On the Hochschild cohomology of Beurling Algebras

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Abstract

Let $G$ be a locally compact group and let $\omega$ be a weight function on $G$. Under a very mild assumption on $\omega$, we show that $L^1(G, \omega)$ is $(2n+1)$-weakly amenable for every $n \in \mathbb{Z}^+$. Also for every odd $n \in \mathbb{N}$ we show that $H^2(L^1(G, \omega), (L^1(G, \omega))^{(n)})$ is a Banach space.

1 introduction

In this paper we shall be concerned with the structure of the first and second cohomology group of $L^1(G, \omega)$ with coefficients in the $n$th dual space $(L^1(G, \omega))^{(n)}$. We begin by recalling some terminology.

Let $\mathcal{A}$ be a Banach algebra, and $X$ be a Banach $\mathcal{A}$-bimodule. The dual space $X'$ is a Banach $\mathcal{A}$-bimodule where the products $a \cdot \lambda$ and $\lambda \cdot a$ are specified by

$$a \cdot \lambda(x) = \lambda(x \cdot a), \quad \lambda \cdot a(x) = \lambda(a \cdot x) \tag{1.1}$$

for all $a \in \mathcal{A}$, $x \in X$ and $\lambda \in X'$. The canonical embedding of $X$ in $X''$ is denoted by $\iota$ or $\hat{\cdot}$. We denote higher duals by $X^{(n+1)} = X^{(n)'}$ for all $n \in \mathbb{N}$; with the convention $X^{(0)} = X$. Then $X^{(n)}$ is also a Banach $\mathcal{A}$-bimodule; the definitions are consistent in the sense that $a \hat{\cdot} x = \hat{a} \cdot \hat{x}$. So that $X^{(n)}$ is a submodule of $X^{(n+2)}$. If $X$ is symmetric, then so is $X^{(n)}$. If $X$ is unital, then so is $X'$. The adjoint of the received by the editors June 2004 - In revised form in September 2004. Communicated by A. Valette. 2000 Mathematics Subject Classification : Primary 43A20; Secondary 46M20. Key words and phrases : weak amenability, cohomology, Beurling algebra.
injective map \( i : X^{(n-1)} \to X^{(n+1)} \) is the projective map \( P : X^{(n+2)} \to X^{(n)} \), defined by \( P(\Lambda) = \Lambda|_{i(X^{(n-1)})} \). Then \( P \) is a \( \mathcal{A} \)-bimodule morphism, and so we may write
\[
X^{(n+2)} = X^{(n)} \oplus \text{Ker} \ P = X^{(n)} \oplus (X^{(n-1)})^\perp,
\]
as Banach \( \mathcal{A} \)-bimodules. We shall also consider the second dual \( \mathcal{A}'' \) of a Banach algebra \( \mathcal{A} \) as a Banach algebra; indeed, two products are defined on \( \mathcal{A}'' \) as follows.

Let \( a \in \mathcal{A} \), \( \lambda \in \mathcal{A}' \) and \( m, n \in \mathcal{A}'' \). Then \( m \cdot \lambda \) and \( \lambda \cdot m \) are defined by
\[
m \cdot \lambda(a) = m(\lambda \cdot a), \quad \lambda \cdot m(a) = m(a \cdot \lambda),
\]
where \( \lambda \cdot a \) and \( a \cdot \lambda \) are defined by (1.1). Next \( m \Box n \) and \( m \circ n \) are defined in \( \mathcal{A}'' \) by
\[
m \Box n(\lambda) = m(n \cdot \lambda), \quad m \circ n(\lambda) = n(\lambda \cdot m).
\]

Then \( \mathcal{A}'' \) is a Banach algebra with respect to each of the products \( \Box \) and \( \circ \), which are called the first and second Arens products on \( \mathcal{A}'' \), respectively. For fixed \( n \) in \( \mathcal{A}'' \), the map \( m \to m \Box n \) is weak* weak* continuous, but map \( m \to n \circ m \) in general is not weak* weak* continuous unless \( m \) is in \( \mathcal{A} \).

The cohomology complex is
\[
0 \longrightarrow X \xrightarrow{\delta^0} C^1(\mathcal{A}, X) \xrightarrow{\delta^1} C^2(\mathcal{A}, X) \xrightarrow{\delta^2} \cdots,
\]
where for \( n \in \mathbb{Z}^+ \), \( C^n(\mathcal{A}, X) \) is the set of all bounded \( n \)-linear maps from \( \mathcal{A} \) to \( X \). The map \( \delta^0 : X \to C^1(\mathcal{A}, X) \) is given by \( \delta^0(x)(a) = a \cdot x - x \cdot a \) and for \( n \in \mathbb{Z}^+ \), the map \( \delta^n : C^n(\mathcal{A}, X) \to C^{n+1}(\mathcal{A}, X) \) is given by
\[
\delta^n T(a_1, \ldots, a_{n+1}) = a_1 \cdot T(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i T(a_1, \ldots, a_ia_{i+1}, \ldots a_{n+1}) + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1},
\]
where \( T \in C^n(\mathcal{A}, X) \) and \( a_1, \ldots, a_{n+1} \in \mathcal{A} \). The space ker \( \delta^n \) of bounded \( n \)-cocycle is denoted by \( Z^n(\mathcal{A}, X) \) and the space \( \text{Im} \delta^{n-1} \) of bounded \( n \)-coboundary is denoted by \( B^n(\mathcal{A}, X) \). We recall that \( B^n(\mathcal{A}, X) \) is a subspace of \( Z^n(\mathcal{A}, X) \) and that the \( n \)th cohomology group \( \mathcal{H}^n(\mathcal{A}, X) \) is defined by the quotient
\[
\mathcal{H}^n(\mathcal{A}, X) = \frac{Z^n(\mathcal{A}, X)}{B^n(\mathcal{A}, X)},
\]
which is called the \( n \)th Hochschild (continuous) cohomology of \( \mathcal{A} \) with coefficients in \( X \).

