Image denoising in the presence of non-Gaussian, power-law noise

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Abstract—In image processing, noise is usually modeled as white Gaussian noise to represent general sensor and environmental clutter, and many effective methods have been developed to remove Gaussian noise. We show here that in many situations, such as Terahertz (THz) images or under-water images distorted by wavy surface, noise may be highly non-Gaussian, and even heavy-tailed with power-law distributions. We perceive that such noise may be ubiquitous, such as in images obtained by radar, LIDAR, satellite, and electro-optical visual cameras, in unsteady environments. We show that such noise cannot be effectively reduced by even the best method (block-matching 3D transformation, BM3D) for removing Gaussian noise. A fundamental issue arises of how to develop a proper framework to aptly deal with such non-Gaussian noise. We propose a viable new approach using power-law analysis, and evaluate its effectiveness using well-known images in computer vision community. We show that the new approach, which we call thresholding-median filtering and BM3D (TM-BM3D), works effective on all known types of noise, Gaussian, salt and pepper, and power-law noise.

I. INTRODUCTION

Image denoising is a fundamental issue in pattern recognition [1]. Despite extensive research in the past several decades, image denoising still poses a significant challenge, due to the great variety of natural images together with many types of noise sources contained in the images.

Among the best image denoising approaches is the transform-domain denoising, which typically assumes that the signal is sparsely represented in the transform domain. Albeit straightforward fixed 2D transform may not achieve good sparsity, the basic idea can be effectively utilized by a technique called block matching 3D transformation (BM3D) [2], which first partitions an image into a few groups composed of similar blocks, then performs 3D transformation on each group. While the BM3D can be slightly improved when an image is severely contaminated by noise [3], overall, BM3D can be considered to be among the best image denoising techniques for Gaussian noise. The basic reasons for this optimality are (1) The block-matching is akin to identifying recurrence patterns in nonlinear dynamics; recurrence time statistics are among the most powerful complexity measures, embodying periodivity, information dimension, and Kolmogorov and Renyi entropies [4], [5], [6], [7], [8]; and (2) the transform domain thresholding, which is the essence of compressive sensing, is aptly utilized on the block-matched image fragments, and thus, the approach may be termed suitable sub-space compressive sensing.

When the noise is not Gaussian, we have the classic median filter at our disposal [9], which is especially good at reducing salt and pepper noise. A fundamental question is, what are the effective denoising approaches if the noise is neither Gaussian nor salt and pepper such that both BM3D and the median filter may not be effective? One can readily perceive that such noises could be very relevant in situations involving radar, LIDAR, satellite, and electro-optical visual cameras, in unsteady environments [10].

One particularly interesting non-Gaussian noise source is the heavy-tailed, power-law distributed noise with diverging moments, and possibly with long-range correlations. Such noise resonates well with the ubiquitous power-law distributions in various areas of science and engineering as reviewed by [15]. Such noise also resonates with the prevalent long-range correlations in sea clutter [16], number theory [17], astronomy [18], [19], [20], [21], communications [22], [23], [24], [25], [26], [27], genetics [28], [29], [30], cognition [31], vision [32], [33], [34], radar analysis [10], [35], [36], [37], physiological signals [38], [39], [40], human response to natural and social phenomena [41], and image fusion [42], [43], [44].

As we will show in the paper, even without considering correlations, such noise, when it is large, is not the same as the salt and pepper noise, and thus cannot be effectively removed by the median filter. Our main goal in this paper is to develop an effective approach to remove such noise. For simplicity, we shall not consider correlations here, since correlations can vary tremendously across images, and thus generic results are hard to obtain. Such a simplification would also enable other researchers to readily repeat our results and apply our technique to their problems involving non-Gaussian noise.

