New analytical method for solving Burgers’ and nonlinear heat transfer equations and comparison with HAM

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A B S T R A C T
In this study, we present a numerical comparison between the differential transform method (DTM) and the homotopy analysis method (HAM) for solving Burgers’ and nonlinear heat transfer problems. The first differential equation is the Burgers’ equation serves as a useful model for many interesting problems in applied mathematics. The second one is the modeling equation of a straight fin with a temperature dependent thermal conductivity. In order to show the effectiveness of the DTM, the results obtained from the DTM is compared with available solutions obtained using the HAM [M.M. Rashidi, G. Domairry, S. Dinarvand, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 708–717; G. Domairry, M. Fazeli, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 489–499] and whit exact solutions. The method can easily be applied to many linear and nonlinear problems. It illustrates the validity and the great potential of the differential transform method in solving nonlinear partial differential equations. The obtained results reveal that the technique introduced here is very effective and convenient for solving nonlinear partial differential equations and nonlinear ordinary differential equations that we are found to be in good agreement with the exact solutions.

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1. Introduction

Most phenomena in real world are described through nonlinear equations. Nonlinear phenomena play important roles in applied mathematics, physics and in engineering problems in which each parameter varies depending on different factors. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations (NLPDEs) in physics and mathematics, it is still a hot spot to seek new methods to obtain new exact or approximate solutions. Large class of nonlinear equations does not have a precise analytic solution, so numerical methods have largely been used to handle these equations. There are also some analytical techniques for nonlinear equations. Some of the classic analytic methods are the Lyapunov’s artificial small parameter method, perturbation techniques and δ-expansion method. In the recent years, many authors mainly had paid attention to study solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM), tanh method, homotopy perturbation method (HPM), sinh–cosh method, HAM, DTM and variational iteration method (VIM).

In 1992, Liao [1] employed the basic ideas of homotopy in topology to propose a general analytic method for nonlinear problems, namely the HAM. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions of perturbation methods. In recent years, the HAM has been successfully employed to solve many types of nonlinear problems, see [2–6] and the references therein.

The concept of differential transform method method was first introduced by Zhou [7] in 1986 and it was used to solve both linear and nonlinear initial value problems in electric circuit analysis. The main advantage of this method is that it can be applied directly to NLPDEs without requiring linearization, discretization, or perturbation. It is a semi analytical–numerical technique that formulizes Taylor series in a very different manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. Not like the traditional high order Taylor series method that requires symbolic computation, the DTM is an iterative procedure for obtaining Taylor series solutions. Another important advantage is that this method reducing the size of computational work while the Taylor series method is computationally taken long time for large orders. This method is well addressed in [23–30].

The Burgers’ equation is a nonlinear partial differential equation of second order. Burgers’ equation was first introduced by Bateman [8] and then treated by Burgers’ [9,10] as a mathematical model for turbulence. This equation has a large variety of applications in modeling of water in unsaturated soil, dynamic of soil water, statistics of flow problems, mixing and turbulent diffusion, cosmol-
ogy and seismology [11–13]. The Burgers’ equation is a nonlinear equation, very similar to the Navier–Stokes equation and there is analogy between the Burgers’ equation and Navier–Stokes equation due to the form of nonlinear terms. This single equation has a convection term, a diffusion term, and a time-dependent term.

In Burgers’ equation, discontinuities may appear in finite time, even if the initial condition is smooth. They give rise to the phenomenon of shock waves with important applications in physics [14]. These properties make Burgers’ equation a proper model for testing numerical algorithms in flows where severe gradients or shocks are anticipated. Several numerical methods to solve this system have been given such as algorithms based on cubic spline function technique [15], the explicit–implicit method [16], Adomian’s decomposition method [17]. The variational iteration method was used to solve the 1D Burgers’ and coupled Burgers’ equations [18]. Rashidi and et al. [19] used the HAM to solve Burgers’ equation.

Fins or extended surfaces are frequently used to enhance the rate of heat transfer from the primary surface. The rectangular fin is widely used, probably, due to simplicity of its design and it is less difficult in manufacturing process. However, it is well-known that the rate of heat transmission from a fin base diminishes along its length. Kern and Kraus [20] made an extensive review on this issue. Aziz and Hug [21] used the regular perturbation method and a numerical solution method to compute a closed form solution for a straight convective fin with temperature-dependent thermal conductivity. The HAM was used by Domainy and Fazeli to solve Rectangular purely convective fin with temperature dependent thermal conductivity [22].

In this paper, we extend the application of the differential transform method to construct analytical approximate solutions of the Burgers’ equation (5) and the modeling equation of a straight fin with a temperature-dependent thermal conductivity equation (15). Then we compare our results with the previously obtained results by using the HAM in [19,22] and exact solutions. With this technique, it is possible to obtain highly accurate results or exact solutions for differential equations.