The \( n \)-cochain \( T \) is called cyclic if
\[
T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_0, a_1, \ldots, a_{n-1})(a_n),
\]
and we denote the linear space of all cyclic \( n \)-cochains by \( C^n_\lambda(\mathcal{A}, \mathcal{A}') \). It is well known (see [9]) that the cyclic cochains \( C^n_\lambda(\mathcal{A}, \mathcal{A}') \) form a subcomplex of \( C^n(\mathcal{A}, \mathcal{A}') \), that is \( \delta^n : C^n_\lambda(\mathcal{A}, \mathcal{A}') \to C^{n+1}_\lambda(\mathcal{A}, \mathcal{A}') \), and so we have cyclic versions of the spaces defined above, which we denote by \( B^n_\lambda(\mathcal{A}, \mathcal{A}') \), \( Z^n_\lambda(\mathcal{A}, \mathcal{A}') \) and \( \mathcal{H}^n_\lambda(\mathcal{A}, \mathcal{A}') \). Note that
it is usual to denote the cyclic cohomology group by $\mathcal{H}^n(A)$, as there is only one bimodule used, namely $A'$.

To show that $\mathcal{H}^n(A, X) = 0$, we must show that every $n$-cocycle from $A$ to $X$ is an $n$-coboundary. In particular case for $n = 1$, $Z^1(A, X)$ is the space of all continuous derivations from $A$ to $X$, and $B^1(A, X)$ is the space of all inner derivations from $A$ to $X$. Thus $\mathcal{H}^1(A, X) = 0$ if and only if each continuous derivation from $A$ to $X$ is inner.

The space $Z^n(A, X)$ is a Banach space, but in general $B^n(A, X)$ is not closed; we regard $\mathcal{H}^n(A, X)$ as a complete seminormed space with respect to the quotient seminorm. This seminorm is a norm if and only if $B^n(A, X)$ is a closed subspace of $C^n(A, X)$, which means that $\mathcal{H}^n(A, X)$ is a Banach space.

There have been very extensive studies devoted to calculation of the cohomology group $\mathcal{H}^1(A, X)$ and the higher dimensional groups $\mathcal{H}^n(A, X)$ for various classes of Banach algebras $A$ and Banach $A$-bimodules $X$. Our purpose here, being particularly concerned with the cohomology groups $\mathcal{H}^1(A, X^{(n)})$ and $\mathcal{H}^2(A, X^{(n)})$ for $n \in \mathbb{N}$.

A Banach algebra $A$ is called $n$-weakly amenable if $\mathcal{H}^n(A, A^{(n)}) = 0$. Note that 1-weakly amenable Banach algebras are called weakly amenable.

It was shown in [13] that $L^1(G)$ is weakly amenable for every locally compact group $G$; see also [6] for a shorter proof. Dales, Ghahramani and Grønbæk [5] showed that $L^1(G)$ is always $(2n + 1)$-weakly amenable for $n \in \mathbb{Z}^+$. Johnson [14] for the free group on two generators, proved that $\mathcal{H}^1(\ell^1(\mathbb{F}_2), (\ell^1(\mathbb{F}_2))^{(n)}) = 0$ for every $n \in \mathbb{N}$ and in [12] he proved that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \mathbb{C}) \neq 0$ which by [19, Theorem 8.3.1] implies that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^1(\mathbb{F}_2)) \neq 0$ and $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^\infty(\mathbb{F}_2)) \neq 0$.

In [11] Ivanov and in [15] Matsumoto and Morita showed that $\mathcal{H}^2(\ell^1(G), \mathbb{C})$ is a Banach space for every discrete group $G$ with trivial action on $\mathbb{C}$. A. Pourabbas [18] showed that the second cohomology group of $L^1(G)$ with coefficients in $L^1(G)^{(2n+1)}$ is a Banach space for every locally compact group $G$ and every $n \in \mathbb{Z}^+$. Meanwhile Soma [20] showed that $\mathcal{H}^3(\ell^1(\mathbb{F}_2), \mathbb{R})$ is not a Banach space. In [4] Burger and Monod showed that for a compactly generated locally compact second countable group $G$, the second continuous cohomology $\mathcal{H}^2_{cb}(G, F)$ is a Banach space, where $F$ is a separable coefficient module.

In this paper for every locally compact group $G$ and every $n \in \mathbb{Z}^+$, first we show that $\mathcal{H}^1(L^1(G, \omega), L^1(G, \omega)^{(2n+1)}) = 0$. Next we show that the second cohomology group of $L^1(G, \omega)$ with coefficients in $L^1(G, \omega)^{(2n+1)}$ is a Banach space, where $\omega$ is a weight function with $\sup \{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$. At the end we will give examples which show dependence of cohomology on the weight $\omega$.

2 The first cohomology group

Let $G$ be a locally compact group. A weight on $G$ is a continuous function $\omega : G \to (0, \infty)$ satisfying $\omega(e) = 1$, $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. We say that the weight $\omega$ is diagonally bounded if $\sup \{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$. Throughout for a diagonally bounded weight $\omega$ we set $Db(\omega) = \sup \{\omega(g)\omega(g^{-1}) : g \in G\}$. The
Beurling algebra $L^1(G,\omega)$ is defined as below,

$$L^1(G,\omega) = \left\{ f : G \to \mathbb{C} : f \text{ is measurable and } \|f\|_1^\omega = \int |f(x)| \omega(x) d(x) < \infty \right\}.$$ 

$L^1(G,\omega)$ is a Banach algebra with convolution product and norm $\|\cdot\|_1^\omega$. The dual space $L^\infty(G,\omega^{-1}) = (L^1(G,\omega))^\prime$ consists of all measurable functions $\varphi$ on $G$ with

$$\|\varphi\|^\infty_\omega = \text{ess sup} \left\{ \frac{|\varphi(g)|}{\omega(g)} : g \in G \right\} < \infty.$$ 

$L^1(G,\omega)$ has a bounded approximate identity $\{e_\alpha\}$, and by [2, Proposition 28.7], the Banach algebra $(L^1(G,\omega)^\prime, \Box)$ has a right identity element $E$ such that $\|E\| \leq M$, where $M = \sup_\alpha \|e_\alpha\|_1^\omega$.

