation. Then there exists a unique quasi-normal fuzzy subset \( X \) of \( U \), such that \( F(x, y) = \mu_X(y) - \mu_X(x) \), for all \( x, y \in X \), where \( \mu_X \) denotes the indicator of \( X \).

Proof. It follows directly from Proposition 3.3, after setting \( k = 1 - b \).

This Section 3 can be concluded by saying that a quasi-normal fuzzy subset of a universe \( U \) is encoded (or can be identified to in a unique way) through a solution of a Sincov’s functional equation \( F : U \times U \to [-1, 1] \).

4. Fuzzy subsets vs. numerical representability of total preorders

4.1. Sincov’s representability of total preorders

Sincov’s functional equations are closely related to the representability of total preorders defined on a universe \( U \). As a matter of fact, given a total preorder defined on a universe \( U \), the kind of numerical representation, by means of a utility function, involved in Definition 2.10 is actually equivalent to another representation that uses a unique quasi-normal fuzzy subset \( X \) of \( U \), accomplishing the Sincov’s functional equation, as the next well-known result shows. (See e.g. Theorem 1 in [7]).

Proposition 4.1. Let \( \preceq \) be a total preorder defined on a universe \( U \). Then \( \preceq \) is representable if and only if there exists a bivariate function \( F : U \times U \to \mathbb{R} \) such that \( F \) satisfies the Sincov’s functional equation and, in addition, \( x \preceq y \Leftrightarrow F(x, y) \geq 0 \) holds for every \( x, y \in U \).

The following definition is inspired by Proposition 4.7, and it is equivalent to Definition 2.10.

Definition 4.2. Let \( \preceq \) be a representable total preorder defined on a universe \( U \). A bivariate function \( F : U \times U \to \mathbb{R} \) satisfying the Sincov’s functional equation, and such that \( x \preceq y \Leftrightarrow F(x, y) \geq 0 \) holds for every \( x, y \in X \), is called a Sincov’s representation of \( \preceq \).
4.2. Representable total preorder induced by a fuzzy subset

Given a universe \( U \), Corollary 3.6 already states that a bivariate function \( F : U \times U \to [-1,1] \) that satisfies the Sincov’s functional equation can be identified to a unique quasi-normal fuzzy subset \( X \) of \( U \). Furthermore, that bivariate function \( F \) also induces a representable total preorder on the universe \( U \), as stated in Proposition 4.1., by just declaring that \( a \succeq b \iff F(a, b) \geq 0 \), for every \( a, b \in U \). Consequently, a quasi-normal fuzzy subset \( X \) of \( U \) induces on the universe \( U \), in a natural way, a representable total preorder.

Remark 4.3. Nevertheless, this last fact is by no means a surprise: indeed, if \( X \) is a fuzzy subset of a universe \( U \), defined by means of the indicator \( \mu_X \), the binary relation \( \preceq \) on \( U \) given by \( x \preceq y \iff \mu_X(x) \leq \mu_X(y) \) is obviously a representable total preorder on \( U \). As a matter of fact, \( X \) is not restricted here to be quasi-normal.

4.3. Retrieving a fuzzy subset from a representable total preorder on a universe

It has just seen in Remark 4.3. that a fuzzy subset \( X \) of a universe \( U \) immediately induces a representable total preorder on \( U \).

Looking for a converse result, let us study now how to build, in some standard way, a fuzzy subset from a representable total preorder \( \preceq \) defined on a universe \( U \).

Proposition 4.4. Any representable total preorder \( \preceq \) defined on a universe \( U \) induces a fuzzy subset on a universe.

Proof. Let \( u : U \to \mathbb{R} \) be a utility function that represents \( \preceq \). Assume without lost of generality\(^4\) that \( u \) takes values in \([0,1]\). Thus a fuzzy subset \( X \) of \( U \) is got by declaring that \( u \) is the membership function \( \mu_X \) of the fuzzy subset \( X \) to be defined.