2. Basic idea of the differential transform method

The basic definitions and fundamental operations of the two-dimensional differential transform are defined in [23–30]. Consider a function of two variables w(x,y) be analytic in the domain Ω and let (x, y) = (x₀, y₀) in this domain. The function w(x, y) is then represented by one series whose centre at located w(x₀, y₀). The differential transform of the function is the form

\[ W(k, h) = \frac{1}{k!h!} \left[ \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} \right]_{(x₀, y₀)}, \]  

where w(x, y) is the original function and W(k, h) is the transformed function. The transformation is called T-function and the lower case and upper case letters represent the original and transformed functions respectively. Then its inverse transform is defined as

\[ w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h)(x-x₀)^k(y-y₀)^h. \]  

The relations (1) and (2) imply that

\[ w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \left[ \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} \right]_{(x₀, y₀)} (x-x₀)^k(y-y₀)^h. \]  

In a real application and when (x₀, y₀) are taken as (0,0), then the function w(x, y) is expressed by a finite series and Eq. (2) can be written as

\[ w(x, y) = \sum_{k=0}^{m} \sum_{h=0}^{n} W(k, h)x^k y^h, \]  

in addition, Eq. (4) implies that \( \sum_{k=m+1}^{\infty} \sum_{h=n+1}^{\infty} W(k, h)x^k y^h \) is negligibly small. Usually, the values of m and n are decided by convergences of the series coefficients.

3. Application

3.1. The Burgers’ equation

Consider the Burgers’ equation [31]

\[ u_t + u u_x - u_{xx} = 0, \quad x \in \mathbb{R}, \]  

with the exact solution [32]

\[ u(x, t) = \frac{1}{2} - \frac{1}{2} \tan \left( \frac{1}{4} \left[ x - \frac{1}{2} \right]^2 \right), \]  

and with the initial condition

\[ u(x, 0) = \frac{1}{2} - \frac{1}{2} \tan \left( \frac{x}{4} \right). \]  

Taking the two-dimensional transform of Eq. (5) by using the related definitions in Table 1, we have

\[ (h+1)U(k, h+1) + \sum_{r=0}^{h} \sum_{s=0}^{k} (r+1)(k-r+1)U(r, h-s)U(k-r+1, s) \]  

\[ - (k+1)(k+2)U(k+2, h) = 0, \]  

by applying the differential transform into Eq. (7), the initial transformation coefficients are thus determined by

<table>
<thead>
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<th>Table 1</th>
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<tr>
<td>The operations for the two-dimensional differential transform method.</td>
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<table>
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<tr>
<th>Original function</th>
<th>Transformed function</th>
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<tbody>
<tr>
<td>w(x₀, y₀) = u(x₀, y₀) + v(x₀, y₀)</td>
<td>W(k, h) = U(k, h) + V(k, h)</td>
</tr>
<tr>
<td>w(x₀, y₀) = λu(x₀, y₀)</td>
<td>W(k, h) = λU(k, h), λ is a constant</td>
</tr>
<tr>
<td>w(x₀, y₀) = \frac{\partial u(x₀, y₀)}{\partial x}</td>
<td>W(k, h) = (k + 1)U(k + 1, h)</td>
</tr>
<tr>
<td>w(x₀, y₀) = \frac{\partial u(x₀, y₀)}{\partial y}</td>
<td>W(k, h) = (h + 1)U(k, h + 1)</td>
</tr>
<tr>
<td>w(x₀, y₀) = u(x₀, y₀)v(x₀, y₀)</td>
<td>W(k, h) = (k + 1)(k + 2)...(k + r)h(h + 1)(h + 2)...(h + s)U(k + r, h + s)</td>
</tr>
<tr>
<td>w(x₀, y₀) = \frac{\partial u(x₀, y₀)}{\partial x}</td>
<td>W(k, h) = \sum_{r=0}^{h} \sum_{s=0}^{k} U(r, h-s)V(k-r, s)</td>
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<tr>
<td>w(x₀, y₀) = \frac{\partial u(x₀, y₀)}{\partial y}</td>
<td>W(k, h) = \sum_{r=0}^{h} \sum_{s=0}^{k} (r+1)(k-r+1)U(r+1, h-s)V(k-r+1, s)</td>
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<tr>
<td>w(x₀, y₀) = \frac{\partial^2 u(x₀, y₀)}{\partial x^2}</td>
<td>W(k, h) = \sum_{r=0}^{h} \sum_{s=0}^{k} (r+1)(k-r+1)(h+1)(h+2)...(h+s)U(k+r, h+s)</td>
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<tr>
<td>w(x₀, y₀) = \frac{\partial^3 u(x₀, y₀)}{\partial x^3}</td>
<td>W(k, h) = \sum_{r=0}^{h} \sum_{s=0}^{k} (r+1)(k-r+1)(h+1)(h+2)...(h+s)U(k-r+1, s)</td>
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<tr>
<td>w(x₀, y₀) = \frac{\partial^3 u(x₀, y₀)}{\partial x^2 \partial y}</td>
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</table>
follows in Table 2. Hence, substituting all have series solution as below