The space $M(G,\omega)$ of all complex, regular Borel measures $\mu$ on $G$ with $\mu \cdot \omega \in M(G)^\prime$ with the convolution product and norm

$$\|\mu\|_\omega = \int \omega(x) d|\mu|(x)$$

is a Banach algebra. The weighted measure algebra $M(G,\omega)$ has a unit element $\delta_e$ and contains $L^1(G,\omega)$ as a closed two sided ideal. Also $M(G,\omega)_s = C_0(G,\omega^{-1})$ consists of all continuous functions on $G$ such that $\frac{f}{\omega} \in C_0(G)$.

**Lemma 2.1.** The multiplier algebra of $L^1(G,\omega)$ is isometrically isomorphic with $M(G,\omega)$.

**Proof.** The proof is similar to the proof $\Delta(L^1(G)) = M(G)$ [10, p. 276].

Let $\{\mu_\alpha\}$ be a net in $M(G,\omega)$ and $\mu \in M(G,\omega)$. We say that $\{\mu_\alpha\}$ tends to $\mu$ in so-topology if for every $f \in L^1(G,\omega)$, we have

$$\mu_\alpha \ast f \to \mu \ast f \quad \text{and} \quad f \ast \mu_\alpha \to f \ast \mu.$$

**Lemma 2.2.** Let $G$ be a locally compact group. Then the so-closed convex span of

$$\left\{ \frac{\lambda}{\omega(g)} \delta_g : g \in G, \lambda \in \mathbb{C}, |\lambda| = 1 \right\}$$

is the unit ball in $M(G,\omega)$.

**Proof.** The proof is the same as the unweighted case [8, 1.1.1-1.1.3].

**Note.** By the previous Lemma every measure $\mu$ in $M(G,\omega)$ is the so-limit of a net $\{\mu_\alpha\}$, where each $\mu_\alpha$ is a linear combination of point masses.

Now for every $n \in \mathbb{Z}^+$ we will show that $L^1(G,\omega)^{(2n+1)}_R$, the real-valued functions in $L^1(G,\omega)^{(2n+1)}$, is a complete lattice in the sense that every non-empty subset of $L^1(G,\omega)^{(2n+1)}$ which is bounded above has a supremum.
**Proposition 2.3.** The Banach space $L^\infty(G, \omega^{-1})$ with the product

$$f \cdot g(x) = \frac{f(x)g(x)}{\omega(x)}, \quad f, g \in L^\infty(G, \omega^{-1})$$

and complex conjugate as involution is a commutative C*-algebra.

**Proof.** Define $\varphi : L^\infty(G, \omega^{-1}) \to L^\infty(G)$ by $\varphi(f) = f\omega^{-1}$. Then $\varphi$ is a *-isometrical isomorphism from $L^\infty(G, \omega^{-1})$ onto $L^\infty(G)$. Thus $L^\infty(G, \omega^{-1})$ is a commutative C*-algebra. 

**Remark 2.4.** Set $X = L^1(G, \omega)$ with $(2^n)$ ($n \geq 1$). We note that $L^1(G, \omega)' = L^\infty(G, \omega^{-1})$ is a commutative C*-algebra. Because the second dual of a commutative C*-algebra is a commutative von Neumann algebra, then $X' = L^1(G, \omega)^{(2n+1)}$ is the underlying space of a commutative von Neumann algebra, and hence it is an $L^\infty$-space. The space $X_{\mathbb{R}}'$ of real-valued functions in $X'$ forms a complete lattice.

Throughout the rest of this section we set $A = L^1(G, \omega)$ and $X = A^{(2n+2)}$, where $n \in \mathbb{Z}^+$. The map

$$\theta : M(G, \omega) \to (A'', \Box), \quad \mu \mapsto E \Box \mu$$

is a continuous embedding. In fact for all $\mu \in M(G, \omega)$ we have

$$\|\theta(\mu)\| \leq \|\mu\|_{\omega} \|E\| \leq \|\mu\|_{\omega} M.$$ 

We write $E_s$ for $E \Box \delta_s$, where $s \in G$ and $E$ is a right identity for $(A'', \Box)$. If $D : A \to X'$ is a continuous derivation, then by [5, Proposition 1.7] $D'' : (A'', \Box) \to X'''$ is a continuous derivation.

**Lemma 2.5.** Let $\omega$ be a diagonally bounded weight on $G$. Then

(i) For every subset $B$ of $X''_{\mathbb{R}}$, and for every $r \in G$, we have

$$E \cdot \sup \{E_r \cdot \Lambda : \Lambda \in B\} = E_r \cdot \sup \{E \cdot \Lambda : \Lambda \in B\}$$

and

$$\sup \{E_r \cdot \Lambda : \Lambda \in B\} \cdot E = \sup \{E \cdot \Lambda : \Lambda \in B\} \cdot E_r.$$ 

(ii) The set $\{E_{s-1} \cdot \text{Re} D''(E_s) : s \in G\}$ is a bounded subset of $X''_{\mathbb{R}}$.

**Proof.** (i) Let $\alpha = \sup \{E \cdot \Lambda : \Lambda \in B\}$ and $\gamma = \sup \{E_r \cdot \Lambda : \Lambda \in B\}$. For all $\Lambda \in B$ we have $E_r \cdot \Lambda = E_r \cdot (E \cdot \Lambda) \leq E_r \cdot \alpha$. So

$$E \cdot \sup \{E_r \cdot \Lambda : \Lambda \in B\} \leq E_r \sup \{E \cdot \Lambda : \Lambda \in B\}.$$ 

Conversely

$$\alpha = \sup \{E \cdot \Lambda : \Lambda \in B\} = \sup \{E_{r^{-1}}(E_r \cdot \Lambda) : \Lambda \in B\} \leq E_{r^{-1}} \cdot E \cdot \gamma.$$
Thus $E_r \alpha \leq E \gamma$. By the same method we have

$$\sup \{E_r \Lambda : \Lambda \in B\} \cdot E = \sup \{E \Lambda : \Lambda \in B\} \cdot E_r.$$  \hspace{1cm}