Remark 4.5. Due to the fact already commented in Remark 2.13., the fuzzy subset \( X \) induced by Proposition 4.4. is not unique, in general. To put an example, take \( U = (0,1) \) endowed with the usual linear order \( \leq \) of the real line. One may immediately observe that both the functions \( u_1, u_2 : U \to (0,1) \), respectively defined by \( u_1(t) = t; u_2(t) = t^2 \), for every \( t \in U \) represent \( \leq \). However, the fuzzy subset \( X_1 \) whose indicator is \( u_1 \) is different from the fuzzy subset \( X_2 \) whose indicator is \( u_2 \), because, for instance, \( u_1(\frac{1}{2}) = \frac{1}{2} \neq \frac{1}{4} = u_2(\frac{1}{2}) \).

Despite not having uniqueness in the fuzzy subsets of a universe \( U \), defined through a representable total preorder \( \preceq \), one could interchangeably be working with solutions of the Sincov’s functional equation \( F : X \times U \to [-1,1] \), utility functions \( u \), representing \( \preceq \), taking values in \([0,1]\) and satisfying that sup\(\{u(t) : t \in U\} = 1 \), and quasi-normal fuzzy subsets of the universe \( U \), as shown in the following Proposition 4.6., with which this Section 4 is closed.

Proposition 4.6. Let \( U \) be a universe. Let \( \preceq \) denote a representable total preorder on \( U \). Fix a utility representation \( u : X \to [0,1] \) of the preorder \( \preceq \) and assume that sup\(\{u(t) : t \in U\} = 1 \). Then the following statements hold true:

1) The function \( u \) is the indicator of a (unique) quasi-normal fuzzy subset \( X \) of \( U \).

2) There is a Sincov’s representation \( F : U \times U \to [-1,1] \) of the total preorder \( \preceq \) such that \( F(x, y) = u(y) - u(x) \), for every \( x, y \in X \).

Proof. Part 1) is obvious by just considering \( u \) as the membership function of a quasi-normal fuzzy subset \( X \) of \( U \).

To prove part 2), define \( F(x, y) = u(y) - u(x) \) for all \( x, y \in U \). Since \( u \) takes values in \([0,1]\) it is clear that \( F \) takes values in \([-1,1]\) and satisfies the Sincov’s functional equation. Moreover \( x \preceq y \iff u(x) \leq u(y) \implies F(x, y) = u(y) - u(x) \geq 0 \) holds for every \( x, y \in U \).

5. Concluding remarks

5.1. Miscellaneous examples

The following two examples illustrate the main ideas discussed in the previous sections.

Example 5.1. Suppose that a firm wants to hire people. To do so, an exam of one hundred items, or tasks to be performed, is put to all the possible candidates. Thus, the score that a candidate could obtain takes
values in \([0, 100]\). The perfect score is 100, and the worst one, namely 0, points out that the candidate is not able to cope with any of the tasks involved. Needless to say, ratios or percentages could be considered at this stage, so working with (relative) scores that take values in \([0, 1]\). Let \(U\) denote the set of candidates. Let \(r(x) \in [0, 1]\) be the ratio corresponding to the candidate \(x \in U\). That is, 100 \(r(x)\) is the number of tasks that the candidate \(x\) is able to do. The function \(r : U \rightarrow [0, 1]\) is then the indicator of a suitable fuzzy subset \(X\) of \(U\), so that all the information corresponding to the scores of the candidates in the exam is furnished by \(X\). The fuzzy subset \(X\) is then normal if at least one of the candidates has got a perfect score.

Suppose now that all this the information is encoded in a different way: each candidate \(x \in U\) is confronted to each other outsider \(y \in U\). Let \(N(x, y)\) be the difference between the number of tasks that \(y\) is able to do and the number of tasks that \(x\) is able to do. (Notice that this difference could be negative). Let \(F : U \times U \rightarrow [-1, 1]\) be given as 100 \(F(x, y) = N(x, y)\) for all \(x, y \in U\). This bivariate function \(F\) pairwise compares the abilities of the candidates. By Corollary 3.6., if the table of values of \(F(x, y)\) \((x, y \in U)\) is given as a database, and provided that it is also known that at least one of the candidates is perfect, then the whole ranking of the candidates could be fully retrieved, assigning to each of them the exact score that she/he has got in the exam.