The remainder of the paper is organized as follows. In Sec. II, we present examples of images with non-Gaussian noises. In Sec. III, we define our problem of denoising involving power-law and present a viable approach. In Sec. IV, we summarize our findings.
II. IMAGE DENOISING WITH NON-GAUSSIAN NOISE: 
MOTIVATIONAL EXAMPLES

Non-Gaussian noise may exist in a great variety of images. In this section, we present a few important scenarios.

The first example shown in Fig. 1 is the terahertz (THz) images [45], [46], [47], [48]. While the images taken at normal direction, which are shown as the 1st and the 3rd images of Fig. 1, have good quality, the 2nd and the 4th images shown there were taken by rotating 30°, and have very poor quality. In fact, even an eye can barely identify the object in the images. It is thus not surprising that the noise in such poor quality THz images cannot be Gaussian. This can be readily verified if one estimates the probability density function (PDF) of the pixel values of the images shown in Fig. 1. More interestingly, if one computes the eigenvalue spectrum of these images, one obtains the power-law spectra shown in the last picture in Fig. 1 — the power-law spectra suggests that a large number of eigenvalues and eigenvectors (or eigen-images) are needed to reconstruct the original images, and therefore, the dimension of these images cannot be reduced substantially.

Our second example is the famous cascade multifractals, which are obtained by recursively splitting a given positive number, e.g. \( w \), in Fig. 2(a), into four parts, \( w_{ir} \), \( j = 1, 2, 3, 4 \), with the rule that \( r_{ij} \), \( j = 1, 2, 3, 4 \) follow the same PDF. For more details of such a cascade multifractal model, we refer to [49] (see also [23], [24], [25], [26], [27]). Such an operation yields a highly non-uniform image, as shown in Fig. 2(b). In fact, when Fig. 2(b) is integrated, a beautiful landscape, which may be used to model tomography, emerges, as shown in Fig. 2(c). What is interesting here is that the pixel values in Fig. 2(b) gives a power-law distribution, as shown in Fig. 2(d). Since the cascade model is a good model for many natural scenes, we may conclude that power-law distribution is relevant to many natural images.

As our third example, we discuss an important type of medical images, the ultrasound image. Fig. 3(left) shows an example. When the pixel values there are used to estimate a PDF, we obtain the picture shown in the right of Fig. 3. The shape suggests a log-normal distribution. In fact, if we take logarithm of the pixel values before we estimate the PDF, then we get a bell-shaped PDF, very similar to the Gaussian distribution. This means that the distribution here is indeed log-normal [49]. Under fairly general conditions, a log-normal distribution can yield a power-law distribution [49]. Hence, we see again that non-Gaussian, or more specifically, log-normal or even power-law distributions are very relevant to ultrasound images.

As our last example, we discuss sub-surface floating mines. As one can readily visualize, Fig. 4(a) is the truth or the mine under a still water surface. When the water surface becomes wavy, we would obtain Fig. 4(b) — depending on the environmental disturbances, the distortion could be even bigger. The differenced image between Fig. 4(a,b) is shown in Fig. 4(c), whose PDF is shown in Fig. 4(d), which follows nicely a power-law distribution. Since this scenario is typical of disturbed natural environments, we can conclude that power-law disturbance is prevalent in natural images obtained in unsteady environments such as the three examples of (1) clothing obscuration in THz imaging, (2) multifractal landscapes, and (3) sea clutter. In addition, we can easily perceive that images of birds or flowers on windy


III. DENOISING POWER-LAW NOISE

A. Scope of the work

The images shown in Sec. II not only contain power-law noise, but also have long-range correlations. In this work, however, we will not discuss the correlations in the noise, since correlation structure would differ vastly among natural images, and thus generic results are difficult to obtain. In order for our work to be easily repeated, we will focus on the popular test images shown in Fig. 4. Specifically, we shall add power-law noise to these images, where a power-law (or Pareto distribution) is defined as

\[
P[X \geq x] = \left( \frac{\sigma}{x} \right)^\alpha, \quad x \geq \sigma > 0
\]  