(ii) Since $\|E_s\| \leq \omega(s)M$ for every $s \in G$, then

$$\|E_{s^{-1}} \cdot \Re D''(E_s)\| = \|\Re(E_{s^{-1}} \cdot D''(E_s))\|
\leq \|E_{s^{-1}} \cdot D''(E_s)\| \leq \|E_{s^{-1}}\| \|D''\| \|E_s\|
\leq \omega(s)\omega(s^{-1}) \|D''\| M^2 \leq Db(\omega) \|D''\| M^2.$$

Thus $\{E_{s^{-1}} \cdot \Re(D''(E_s)) : s \in G\}$ is a bounded subset of $X''_R$. \hspace{1cm} \blacksquare

**Theorem 2.6.** Let $G$ be a locally compact group. Then $L^1(G, \omega)$ is a $(2n + 1)$-weakly amenable for every $n \in \mathbb{Z}^+$, whenever $\omega$ is a diagonally bounded weight on $G$.

**Proof.** Set $\mathcal{A} = L^1(G, \omega)$ and $X = L^1(G, \omega)^{(2n)}$. The result in [17] establishes the case $n = 1$ and we may suppose that $n \in \mathbb{N}$. Let $\{e_n\}$ be a bounded approximate identity for $\mathcal{A}$. Then there exists a right identity $E$ for $(\mathcal{A}''$, $\square)$ such that $\|E\| \leq M$.

Since $\mathcal{A}$ is a closed ideal of $M(G, \omega)$, then by [7] $(\mathcal{A}''$, $\square)$ is a closed ideal of $(M(G, \omega)''$, $\square)$. Let $D \in L^1(A, X')$. Then $D'' : (\mathcal{A}''$, $\square) \rightarrow X''$ is a continuous derivation. For $r, s \in G$ we have

$$D''(E_{st}) = D''(E_s) \cdot E_t + E_s \cdot D''(E_t)$$

and so

$$E_{(st)^{-1}} \cdot D''(E_{st}) = E_{t^{-1}} \cdot (E_{s^{-1}} \cdot D''(E_s)) \cdot E_t + E_{t^{-1}} \cdot D''(E_t). \hspace{1cm} (2.1)$$

By Lemma 2.5(ii) the set $\{E_{s^{-1}} \cdot \Re D''(E_s) : s \in G\}$ is bounded in $X''_R$. Since $X''_R$ is a complete lattice, then

$$\phi_r = \sup \{E_{s^{-1}} \cdot \Re(D''(E_s)) : s \in G\} \hspace{1cm} (2.2)$$

exists in $X''_R$. Let $t \in G$. Then from (2.1), (2.2) and Lemma 2.5(i) we have

$$E \cdot \phi_r \cdot E = E_{t^{-1}} \cdot \phi_r \cdot E_t + E_{t^{-1}} \cdot \Re D''(E_t) \cdot E.$$  \hspace{1cm}

Hence

$$E \cdot \Re D''(E_t) \cdot E = E_{t} \cdot \phi_t \cdot E - E \cdot \phi_r \cdot E_t.$$  \hspace{1cm}

Similarly, by considering imaginary parts we obtain $\phi_t \in X''_R$ such that

$$E \cdot \Im D''(E_t) \cdot E = E_{t} \cdot \phi_t \cdot E - E \cdot \phi_t \cdot E_t.$$  \hspace{1cm}

Thus if we define $\phi = \phi_r + \phi_t$, then $\phi \in X''_R$ and for all $t \in G,$

$$E \cdot D''(E_t) \cdot E = E_t \cdot \phi \cdot E - E \cdot \phi \cdot E_t.$$
If \( \nu \) is a linear combination of point masses and \( f, g \in \mathcal{A} \), then we have

\[
f \cdot D''(E \square \nu) \cdot g = (f * \nu) \cdot \phi \cdot g - f \cdot \phi \cdot (\nu * g).
\]

(2.3)

Now take \( h \in \mathcal{A} \). Then there is a net \( \{\nu_\alpha\} \) of linear combination of point masses such that \( \nu_\alpha \to h \) in the strong operator topology on \( \mathcal{A} \), that is, \( \lim_\alpha (f * \nu_\alpha) = f * h \) and \( \lim_\alpha (\nu_\alpha * g) = h * g \) for every \( f, g \in \mathcal{A} \).

Let \( f, g \in \mathcal{A} \). Then

\[
\lim_{\alpha} f \cdot D''(E \square \nu_\alpha) \cdot g = \lim_{\alpha} (D''(f * \nu_\alpha) \cdot g - D''(f) \cdot (\nu_\alpha * g)) = f \cdot D''(h) \cdot g.
\]

So, from (2.3) we have

\[
f \cdot D''(h) \cdot g = (f * h) \cdot \phi \cdot g - f \cdot \phi \cdot (h * g) = f \cdot (h \cdot \varphi - \varphi \cdot h) \cdot g.
\]

Let \( P : X'' \to X' = \mathcal{A}^{(2k+1)} \) be the natural projection, so that \( P \) is an \( \mathcal{A} \)-bimodule morphism. We have \( D = P \circ D'' \). Set \( \phi_0 = P(\phi) \). Then

\[
f \cdot D(h) \cdot g = f \cdot (h \cdot \phi_0 - \phi_0 \cdot h) \cdot g
\]

for every \( f, g, h \in \mathcal{A} \), and so

\[
D(h)(f \cdot x \cdot g) = (h \cdot \phi_0 - \phi_0 \cdot h)(f \cdot x \cdot g)
\]

for every \( f, g, h \in \mathcal{A} \) and \( x \in X \). Now by [5, proposition 1.17] we have \( D(h)(x) = (h \cdot \phi_0 - \phi_0 \cdot h)(x) \). Then \( D \) is an inner derivation and so \( \mathcal{A} \) is \((2k+1)-\)weak amenable.

\[ \blacksquare \]

### 3 The second cohomology group

In this section firstly we prove that \( \mathcal{H}^2(\ell^1(G, \omega), \ell^1(G, \omega)^{(2n+1)}) \) is a Banach space for every discrete group \( G \). Secondly we will generalize this method to show that \( \mathcal{H}^2(L^1(G, \omega), (L^1(G, \omega))^{(2n+1)}) \) is a Banach space for every locally compact group \( G \). Recall that we set \( D \circ \omega = \sup \{ \omega(g)\omega(g^{-1}): g \in G \} \).