**Example 5.2.** Let \(U\) be a certain universe of currently living animal species, all of them directly coming from a mother species \(a\) \(\in U\) along all the last ten million years. By comparing the genetic DNA code of any species \(t \in U\) to that of \(a\), the genetic distance (i.e.: number of speciation events, or mutations) between \(a\) and \(t\) is obtained for any \(t \in U\). Denote it by \(G_t(a, t)\) \((t \in U)\). Assume the existence of an adjusting constant \(k > 0\) such that \(kG(a, t) = 1\) holds for any species \(x \in U\) that swerved from \(a\) exactly ten million years ago. The bivariate function \(F : U \times U \rightarrow [-1, 1]\) defined by \(F(x, y) = k(G_t(a, t) - G_t(a, x))\) \((x, y \in U)\) satisfies the Sincov’s functional equation. Using Theorem 3.1., one may define a fuzzy subset \(X\) of \(U\) such that \(F(x, y) = \mu_X(y) - \mu_X(x)\) holds for every \(x, y \in U\). The crucial fact here is that this subset \(X\) can be used as a helpful tool to estimate the approximate age of the living species that belong to \(U\). Indeed, one may admit that a species \(x \in U\) appeared on planet Earth \((10 - 10\mu_X(x))\) million years ago.

**5.2. An application: defining operations**

Let \(U\) be a universe. Let \(F_1, F_2 : U \times U \rightarrow [-1, 1]\) be two bivariate functions, such that each of them satisfies the Sincov’s functional equation. It is straightforward to see that the function \(G : U \times U \rightarrow [-1, 1]\), defined by \(2G(x, y) = F_1(x, y) + F_2(x, y)\) for every \(x, y \in U\), satisfies the Sincov’s functional equation, too. In addition, again by Corollary 3.6., the function \(F_1\) (respectively \(F_2\)) has assigned a unique quasi-normal fuzzy subset \(X_1\) (respectively \(X_2\), \(Y\)) of the universe \(U\). The fact of \(G\) appearing as the arithmetic mean of \(F_1\) and \(F_2\) suggests the definition of an operation, that could also be called the arithmetic mean, directly acting on the quasi-normal fuzzy subsets of the universe \(U\), and so that the arithmetic mean of \(F_1\) and \(F_2\) is \(G\).

Conversely, given two fuzzy subsets \(X_1\) and \(X_2\) of a universe \(U\), some suitable aggregation operator, typical in the fuzzy setting (e.g.: a triangular conorm, see [24]) gets new fuzzy subsets of \(U\) from \(X_1\) and \(X_2\), and preserves the quasi-normality. For instance, if \(\mu_X\) (respectively \(\mu_Y\)) is the membership function of the quasi-normal fuzzy subset \(X_1\) (respectively, \(X_2\)) of the universe \(U\), one may define another quasi-normal fuzzy subset \(Y\) by means of the indicator \(\mu_Y : U \rightarrow [0, 1]\) given by \(\mu_Y(t) = \max(\mu_X(t), \mu_Y(t))\) for every \(t \in U\). These operators acting on \(X_1\) and \(X_2\) could be then translated into operations acting directly on Sincov’s functional equations \(F_1\) and \(F_2\) that respectively correspond to \(X_1\) and \(X_2\) in the spirit of Corollary 3.6. The output would also be a Sincov’s functional equation on \(U\) and taking values in \([-1, 1]\).

Furthermore, one may pass to consider representative total preorders (or their corresponding utility representations) coming from quasi-normal fuzzy subsets of an universe, so that operations on the fuzzy subsets could also induce operations defined, directly, on representable total preorders, and giving rise, as the final output, to new representable total preorders.