\[
\sum_{k=0}^{\infty} U(k, 0)x^k = \frac{1}{2} \frac{x}{8} + \frac{x^3}{384} - \frac{x^5}{15360} + \frac{17x^7}{10321920} - \frac{31x^9}{743178240} + \frac{691x^{11}}{653996851200} + \cdots
\]  

(9)

Hence from Eq. (9)

\[
U(k, 0) = 0, \quad \text{if } k = 2, 4, 6, \ldots.
\]

\[
U(1, 0) = \frac{1}{8}, \quad U(3, 0) = \frac{1}{384},
\]

\[
U(5, 0) = -\frac{1}{15360}, \quad \ldots.
\]  

(10)

Substituting Eq. (10) in Eq. (8), and by recursive method we can calculating another values of \(U(k, h)\), some results are listed as follows in Table 2. Hence, substituting all \(U(k, h)\) into Eq. (4) we have series solution as below

\[
u(x, t) \cong \sum_{k=0}^{m} \sum_{h=0}^{n} U(k, h)x^kt^h
\]

\[
= U(0, 0) + U(1, 0)x + U(0, 1)t
\]

\[
+ U(1, 1)x + \cdots + U(m, n)x^mt^n
\]

\[
= \frac{1}{2} + \frac{t}{16} - \frac{t^3}{3072} + \frac{t^5}{491520} - \frac{t^7}{8} + \frac{t^2}{512} - \frac{t^4}{49152} - \frac{t^6}{256} + \frac{t^3x^2}{12888} + \frac{t^5x^2}{15728640} + \frac{x^3}{384} + \frac{t^2x^3}{6144} + \frac{t^4x^3}{4718592} + \frac{t^6x^3}{6144}
\]

\[
- \frac{2359296}{15360} + \frac{188743680}{15360} - \frac{31t^4x^4}{x^5} + \frac{1966080}{94371840} + \cdots
\]  

(11)

Our approximation has one more interesting property, if we expand exact solution (6) using Taylor’s expansion about \((0, 0)\), we have the series same as the our approximation (11).

In Fig. 1, we study the diagrams of the results obtained by the DTM for \(m = 50, n = 50\) in comparison with the HAM [19] and exact solution (6).

Note that the solution series obtained by the HAM contains the auxiliary parameter \(h\), which provides us with a simple way to adjust and control the convergence of the solution series. As pointed

<table>
<thead>
<tr>
<th>Example 3.1</th>
<th>Exact</th>
<th>HAM(h=0.6)</th>
<th>HAM(h=1)</th>
<th>DTM</th>
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<tr>
<td>(x = 0.25)</td>
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<td>(x = 0.75)</td>
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<td>(x = 1)</td>
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<td>(x = 1.5)</td>
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</table>

Fig. 1. The results obtained by the DTM \((m = 50, n = 50)\) Eq. (11) and the HAM [19] by 9th-order approximate solution, in comparison with the exact solution (6), when \(5 \leq t \leq 12\), (a) \(x = 0.25\); (b) \(x = 0.75\); (c) \(x = 1\); (d) \(x = 1.5\).
\[ h = -\bar{h} x (\text{HAM obtained by HPM, which proposed in 1998 by Dr. He}) \]

Therefore, the value of the HAM logically contains the HPM. We can find that the best solution for the case of \( t \leq 6 \) is insulated. The one-dimensional energy balance equation is given as:

\[ \frac{d}{dx} \left[ \frac{1}{\lambda} \frac{dT}{dx} \right] - Ph(T_b - T_a) = 0. \]  

In [19], the valid region of \( h \) in this case is \(-1.6 < h < -0.4\). When \( h = -1 \), the solution by the HAM is the same solution series obtained by HPM, which proposed in 1998 by Dr. He [33]. Therefore, the HAM logically contains the HPM. We can find that the best value of \( h \) in this case is \(-0.6\).

In Table 3, we present a numerical comparison between the HAM (\( h = -0.6 \)) by 9th-order approximate solution, the DTM (\( m = 50, n = 50 \)), and exact solution (6) in this case of \( x = 0.5, x = 1.5 \) and \( 0 \leq t \leq 12 \).