**Theorem 3.1.** \( \mathcal{H}^2(\ell^1(G, \omega), \ell^1(G, \omega)^{(2n+1)}) \) is a Banach space for every discrete group \( G \) and for every diagonally bounded weight \( \omega \).

**Proof.** Set \( X = \ell^1(G, \omega)^{(2n)} \). Let \( \psi \in C^1(\ell^1(G, \omega), X') \). Then for every \( g, h \in G \) and \( s \in X \) with \( \|s\| \leq 1 \) we have

\[
|\delta \psi(g, h)(s)| = |\psi(g)(hs) - \psi(gh)(s) + \psi(h)(sg)| \leq ||\delta \psi|| \omega(g)\omega(h).
\]

(3.1)

Since the set \( \{\text{Re} \psi(g) \cdot g^{-1} : g \in G\} \) is bounded above by \( ||\psi|| D \circ \omega \) in \( X' \). Then

\[
f_r(s) = \sup_{g \in G} \left\{ \text{Re} \psi(g)(g^{-1} s) \right\}
\]
exists in $X_R$. For every $h \in G$ by (3.1) we have

$$f_r(hs) = \sup_{g \in G} \left\{ \text{Re} \psi(g)(g^{-1}hs) \right\}$$

$$= \sup_{g \in G} \left\{ \text{Re} \psi(hg)(g^{-1}s) \right\}$$

$$\leq \sup_{g \in G} \left\{ \text{Re} \psi(h)(s) + \text{Re} \psi(g)(g^{-1}sh) + \|\delta \psi\| \omega(g)\omega(g^{-1})\omega(h) \right\}$$

(3.2)

$$= \text{Re} \psi(h)(s) + \sup_{g \in G} \left\{ \text{Re} \psi(g)(g^{-1}sh) \right\} + \|\delta \psi\| \omega(h)Db(\omega)$$

$$= \text{Re} \psi(h)(s) + f_r(sh) + \|\delta \psi\| \omega(h)Db(\omega).$$

On the other hand

$$f_r(hs) = \sup_{g \in G} \left\{ \text{Re} \psi(g)(g^{-1}hs) \right\}$$

$$\geq \text{Re} \psi(h)(s) + f_r(sh) - \|\delta \psi\| \omega(h)Db(\omega).$$

(3.3)

From (3.2) and (3.3) we have

$$|h \cdot f_r(s) - f_r \cdot h(s) + \text{Re} \psi(h)(s)| \leq \|\delta \psi\| \omega(h)Db(\omega).$$

Similarly, by considering imaginary parts we have

$$|h \cdot f_i(s) - f_i \cdot h(s) + \text{Im} \psi(h)(s)| \leq \|\delta \psi\| \omega(h)Db(\omega).$$

By putting $f = f_r + if_i$ we obtain

$$|h \cdot f(s) - f \cdot h(s) + \psi(h)(s)| \leq 2 \|\delta \psi\| \omega(h)Db(\omega).$$

Now let us define

$$\tilde{\psi}(h)(s) = (\delta f)(h)(s) + \psi(h)(s),$$

so $\delta \tilde{\psi} = \delta \psi$ and $|\tilde{\psi}(h)(s)| \leq 2 \|\delta \psi\| \omega(h)Db(\omega)||s||$ for every $h \in G$ and $s \in X$. Thus $\|\tilde{\psi}\| \leq 2 \|\delta \psi\| Db(\omega)$ and this finishes the proof.

Lemma 3.2. The cyclic cohomology group $\mathcal{H}_3^2(\ell^1(G, \omega))$ is a Banach space for every discrete group $G$ and for every diagonally bounded weight $\omega$.

Proof. Let $\psi \in C^1(\ell^1(G, \omega), \ell^\infty(G, \omega^{-1}))$ such that $\psi(h)(g) = -\psi(g)(h)$ for $g, h \in G$, and let us consider $\tilde{\psi}(h)(g) = (\delta f)(h)(g) + \psi(h)(g)$ as in Theorem 3.1. Then $\delta \tilde{\psi} = \delta \psi$ and $\|\tilde{\psi}\| \leq 2 \|\delta \psi\| Db(\omega)$, further

$$\tilde{\psi}(h)(g) = (\delta f)(h)(g) + \psi(h)(g)$$

$$= - (\delta f)(g)(h) - \psi(g)(h)$$

$$= -\tilde{\psi}(g)(h).$$

Hence $\mathcal{H}_3^2(\ell^1(G, \omega))$ is a Banach space.
We can now state the final result of this paper, we show that the cohomology group \( H^2(L^1(G, \omega), L^1(G, \omega)^{(2n+1)}) \) is a Banach space for every locally compact group \( G \) and every diagonally bounded weight \( \omega \).

We recall a construction that shows that \( L^\infty(G, \omega^{-1}) \) is an \( M(G, \omega) \)-bimodule. For \( f \in L^\infty(G, \omega^{-1}), a \in L^1(G, \omega) \) and \( \mu \in M(G, \omega) \) define the module actions by
\[
(f \mu)(a) = f(\mu * a) \quad \text{and} \quad (\mu f)(a) = f(a * \mu).
\]

Throughout this section the notations \( \limsup \) and \( \liminf \) are frequently simplified to \( \lim \) and \( \lim \). We denote by so-\( \lim \mu_n \) the limits of measures in the strong operator topology.