(1)

where \( \sigma \) and \( \alpha > 0 \) are parameters. When \( \alpha < 2 \), the distribution has infinite variance. When \( \alpha \leq 1 \), even the mean becomes infinite. In this study, we only consider \( \alpha \leq 2 \). Note that \( \sigma \) is not related to variance of the random variable \( X \), since theoretically variance is infinite here. Because of this, the sample variance for the noise in an corrupted image would change from one realization to another. Therefore, a variable such as the sample variance of the noise in an image is not a good metric to quantify the amount of noise. In the following, we will always utilize the original parameters \( \sigma \) and \( \alpha \).

B. Ineffectiveness of BM3D and the median filter for denoising power-law noise in images

In order to develop suitable filters to denoise power-law noise in images, it is important first to examine how effective BM3D, the best Gaussian noise denoising filter, and median filter may reduce power-law noise in images. We first check the BM3D approach to the text images with additive power-law noise [2].

BM3D is based on an enhanced sparse representation in the transform-domain. The enhancement of the sparsity is achieved by grouping similar 2D image fragments or blocks into 3D data arrays which are called groups by the original authors. One important element in BM3D is the collaborative filtering, which is a special procedure developed to deal with these 3D groups. It consists of three successive steps: 3D transformation of 3D group, shrinkage of transform spectrum, and an inverse 3D transformation. The result is a 3D estimate that consists of the jointly filtered image blocks. The filtered blocks are then returned to their original positions. Note there is overlapping, and thus redundancy, in image blocks. This redundancy may be exploited through aggregation, and using a Wiener filter.

Fig. 6 shows an example of power-law noise denoising using BM3D, where the left picture is the noisy image, and the right the filtered image. We observe that filtering using BM3D, while not reducing the power-law noise, has further distorted the noisy image. Therefore, BM3D is not effective in reducing power-law noise. Note that the BM3D does not claim to reduce all types of noise and we have just illustrated that while it is good for Gaussian noise reduction, it is probably not as effective for natural scene noise analysis.

Next we examine whether median filter may be able to
reduce power-law noise in images. It turns out if power-law noise is small, the median filter is quite effective. However, when noise is large, the median filter is not an effective choice. An example is shown in Fig. 7, where the left picture is the noisy image, and the right one is the denoised image. Overall, we may conclude that a small amount of power-law noise might be treated similarly to salt and pepper noise but a large amount of power-law could not be removed.

C. A 3-stage filter for denoising power-law noise

The success and failure of the median filter in reducing small and large amounts of power-law noise in images, respectively, motivates us to develop a multi-step filtering procedure that first manages to make power-law noise more like salt and pepper noise, then apply a median filter. Indeed, these are the two basic steps. However, these are not enough. We still need some elements from BM3D. It turns out that Wiener filtering, which is a crucial element in BM3D, is not useful in dealing with power-law noise. Therefore, that part is left out. To summarize, our filtering procedure, which we call thresholding-median filtering and BM3D (TM-BM3D), consists of three steps:

- Step 1: Assign zero to a pixel if its value is larger than a chosen threshold. It turns out there is a generic choice for the threshold, independent of the testing images. For normalized clean images with pixel values ranging from 0 to 1, the generic threshold value for noisy images is 2, noting that with unbounded power-law noise, the pixel values in the resulting noisy image can be much larger than 2.
- Step 2: Apply the median filter
- Step 3: Apply the first stage filter of BM3D. This consists of (i) block-wise estimates, which finds similar image blocks and stacks them together in a 3D array; (ii) application of a 3D transform to the formed group, hard-thresholding of the transform coefficients, and inversion of the 3D transform to produce estimates of all grouped blocks; and (iii) returning of the estimates of the blocks to their original positions. The basic operations involved in this step are summarized in the schematic shown in Fig. 8.