Table 3 indicates that the results obtained by the DTM for the case of \( 0 \leq t \leq 8 \) have nine digits precision with the exact solutions.

In Figs. 2 and 3, we present the comparison of the errors in answers results by the DTM (\( m = 50, n = 50 \)) and the HAM (\( h = -0.6 \)) by 9th-order approximate solution at \( t = 10 \) and \( t = 11 \) respectively for the case of \(-1 \leq x \leq 1\). Considering these two figures, we find out that errors of the DTM are very less than those of the HAM even for large \( t \).

### 3.2. Rectangular purely convective fin with temperature dependent thermal conductivity

Consider a straight fin with a temperature-dependent thermal conductivity, arbitrary constant cross-sectional area \( A_c \), perimeter \( P \), and length \( b \) (see Fig. 4). The fin is attached to a base surface of temperature \( T_b \), extends into a fluid of temperature \( T_a \) and its tip is insulated. The one-dimensional energy balance equation is given as:

\[ \frac{d}{dx} \left[ \frac{1}{\lambda} \frac{dT}{dx} \right] - Ph(T_b - T_a) = 0. \]  

The thermal conductivity of the fin material is assumed a linear function of temperature according to

\[ k(T) = k_a \left[ 1 + \lambda(T - T_a) \right]. \]  

where \( k_a \) is the thermal conductivity at the ambient fluid temperature of the fin and \( k \) is the parameter describing the variation of the thermal conductivity.

Employing the following dimensionless parameters

\[ \theta = \frac{T - T_a}{T_b - T_a}, \quad \zeta = \frac{x}{b}, \quad \beta = \lambda(T_b - T_a), \quad \psi = \left( \frac{h P b^2}{k_a A_c} \right)^{1/2}. \]  

The formulation of the problem reduces to

\[ \frac{d^2\theta}{d\zeta^2} + \frac{d\theta}{d\zeta} + \beta \left( \frac{d\theta}{d\zeta} \right)^2 - \psi^2 \theta = 0. \]  

Fig. 2. The comparison of the errors in answers results by the DTM (\( m = 50, n = 50 \)) and the HAM (\( h = -0.6 \)) by 9th-order approximate solution at \( t = 10 \).

Fig. 3. The comparison of the errors in answers results by the DTM (\( m = 50, n = 50 \)) and the HAM (\( h = -0.6 \)) by 9th-order approximate solution at \( t = 11 \).

Fig. 4. Geometry of a straight fin.
Related definitions, we have

\[ \Theta(\zeta) = \frac{\psi}{\zeta} = \theta = 1 \]

... (20)

The comparison of the dimensionless temperature errors in answers results by the DTM and the HAM [22] were considered for \( \psi = 0.5 \) and \( \psi = 1 \). Note that the solution series obtained by the HAM contains the auxiliary parameter \( h \), which influence its convergence region and rate. We should therefore focus on the choice of \( h \) by plotting of errors in answers results by the HAM for some values of \( h \).

As pointed in [22], the valid region of \( h \) for the case of \( \psi = 0.5 \) and constant thermal conductivity is \(-1.3 < h < 0\) and for the case of \( \psi = 1 \) and constant thermal conductivity is \(-1.6 < h < -0.1\).

In Fig. 5, we present the comparison of the errors in answers results by the DTM \( (m = 15) \) and the HAM \( (h = -0.78, \bar{h} = -0.785) \).
and $h = -0.791$) by 12th-order approximate solution for the case of the constant thermal conductivity.

From Table 4 and Fig. 5, it can be concluded that approximate the DTM expression presented as a solution in this study gives better results than the HAM approximation. We observe that the best value of $h$ is $-0.791$ for the case of $(\psi = 1, \beta = 0)$ Fig. 5 and $-0.92$ for the case of $(\psi = 0.5, \beta = 0)$.

When conductivity varies with temperature, Eq. (15) becomes a nonlinear equation for which analytical solution is not available. Hence, in this paper a second analysis is also conducted via the classical fourth-order Runge–Kutta for the purpose of testing this method. In Fig. 6, dimensionless temperature distribution is compared for $\psi = 1$ with various $\beta$ values. From Fig. 6 it is seen that, when the problem becomes nonlinear the obtained results by those the DTM and the HAM agreement with numerical results.

4. Conclusions

In this paper, we presented a reliable algorithm based on the DTM to solve some nonlinear equations. Some examples are given to illustrate the validity and accuracy of this procedure. The present method reduces the computational difficulties of the other methods (same as the HAM, VIM, ADM and HPM) and all the calculations can be made simple manipulations. The accuracy of the method is very good. The method has been applied directly without requiring linearization, discretization, or perturbation. The obtained results demonstrate the reliability of the algorithm and gives it a wider applicability to nonlinear differential equations.

References