**Proposition 3.3.** Set \( X = L^1(G, \omega)^{(2n)} \). Let \( \psi \in C^1(L^1(G, \omega), X') \). Then there is a \( \tilde{\psi} \in C^1(M(G, \omega), X') \) with

(i) \( \tilde{\psi}|_{L^1(G, \omega)} = \psi \) and \( \delta \tilde{\psi}|_{L^1(G, \omega) \times L^1(G, \omega)} = \delta \psi. \)

(ii) Let \( \mu \) be in \( M(G, \omega) \) with \( \|\mu\|_\omega \leq 1 \) and let \( x \) be in \( X \) with \( \|x\| \leq 1 \) and \( a, b \in L^1(G, \omega) \) with \( \|a\|_1 \leq 1 \) and \( \|b\|_1 \leq 1 \). If \( \{\mu_n\} \) is a net in \( M(G, \omega) \) with \( \|\mu_n\|_\omega \leq 1 \) such that so-\( \lim \mu_n = \mu \), then
\[
\left( \liminf_n \ Re \tilde{\psi}(\mu_n)(a \cdot x \cdot b) + i \liminf_n \ Im \tilde{\psi}(\mu_n)(a \cdot x \cdot b) - \tilde{\psi}(\mu)(a \cdot x \cdot b) \right) \leq 3 \|\delta \tilde{\psi}\|.
\]

**Proof.** (i) We follow the proof of [12, Lemma 1.10] for this particular case. Let \( \mu \in M(G, \omega) \) and let \( \{e_\alpha\} \) be a bounded approximate identity for \( L^1(G, \omega) \) with bound \( M \). Defining
\[
\tilde{\psi}_\alpha(\mu) = \psi(\mu * e_\alpha)
\]
we see that \( \tilde{\psi}_\alpha \) is a bounded net in \( C^1(M(G, \omega), X') \) and so has a cofinal subnet \( \tilde{\psi}_\beta \) convergent to a limit \( \tilde{\psi} \) in the weak*-topology induced by identifying \( C^1(M(G, \omega), X') \) with \( C_1(M(G, \omega), X') \). Thus
\[
\lim_\beta \psi(\mu * e_\beta)(x) = \tilde{\psi}(\mu)(x)
\]
for all \( \mu \in M(G, \omega), x \in X \). Since for all \( a \in L^1(G, \omega) \), \( \psi(a * e_\beta) \to \psi(a) \) in norm, \( \tilde{\psi}|_{L^1(G, \omega)} = \psi \). Also \( \delta \tilde{\psi}|_{L^1(G, \omega) \times L^1(G, \omega)} = \delta \psi. \)

To prove (ii) let us consider \( \mu, \nu \in M(G, \omega) \) with \( \|\mu\|_\omega, \|\nu\|_\omega \leq 1 \) and \( x \in X \) with \( \|x\| \leq 1 \). Then
\[
\left| \delta \tilde{\psi}(\mu, \nu)(x) \right| = \left| \mu \cdot \tilde{\psi}(\nu)(x) - \tilde{\psi}(\mu * \nu)(x) + \tilde{\psi}(\mu) \cdot \nu(x) \right| \leq \|\delta \tilde{\psi}\| . \tag{3.4}
\]

For \( a, b \in L^1(G, \omega) \) with \( \|a\|_1 \leq 1, \|b\|_1 \leq 1 \) and \( x \in X \) with \( \|x\| \leq 1 \) by (3.4)
\[
- Re \tilde{\psi}(\mu_\alpha)(a \cdot x \cdot b) = - Re \tilde{\psi}(\mu_\alpha) \cdot a(x \cdot b)
\leq Re \mu_\alpha \cdot \psi(a)(x \cdot b) - Re \psi(\mu_\alpha * a)(x \cdot b) + \|\delta \tilde{\psi}\|
\]
and so
\[
- \lim Re \tilde{\psi}(\mu_\alpha)(a \cdot x \cdot b) \leq \lim \left\{ Re \mu_\alpha \cdot \psi(a)(x \cdot b) - Re \psi(\mu_\alpha * a)(x \cdot b) + \|\delta \tilde{\psi}\| \right\}
= Re \mu \cdot \psi(a)(x \cdot b) - Re \psi(\mu * a)(x \cdot b) + \|\delta \tilde{\psi}\|.
\]
On the other hand
\[- \lim \operatorname{Re} \hat{\psi} (\mu_a) (a \cdot x \cdot b) \geq \operatorname{Re} \mu \cdot \psi (a) (x \cdot b) - \operatorname{Re} \psi (\mu * a) (x \cdot b) - \| \delta \hat{\psi} \| .\]

Hence
\[\left| \mu \cdot \operatorname{Re} \psi (a) (x \cdot b) - \operatorname{Re} \psi (\mu * a) (x \cdot b) + \lim \operatorname{Re} \hat{\psi} (\mu_a) (a \cdot x \cdot b) \right| \leq \| \delta \hat{\psi} \|.\]

Similarly for imaginary parts we have
\[\left| \mu \cdot \operatorname{Im} \psi (a) (x \cdot b) - \operatorname{Im} \psi (\mu * a) (x \cdot b) + \lim \operatorname{Im} \hat{\psi} (\mu_a) (a \cdot x \cdot b) \right| \leq \| \delta \hat{\psi} \|.\]

Therefore
\[\left| \mu \cdot \psi (a) (x \cdot b) - \psi (\mu * a) (x \cdot b) + \left( \lim \operatorname{Re} \hat{\psi} (\mu_a) + i \lim \operatorname{Im} \hat{\psi} (\mu_a) \right) (a \cdot x \cdot b) \right| \leq 2 \| \delta \hat{\psi} \|. \tag{3.5}\]

but from (3.4) we also have
\[\left| \mu \cdot \psi (a) (x \cdot b) - \psi (\mu * a) (x \cdot b) + \psi (\mu) (a \cdot x \cdot b) \right| \leq \| \delta \hat{\psi} \|. \tag{3.6}\]