Now let us examine the effectiveness of the 3-stage filter for reducing power-law noise in images. Fig. 9 shows an example of small amount of power-law noise, where the left picture is the noisy image, and the right one the cleaned image. We observe that the filter is very effective. Note that even though the median filter alone is also effective in this case, it is not as good as the 3-stage filter proposed here.

Next, we consider denoising large amount of power-law noise. An example is shown in Fig. 10, where the left and right are the noisy and cleaned images, respectively. We observe that the 3-stage filter is again quite effective. In particular, comparing with the image filtered by the median filter, which is shown in Fig. 7(right), we clearly see that the image filtered by the 3-stage filter, TM-BM3D, is much better than that filtered by the median filter.

D. Performance evaluation

It is important to systematically study the performance of the 3-stage filter proposed here for reducing power-law noise in images. As pointed out, sample variance is not a good measure of the amount of noise, since the theoretical variance diverges and the sample variance can vary substantially from one realization of the noise to another. We thus will use the two original parameters, $\alpha$ and $\sigma$ to quantify the strength of noise. To describe the effect of filtering, we shall employ the commonly used Peak signal to Noise Ratio (PSNR) described...
the performance of BM3D in the case of Gaussian noise to eight different test images shown in Fig. 5. For comparison, that eight difference curves are plotted together, corresponding \( \alpha \) with \( \text{MSE} \) where \( \text{MAX} \) is the maximum possible value of an image filtered by BM3D is also plotted in Fig. 11. Again, there are eight curves there, corresponding to eight test images. The fact that the eight curves collapse tightly on each other indicates that the performance with Gaussian noise is basically independent of the test images. This feature is basically preserved with power-law noise, except for small \( \sigma \). Note that in the case of Gaussian noise, \( \sigma \) is indeed the standard deviation of the noise, while for the case of power-law noise, \( \sigma \) is a scale parameter defined in Eq. 1.

From Fig. 11, we clearly observe that the performance with power-law noise is much worse than the case of Gaussian noise. Recalling that the theoretical power-law noise variance is infinite, and observing that the curves in Fig. 11(right) are basically parallel to those for Gaussian noise, we have a good reason to conjecture that the 3-stage filter is close to be optimal for reducing power-law noise.

We have also examined how the performance of the 3-stage filter depends on the \( \alpha \) parameter. The results are shown in Fig. 12. We observe that for small \( \sigma \), the performance is worse with decreasing \( \alpha \). This makes sense, since the moments for the power-law noise with order equal to or larger than \( \alpha \) diverges, therefore, more moments diverge with decreasing \( \alpha \). However, overall, the effect of \( \alpha \) is not large. Moreover, as with the case of \( \alpha = 1.5 \) shown in Fig. 11 (red curves), the performance is fairly independent of the test images. This means similar performance results are likely to be observed with other test images.

**IV. CONCLUSIONS AND DISCUSSIONS**

Power-law distributions and fractal behavior in science and engineering are ubiquitous, and thus are likely to be observed in a great variety of images that can be obtained in nature, especially in situations including radar, LIDAR, satellite, and electro-optical visual cameras, in unsteady environments. This consideration has motivated us to study denoising of power-law noise in images. We have shown the BM3D, the best filter for removing Gaussian noise, is not effective at reducing power-law noise. While the median filter performs better than BM3D in this scenario, the results are far from satisfactory. We thus have proposed a 3-stage filter, Thresholding-Median filtering and BM3D (TM-BM3D), which consists of (1) physical space hard-thresholding to make noise salt and pepper like, (2) application of the median filter, and (3) application of the 1st stage of BM3D filter. Dependence of the performance of the 3-stage filter on the parameters of power-law noise has been systematically examined. Albeit the performance is worse than the case of Gaussian noise, we have good reason to conjecture that the performance is likely close to optimal, given the difficulty of removing power-law noise. Therefore, the technique proposed here may be very useful in a variety of situations involving information retrieving from noisy images, which would enhance further processing such as target and pattern recognition.

**REFERENCES**