**Example 5.3.** Let \(U = \{a, b, c, d\}\). Define the fuzzy subsets \(X_1\) and \(X_2\) of the universe \(U\) by means of the membership functions \(\mu_X\) and \(\mu_Y\) respectively defined by \(\mu_X(a) = \mu_X(b) = 1, \mu_X(c) = \mu_X(d) = 0\) and \(\mu_Y(a) = 1, \mu_Y(b) = \mu_Y(c) = \mu_Y(d) = 0\). Associated to these indicators, each of the bivariate functions \(F_1, F_2 : U \times U \rightarrow [-1, 1]\), given by \(F_1(x, y) = \mu_Y\),
Let $U$ be a universe. It is well-known that a metric (or a distance) defined on $U$ is a function $d: U \times U \to \mathbb{R}$ satisfying the following properties (for every $x, y, z \in U$):

i) $d(x, y) \geq 0$.

ii) $d(x, y) = d(y, x)$.

iii) $d(x, y) = 0$ if and only if $x = y$.

iv) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

A set $T = \{x, y, z\}$ of three pairwise different points of $U$ is said to be a triangle in $U$. Then the points $x, y, z \in U$ are said to be the vertices of the triangle $T$. If $d$ is a metric defined on $U$, a triangle $T = \{x, y, z\}$ is said to be degenerate with respect to the metric $d$ if there exists a permutation $\{x_1, x_2, x_3\}$ of the vertices of $T$ such that $d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3)$ (see e.g. [23, 27] for further details).

Suppose now that $U$ is a universe. Let $F: U \times U \to \mathbb{R}$ be a bivariate function that satisfies the Sincov's functional equation. Assume in addition that $F(x, y) = 0$ if and only if $x = y$ holds for every $x, y \in U$. With these restrictions, it is straightforward to see that the function $d: U \times U \to [0, 1]$ given by $d(x, y) = |F(x, y)|$ for every $x, y \in U$ is actually a distance on $U$. Given $x, y, z \in U$ it holds that $F(x, y) + F(y, z) = F(x, z)$ because $F$ satisfies the Sincov's functional equation. But this last fact implies that any triangle in $U$ is degenerate as regards $d$. To see this, let $T = \{x, y, z\}$ be a triangle in $U$. Since $x \neq y$ it follows by hypothesis that $F(x, y) \neq 0$. We have that $F(x, y) = F(y, x) = F(x, z) = 0$ since $F$ satisfies the Sincov's functional equation. Therefore $F(x, y) = -F(y, z)$, so that we may assume that $d(x, y) = F(x, y) > 0$ without loss of generality. Three cases may occur now:

**Cases 1:** $F(y, z) > 0$.

In this case we have that $d(y, z) = F(y, z)$. Moreover, $F(x, z) = F(x, y) + F(y, z) > 0$. Hence $d(x, z) = F(x, z)$ and $d(x, z) = d(x, y) + d(y, z)$, so that $T$ is degenerate.

**Cases 2:** $F(y, z) < 0$ and $F(x, z) > 0$.

In this case we have that $F(z, y) = -F(y, z) > 0$. Thus $d(z, y) = d(y, z) = F(z, y)$. Also $d(x, z) = F(x, z) > 0$. Hence $F(x, z) = F(x, y) - F(y, z) = d(x, y) - d(y, z)$ so that $d(x, y) = d(x, z) + d(y, z)$ and $d(x, z) = d(x, y) + d(y, z)$ and $T$ is also degenerate.

**Cases 3:** $F(y, z) < 0$ and $F(x, z) < 0$.

In this last case we have again that $F(z, y) = -F(y, z) > 0$. Thus $d(z, y) = d(y, z) = F(z, y)$. In addition, $F(x, z) = -F(x, y) > 0$. Thus $d(x, z) = d(x, y) = F(x, y)$. Consequently, $d(y, z) = F(x, y) + F(x, z) = d(x, z) + d(x, y)$ and $T$ is degenerate, too.

Therefore, given a universe $U$, any bivariate function $F: U \times U \to [-1, 1]$ that satisfies the Sincov's functional equation such that $F(x, y) \neq 0$ if $x \neq y$ (and $x, y \in U$) induces on $U$, in a natural way, a distance for which all the triangles are degenerate.