Hence (3.5) and (3.6) imply that
\[\left| \left( \lim \operatorname{Re} \hat{\psi} (\mu_a) + i \lim \operatorname{Im} \hat{\psi} (\mu_a) \right) (a \cdot x \cdot b) - \psi (\mu) (a \cdot x \cdot b) \right| \leq 3 \| \delta \hat{\psi} \|. \]

\[\tag*{\blacksquare}\]

**Proposition 3.4.** [18, Proposition 3.1] Let $A$ be a Banach algebra with a bounded approximate identity, and let $X$ be a Banach $A$-bimodule. Let $\hat{\psi} \in C^1 (A, X')$ such that $| \psi(a)(b \cdot x \cdot c)| \leq \| \hat{\psi} \|$ for every $x \in X$ with $\| x \| \leq 1$ and $a, b, c \in A$ with $\| a \| \leq 1$, $\| b \| \leq 1$ and $\| c \| \leq 1$. Then there exists $\hat{\psi} \in X'$ such that
\[| \psi(a)(x) - \hat{\psi}(a)(x)| \leq 5 \| \hat{\psi} \|.\]

**Theorem 3.5.** Let $G$ be a locally compact group, and let $\omega$ be a diagonally bounded weight on $G$. Then $H^2 (L^1 (G, \omega), L^1 (G, \omega)^{(2n+1)})$ is a Banach space for every $n \in \mathbb{Z}^+$. 

**Proof.** Set $X = L^1 (G, \omega)^{(2n)}$. Let $\phi \in C^1 (L^1 (G, \omega), X')$ and let us consider $\hat{\phi} \in C^1 (M(G, \omega), X')$ as in Proposition 3.3. Set
\[S = \left\{ \operatorname{Re} \delta_g \cdot \hat{\phi}(\delta_g) : g \in G \right\};\]

Since $S$ is bounded above by $\| \hat{\phi} \| Db(\omega)$ in $X'_R$, the complete vector lattice of real valued functions in $X'$, then $\psi_r = \sup_{g \in G} S$ exists in $X'_R$.

For every $h \in G$ and $x \in X$ with $\| x \| \leq 1$ by (3.4) we have
\[
\delta_h \cdot \psi_r (x) = \sup_{k \in G} \left\{ \operatorname{Re} (\delta_h * \delta_{k-1}) \cdot \hat{\phi}(\delta_k)(x) \right\} = \sup_{g \in G} \left\{ \operatorname{Re} \delta_g \cdot \hat{\phi}(\delta_g \cdot \delta_h)(x) \right\}
\leq \sup_{g \in G} \left\{ \operatorname{Re} (\delta_{g-1} * \delta_g) \cdot \hat{\phi}(\delta_h)(x) + \operatorname{Re} \delta_{g-1} \cdot \hat{\phi}(\delta_g) \cdot \delta_h(x) \right\} + \| \hat{\phi} \| Db(\omega) \omega(h)
\leq \operatorname{Re} \hat{\phi}(\delta_h)(x) + \psi_r \cdot \delta_h(x) + \| \hat{\phi} \| Db(\omega) \omega(h),
\]
where $hk^{-1} = g^{-1}$. On the other hand,
\[ \delta_h \cdot \psi_r(x) \geq \Re \tilde{\phi}(\delta_h)(x) + \psi_r \cdot \delta_h(x) - \| \delta \tilde{\phi} \| Db(\omega)\omega(h). \]

Therefore,
\[ \| \delta_h \cdot \psi_r(x) - \psi_r \cdot \delta_h(x) - \Re \tilde{\phi}(\delta_h)(x) \| \leq \| \delta \tilde{\phi} \| Db(\omega)\omega(h). \]

Now if $\mu_\alpha = \sum_{i=1}^n \alpha_i \delta_{h_i}$, then by (3.7)
\[ |\mu_\alpha \cdot \psi_r(x) - \psi_r \cdot \mu_\alpha(x) - \Re \tilde{\phi}(\mu_\alpha)(x)| \]
\[ \leq \sum_{i=1}^n |\alpha_i| \| \delta_{h_i} \cdot \psi_r(x) - \psi_r \cdot \delta_{h_i}(x) - \Re \tilde{\phi}(\delta_{h_i})(x) \| \]
\[ \leq \sum_{i=1}^n |\alpha_i| \| \delta \tilde{\phi} \| Db(\omega)\omega(h_i) \leq \| \delta \tilde{\phi} \| Db(\omega) \| \mu_\alpha \|_\omega. \] (3.8)

Similarly, by considering imaginary parts we obtain $\psi_\iota$ such that
\[ |\mu_\alpha \cdot \psi_\iota(x) - \psi_\iota \cdot \mu_\alpha(x) - \Im \tilde{\phi}(\mu_\alpha)(x)| \leq \| \delta \tilde{\phi} \| Db(\omega) \| \mu_\alpha \|_\omega. \] (3.9)

Since every $h$ in $L^1(G,\omega)$ with $\| h \|_\omega^r \leq 1$ is the so-limit of a net $\{ \mu_\alpha \}$ with $\| \mu_\alpha \|_\omega \leq 1$, where every $\mu_\alpha$ is a linear combination of point masses, then by (3.8) and (3.9) for every $x \in X$ with $\| x \| \leq 1$ and $a, b \in L^1(G,\omega)$ with $\| a \|_\omega^r \leq 1$ and $\| b \|_\omega^r \leq 1$ we have
\[ \| (h \cdot \psi - \psi \cdot h)(a \cdot x \cdot b) - \left( \limsup \Re \tilde{\phi}(\mu_\alpha) + i \liminf \Im \tilde{\phi}(\mu_\alpha) \right)(a \cdot x \cdot b) \| \leq 2 \| \delta \tilde{\phi} \| Db(\omega) \]
where $\psi = \psi_r + i \psi_\iota$. Now by Proposition 3.3 (ii), we have
\[ \left\| \left( \limsup \Re \tilde{\phi}(\mu_\alpha) (a \cdot x \cdot b) + i \liminf \Im \tilde{\phi}(\mu_\alpha) (a \cdot x \cdot b) \right) - \phi(h)(a \cdot x \cdot b) \right\| \leq 3 \| \delta \tilde{\phi} \|. \]

Thus
\[ \| (h \cdot \psi - \psi \cdot h)(a \cdot x \cdot b) - \phi(h)(a \cdot x \cdot b) \|
\leq \| (h \cdot \psi - \psi \cdot h)(a \cdot x \cdot b) - \left( \limsup \Re \tilde{\phi}(\mu_\alpha) + i \liminf \Im \tilde{\phi}(\mu_\alpha) \right) (a \cdot x \cdot b) \|
\]
\[ + \left\| \left( \limsup \Re \tilde{\phi}(\mu_\alpha) + i \liminf \Im \tilde{\phi}(\mu_\alpha) \right)(a \cdot x \cdot b) - \phi(h)(a \cdot x \cdot b) \right\| \]
\[ \leq \| \delta \tilde{\phi} \| (2Db(\omega) + 3). \]

Now by Proposition 3.4 there exist $\hat{\phi} \in X'$ such that
\[ \| (h \cdot \psi - \psi \cdot h)(x) - \delta \hat{\phi}(h)(x) - \phi(h)(x) \| \leq 5 \| \delta \tilde{\phi} \| (2Db(\omega) + 3) \]

Define
\[ \tilde{\psi}(h)(x) = -\delta \psi(h)(x) - \delta \hat{\phi}(h)(x) + \phi(h)(x). \]

Then $\delta \tilde{\psi} = \delta \hat{\phi}$ and $\| \tilde{\psi}(h)(x) \| \leq 5 \| \delta \tilde{\phi} \| (2Db(\omega) + 3)$ for every $h \in L^1(G,\omega)$ with $\| h \|_\omega^r \leq 1$ and $x \in X$ with $\| x \| \leq 1$. So $\| \tilde{\psi} \| \leq 5 \| \delta \tilde{\phi} \| (2Db(\omega) + 3)$ and this completes the proof. \[\square\]
**Theorem 3.6.** \( \mathcal{H}^2(L^1(G, \omega)) \) is a Banach space for every locally compact group \( G \) and for every diagonally bounded weight \( \omega \).

**Proof.** Let \( \phi \in C^1(L^1(G, \omega), L^\infty(G, \omega^{-1})) \) be such that for \( a, b \in L^1(G, \omega) \)

\[
\phi(a)(b) = -\phi(b)(a).
\]

By the proof of Theorem 3.5 there exists \( \tilde{\psi} \in C^1(L^1(G, \omega), L^\infty(G, \omega^{-1})) \) defined by

\[
\tilde{\psi}(b)(a) = -\delta \tilde{\psi}(b)(a) + \phi(b)(a)
\]

such that \( \delta \tilde{\psi} = \delta \phi \) and for a constant \( M \),

\[
\|\tilde{\psi}\| \leq M \|\delta \phi\|
\]

and obviously \( \tilde{\psi}(b)(a) = -\tilde{\phi}(a)(b) \).

**Example 3.7.** [17, Example 3.15] It is well known that for \( F_2 \), the free group on two generators, the second unbounded cohomology \( H^2(F_2, \mathbb{R}) \) is trivial [3, Example 4.3 and Example 1 on page 58]. So all bounded 2-cocycles have the form \( \phi(g, h) = \psi(g) - \psi(gh) + \psi(h) \) for some possibly unbounded \( \psi \). We define

\[
\omega(g) = \begin{cases} 
\exp(K - \psi(g)) & \text{if } g \neq e \\
1 & \text{otherwise},
\end{cases}
\]

where \( K \) is a bound for \( \phi \), we get a weight on \( F_2 \) such that \( \sup\{\omega(g)\omega(g^{-1})\} < \infty \). Thus \( \mathcal{H}^2(\ell^1(F_2, \omega), \ell^\infty(F_2, \omega^{-1})) \) is a Banach space. In the case \( \omega = 1 \) as noted in the Introduction \( \mathcal{H}^2(\ell^1(F_2), \ell^\infty(F_2)) \neq 0 \) and by [18] it is a Banach space.

**Example 3.8.** Bade et al. [1] studied the Beurling algebra \( \ell^1(\mathbb{Z}, \omega_\alpha) \). They defined a weight \( \omega_\alpha \) on \( \mathbb{Z} \) by \( \omega_\alpha(n) = (1 + |n|)^\alpha \) and they proved

(i) If \( \alpha > 0 \), then \( \ell^1(\mathbb{Z}, \omega_\alpha) \) is not amenable.

(ii) If \( 0 \leq \alpha < 1/2 \), then \( \ell^1(\mathbb{Z}, \omega_\alpha) \) is weakly amenable.

(iii) If \( \alpha \geq 1/2 \), then \( \ell^1(\mathbb{Z}, \omega_\alpha) \) is not weakly amenable.

Note that if \( \alpha = 0 \), then \( \omega = 1 \) and \( \ell^1(\mathbb{Z}, \omega_\alpha) = \ell^1(\mathbb{Z}) \) is an amenable algebra [2, §43.3]. Thus by [12] \( \mathcal{H}^n(\ell^1(\mathbb{Z}), X') = 0 \) for every Banach \( \ell^1(\mathbb{Z}) \)-bimodule \( X \) and every \( n \geq 1 \). In [16] the second author showed that \( \mathcal{H}^2(\ell^1(\mathbb{Z}, \omega_\alpha), \mathbb{C}) \neq 0 \) for every \( \alpha > 0 \), then by [19] \( \mathcal{H}^2(\ell^1(\mathbb{Z}, \omega_\alpha), \ell^\infty(\mathbb{Z}, \omega_\alpha)) \neq 0 \). Note that \( \omega_\alpha \) is not diagonally bounded. So Theorem 3.5 is not applicable. We do not know whether \( \mathcal{H}^2(\ell^1(\mathbb{Z}, \omega_\alpha), \ell^\infty(\mathbb{Z}, \omega_\alpha)) \) is a Banach space or not.

**Acknowledgment.** The authors express their thanks to the referee for his valuable comments and bringing references [4] and [20] to the authors attention.
References


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