**Example 5.4.** Let $U = \{a, b, c\}$. Let $F : U \times U \to [-1, 1]$ be given by $F(a, a) = F(b, b) = F(c, c) = 0$; $F(a, b) = F(b, c) = -F(b, a) = -F(c, b) = \frac{1}{4}$; $F(a, c) = -F(a, c) = 1$. By definition, the bivariate function $F$ satisfies the Sincov's functional equation.
The associated distance $d(x, y) = |F(x, y)|$ for every $x, y \in U$ is actually given by $d(a, a) = d(b, b) = d(c, c) = 0$; $d(a, b) = d(b, a) = d(b, c) = d(c, b) = \frac{1}{2}$; $d(a, c) = d(c, a) = 1$. The only triangle is the whole set $U$, and it is indeed degenerate since $d(a, c) = 1 = \frac{1}{2} + \frac{1}{2} = d(a, b) + d(b, c)$.

Throughout the identification between solutions of the Sincov’s functional equation and quasi-normal fuzzy subsets of a universe, already stated in Corollary 3.6, given a quasi-normal fuzzy subset $F$ of a universe $U$, it follows that the map $d : U \times U \to [0, 1]$ given by $d(x, y) = |\mu_F(x) - \mu_F(y)|, (x, y \in U)$ is a metric on $U$ satisfying the additional property of all the triangles of the universe $U$ being degenerate with respect to $d$.

5.4. Final comments, and suggestions for further studies

Summarizing, the original definition of a fuzzy subset has involved other additional structures. In the present manuscript, the structure furnished by a Sincov’s functional equation that comes naturally from the membership function of a fuzzy subset has been discussed.

A suggestion to have in mind for a possible continuation of the approach considered here could be the study of continuity as regards Sincov’s functional equations or else fuzzy subsets defined on a universe $U$ endowed with a topology $\tau$, so that if a membership function $\mu_F : U \to [0, 1]$ of a fuzzy subset $F$ of $U$, is continuous with respect to the topology $\tau$ on $U$ and the usual topology on $[0, 1]$, then it is said that $X$ is a $\tau$-continuous fuzzy subset of $U$. In the same way, one may study bivariate functions $F : U \times U \to [-1, 1]$ that satisfy the Sincov’s functional equation and are continuous with respect to the product topology $\tau \times \tau$ on $U \times U$ and the usual topology on $[-1, 1]$. This setting would obviously be related to the continuous representability of total preorder relations (see [9, 18, 19]), following the steps of Proposition 4.6.

Another suggestion for further research in a near future could be to explore other alternative structures that also come from the original definition (see [30]) of a fuzzy subset of a universe. For instance, one may be interested in analyzing some additional structure arising in terms of measure theory (see e.g. [14, 15]). To start with, one may notice that the Lebesgue measure on $(0, 1)$ immediately induces a measure on every fuzzy subset of a given universe.

Another possibility (see [3, 5, 21]) could be to explore the definition of a fuzzy subset in topological terms, starting from the nested topology induced by the $\alpha$-cuts of a fuzzy subset.

Finally, other functional equations somehow related to the Sincov’s functional equation could also be interpreted in terms of fuzzy subsets. In our opinion, if $U$ is a universe, a solution $F : U \times U \to [-1, 1]$ of the so-called separability functional equation, namely $F(x, y) + F(y, t) = F(x, t) + F(y, y)$ for every $x, y, t \in U$ (see e.g. [1, 7, 25] as well as subsection 5.3 in [22] for further details) is closely related to the existence of suitable pairs $(X, Y)$ of fuzzy subsets of $U$. The grounds are that an straightforward strengthening of the proof of Proposition 2.5. shows that $F$ can be decomposed as $F(x, y) = G(x) - H(y), (x, y \in U)$ for some functions $G, H : U \to \mathbb{R}$ that only depend on one single variable. Then, as in Theorem 3.1., one could try to find additional conditions in order to guarantee that $G$ and $H$ may be chosen taking values in $[0, 1]$ corresponding, each, to the membership functions of two fuzzy subsets $X$ and $Y$ of the universe $U$.

By similar reasons, perhaps the functional equation of separability could also be related to the existence of suitable rough sets (see [25]) of the universe $U$, since a rough set can be interpreted as a suitable pair of (crisp) sets which give a lower and an upper approximation of the original set $U$.

References